

Functional Analysis
Princeton University MAT520
HW4, Due Oct 6th 2023 (auto extension until Oct 8th 2023)

October 1, 2023

1 Weak stuff

Solve *at least two* of the following six problems.

1. Prove that any norm-closed convex bounded subset of a reflexive Banach space is weakly compact.
2. Define the Banach space $\ell^\infty(\mathbb{N}) := \{a : \mathbb{N} \rightarrow \mathbb{C} \mid \|a\|_\infty < \infty\}$ with the norm $\|a\|_\infty \equiv \sup_n |a_n|$. Also define

$$\ell^1(\mathbb{N}) := \{a : \mathbb{N} \rightarrow \mathbb{C} \mid \|a\|_1 < \infty\}$$

with the norm $\|a\|_1 := \sum_{n \in \mathbb{N}} |a_n|$.

Our goal here is to use the Banach-Alaoglu theorem to exhibit an element of $(\ell^\infty)^*$ which is not in ℓ^1 .

- (a) Define $\{\mu_n\}_n \subseteq (\ell^\infty)^*$ via

$$\mu_n(a) := \frac{1}{n} \sum_{j=1}^n a_j \quad (a \in \ell^\infty, n \in \mathbb{N}).$$

Show that $\mu_n \in (\ell^\infty)^*$ indeed and $\|\mu_n\| \leq 1$.

- (b) Show there exists some $\mu \in (\ell^\infty)^*$ that is the limit of $\{\mu_n\}_n$.

- (c) Show that $(\ell^1)^* = \ell^\infty$.

- (d) Hence we may think of $J(\ell^1) \subseteq (\ell^\infty)^*$ where J is the natural isometric injection. Show that the limit μ constructed above does *not* lie in $J(\ell^1)$. That is, show that for any $x \in \ell^1$, $J(x) \neq \mu$.

3. Let $\{f_n\}_n$ be given by

$$f_n(t) := e^{int} \quad (t \in [-\pi, \pi]).$$

Show that if $p \in [1, \infty)$ then $f_n \rightarrow 0$ weakly in $L^p([-\pi, \pi])$, but not in the norm topology of $L^p([-\pi, \pi])$.

4. Consider $L^\infty([0, 1])$ with its norm topology (the essential supremum norm), and, since $(L^1([0, 1]))^* = L^\infty([0, 1])$, the weak-star topology on $(L^1([0, 1]))^*$, which is a topology on $L^\infty([0, 1])$. Show that $C([0, 1])$ (the space of all continuous functions) is dense in L^α but not in L^β , for either $(\alpha, \beta) = (1, \infty)$ or $(\alpha, \beta) = (\infty, 1)$.

5. Let X be an infinite-dimensional Banach space and define

$$S := \{x \in X \mid \|x\| = 1\}.$$

Show that the weak-closure of S is

$$B := \{x \in X \mid \|x\| \leq 1\}.$$

6. Let X be a Banach space, and $\{L_n\}_n \subseteq X^*$ be a sequence which converges to some $L \in X^*$ in the weak-star sense. Assume that $\{x_n\}_n \subseteq X$ converges to some $x \in X$ in norm. It is true that $L_n(x_n) \rightarrow L(x)$ in \mathbb{C} ?

2 Banach algebras

Solve *at least ten* of the following dozen problems.

Here \mathcal{A} is a Banach algebra and x, y, \dots are elements in it; $\mathcal{G}_{\mathcal{A}}$ is the set of invertible elements and $r : \mathcal{A} \rightarrow [0, \infty)$ is the spectral radius.

7. Use $(xy)^n = x(yx)^{n-1}y$ to show that $r(xy) = r(yx)$.
8. Show that if $x, xy \in \mathcal{G}_{\mathcal{A}}$ then $y \in \mathcal{G}_{\mathcal{A}}$.
9. Show that if $xy, yx \in \mathcal{G}_{\mathcal{A}}$ then $x, y \in \mathcal{G}_{\mathcal{A}}$.
10. On the Banach space $\ell^2(\mathbb{N} \rightarrow \mathbb{C})$, define the right shift operator $R \in \mathcal{B}(\ell^2(\mathbb{N} \rightarrow \mathbb{C}))$:

$$(Ra)_n := \begin{cases} a_{n-1} & n \geq 2 \\ 0 & n = 1 \end{cases}$$

and the right shift operator

$$(La)_n := a_{n+1} \quad (n \in \mathbb{N}).$$

Calculate RL and LR . Conclude that one may have $xy = \mathbf{1}$ but $yx \neq \mathbf{1}$ in a Banach algebra.

11. Show that $xy - \mathbf{1}$ is invertible iff $yx - \mathbf{1}$ is invertible.
12. Show that if $z \in \mathbb{C} \setminus \{0\}$ then $z \in \sigma(xy)$ iff $z \in \sigma(yx)$. I.e.,

$$\sigma(xy) \cup \{0\} = \sigma(yx) \cup \{0\}.$$

Find an example where $\sigma(xy) \neq \sigma(yx)$.

13. Define $\mathcal{A} := C^2([0, 1] \rightarrow \mathbb{C})$, the space of functions with continuous second derivative. Define, for $a, b > 0$,

$$\|f\| := \|f\|_{\infty} + a\|f'\|_{\infty} + b\|f''\|_{\infty}.$$

Show that \mathcal{A} is a Banach space. Show that \mathcal{A} is a Banach algebra (with pointwise multiplication) iff $a^2 \geq 2b$. You may consider the functions $x \mapsto x$ and $x \mapsto x^2$.

14. Show that if $z \in \partial\sigma(x)$ then $x - z\mathbf{1} \in \partial\mathcal{G}_{\mathcal{A}}$.
15. Let $x \in \partial\mathcal{G}_{\mathcal{A}}$. Show there exists some $\{y_n\}_n \subseteq \mathcal{A}$ with $\|y_n\| = 1$ and

$$\lim_{n \rightarrow \infty} xy_n = \lim_{n \rightarrow \infty} y_nx = 0.$$

Try to characterize the type of Banach algebras in which there are such elements x (which are called *topological divisor of zero*).

16. On $\ell^2(\mathbb{N} \rightarrow \mathbb{C})$ define $T \in \mathcal{B}(\ell^2(\mathbb{N} \rightarrow \mathbb{C}))$ via

$$T(a_1, a_2, a_3, a_4, \dots) := (-a_2, a_1, -a_4, a_3, \dots).$$

Calculate $\sigma(T)$.

17. Show that if $x \in \mathcal{A}$ is nilpotent (i.e. $\exists n \in \mathbb{N}$ with $x^n = 0$) then $\sigma(x) = \{0\}$.
18. Show that r is upper semicontinuous.