Functional Analysis
Princeton University MAT520
HW3, Due Sep 29th 2023 (auto extension until Oct 1st 2023)

September 23, 2023

1. Prove the $\mathbb{C}$-Hahn-Banach theorem using the $\mathbb{R}$-Banach theorem. In particular you have to setup the forgetful functor which maps a $\mathbb{C}$-vector space to its underlying $\mathbb{R}$-vector space to show:
   Let X be a $\mathbb{C}$-vector space, $p : X \to \mathbb{R}$ be given such that
   $$p(\alpha x + \beta y) \leq |\alpha| p(x) + |\beta| p(y) \qquad (x, y \in X; \alpha, \beta \in \mathbb{C} : |\alpha| + |\beta| = 1) .$$
   Let $\lambda : Y \to \mathbb{C}$ linear where $Y \subseteq X$ is a subspace, and such that
   $$|\lambda(x)| \leq p(x) \qquad (x \in Y) .$$
   Then there exists $\Lambda : X \to \mathbb{C}$ linear such that $\Lambda|_Y = \lambda$ and such that
   $$|\Lambda(x)| \leq p(x) \qquad (x \in X) .$$

2. A Banach space is called *reflexive* iff $X \cong X^{**}$. Show that a Banach space X is reflexive iff X^* is reflexive.

3. A pair of Banach spaces are called strictly dual iff $\exists$ map $f : X \to Y^*$ which is isometric, so that the induced map $f^* : Y \to X^*$ is also isometric. Prove that if X and Y are strictly dual and X is reflexive, then $Y = X^*$ and $X = Y^*$ using the Hahn-Banach theorem.

4. Let $S \subseteq L^1([0,1] \to \mathbb{C})$ be a closed linear subspace. Suppose that S is such that $f \in S$ implies $f \in L^p([0,1] \to \mathbb{C})$ for some $p > 1$. Show that $S \subseteq L^p([0,1] \to \mathbb{C})$ for some $p > 1$.

5. [In this question we use the $\mathbb{R}$-Hahn-Banach] Let L be the (unilateral) left shift operator on $\ell^\infty(\mathbb{N} \to \mathbb{R})$:
   $$(L\psi)(n) \equiv \psi(n+1) \qquad (n \in \mathbb{N}) .$$
   Prove that there exists *a Banach limit, i.e. some* $\Lambda : \ell^\infty(\mathbb{N} \to \mathbb{R}) \to \mathbb{R}$ *linear such that: (a)* $\Lambda L = \Lambda$, *(b)*
   $$\liminf_n \psi(n) \leq \Lambda\psi \leq \limsup_n \psi(n) \qquad (\psi \in \ell^\infty) .$$
   Suggestion: Define the functional Λ_n via $\Lambda_n \psi := \frac{1}{n}\sum_{j=1}^n \psi(n)$, the space $M := \{ \psi \in \ell^\infty \mid (\lim_{n\to\infty} \Lambda_n \psi) \exists \}$ and the convex function $p(\psi) := \limsup_n \Lambda_n \psi$.

6. Prove that the closed unit ball of an infinite-dimensional Banach space is not compact.

7. Prove that an infinite-dimensional Banach space cannot be spanned, as a vector space, by a countable subset.