MAT 520 HW3

1. See Theorem 4.5 in the lecture note.

2. Suppose X^* is reflexive but $X \neq X^{**}$. Denote $J : X \to X^{**}$ the canonical map. Let $f \in X^{**} \setminus J(X)$, and define $\ell : J(X) + \mathbb{C}f \to \mathbb{C}$ as $\ell(J(x) + tf) = t$. Note ℓ is bounded since $J(X)$ is closed in X^{**}, where $\inf_x \|J(x) + f\| = \delta > 0$ and $\inf_x \|J(x) + tf\| = |t|\delta$. Extend ℓ using Hahn-Banach to a bounded linear functional $\tilde{\ell}$ on X^{**}. Write $\psi : X^* \to X^{***}$. Since $X^* = X^{***}$, there is $\lambda \in X^*$ such that $\psi(\lambda) = \tilde{\ell}$. In particular $\psi(\lambda)(J(x)) = \tilde{\ell}(J(x)) = \ell(J(x)) = 0$, and this implies
$$0 = \psi(\lambda)(J(x)) = J(x)(\lambda) = \lambda(x)$$
for all $x \in X$. Thus $\lambda = 0$. However, $\tilde{\ell} \neq 0$, which is a contradiction.

3. This is similar to the previous one. Suppose $f^* : Y \to X^*$ is not surjective. Choose $\lambda_0 \in X^* \setminus f^*(Y)$, we define $\ell : f^*(Y) + \mathbb{C}\lambda_0$ as $\ell(f^*(y) + t\lambda_0) = t$. Extend ℓ using Hahn-Banach to $\tilde{\ell} \in X^{**}$. Since X is reflexive, there is x such that $J(x) = \tilde{\ell}$, where $J : X \to X^{**}$ is the canonical map. Then
$$J(x)(f^*(y)) = f^*(y)(x) = f(x)(y) = 0$$
for all $y \in Y$. Since $f : X \to Y^*$ is isometric and hence injective, then $f(x) = 0$ and hence $x = 0$. Thus $\tilde{\ell} = 0$, which is a contradiction. We must have $Y \cong X^*$ and $Y^* \cong X^{**} \cong X^*$.

4. Apply Baire's category theorem to $S = \bigcup_n (L^{1+1/n}([0,1]) \cap S)$ to conclude that there is $L^{1+1/n}([0,1]) \cap S$ that has nonempty interior in S. There is a ball B in $L^1([0,1])$ such that $B \cap S \subset L^{1+1/n}([0,1]) \cap S$. This implies that $S = L^{1+1/n}([0,1])$.

5. Define the subspace $M \subset \ell^\infty$ and the convex function p as in the problem. Let $\ell : M \to \mathbb{R}$ be the linear functional defined as $\ell(\psi) = \lim_n \Lambda_n \psi$. Use the Hahn-Banach theorem to extend ℓ to a linear functional $\Lambda : \ell^\infty \to \mathbb{R}$ such that $\Lambda(\psi) \leq p(\psi)$ for any $\psi \in \ell^\infty$. We have $L\psi - \psi \in M$ and hence $\Lambda(L\psi - \psi) = \ell(L\psi - \psi) = 0$ which implies $\Lambda L = \Lambda$. Indeed
$$\Lambda_n(L\psi - \psi) = \frac{\psi(2) + \cdots + \psi(n+1)}{n} - \frac{\psi(1) + \cdots + \psi(n)}{n} = -\frac{\psi(1)}{n} + \frac{\psi(n+1)}{n} \to 0$$
We show that $p(\psi) \leq \limsup_n \psi(n)$ for all ψ. Suppose for contradiction that $p(\psi) - \epsilon > \limsup_n \psi(n)$ for some ψ. Then $p(\psi) - \epsilon \geq \psi(n)$ for all $n \geq N$ for

some N. Now

$$\frac{1}{n}\sum_{j=1}^{n}\psi(j) = \frac{1}{n}\sum_{j=1}^{N-1}\psi(j) + \frac{1}{n}\sum_{j=N}^{n}\psi(j) \le \frac{1}{n}\sum_{j=1}^{N-1}\psi(j) + \frac{n-N}{n}(p(\psi)-\epsilon)$$

Thus $p(\psi) = \limsup \Lambda_n\psi \le p(\psi) - \epsilon$ which is a contradiction.

6. See Claim 5.16 in the lecture note.

7. See HW2 problem 15.