Functional Analysis
Princeton University MAT520
HW5, Due Oct 13th 2023 (auto extension until Oct 15th 2023)

October 10, 2023

1 Banach algebras and the spectra of elements in it

In the following, $\mathcal{A}$ is a $\mathbb{C}$-Banach algebra.

1. Prove Fekete's lemma: If $\{ a_n \}_n \subseteq \mathbb{R}$ is sub-additive then $\lim_{n\to\infty} \frac{1}{n} a_n$ exists and equals $\inf \frac{1}{n} a_n$.

2. Let $R : \mathbb{C} \to \mathbb{C}$ be a rational function, i.e.,

$$R(z) = p(z) + \sum_{k=1}^{n} \sum_{l=1}^{q} c_{k,l} (z - z_k)^{-l}$$

where p is a polynomial, $n \in \mathbb{N}$, and $\{ z_k \}_k, \{ c_{k,l} \}_{k,l} \subseteq \mathbb{C}$. Let now $a \in \mathcal{A}$ such that $\{ z_k \}_{k=1}^{n} \subseteq \rho(a)$. Assume further that we choose some $\sigma(a) \subseteq \Omega \in \text{Open}(\mathbb{C})$ such that R is holomorphic on Ω, and $\gamma_j : [a,b] \to \Omega$, $j = 1, \dots, m$ a collection of m oriented loops which surround $\sigma(a)$ within Ω, such that

$$\frac{1}{2\pi \mathrm{i}} \sum_{j=1}^{m} \oint_{\gamma_j} \frac{1}{z - \lambda} \mathrm{d}z = \begin{cases} 1 & \lambda \in \sigma(a) \\ 0 & \lambda \notin \Omega \end{cases} .$$

Using Lemma 6.26 in the lecture notes (= Lemma 10.24 in Rudin) show that $R(a)$ obeys the Cauchy integral formula, in the sense that

$$p(a) + \sum_{k=1}^{n} \sum_{l=1}^{q} c_{k,l} (a - z_k)^{-l} = \frac{1}{2\pi \mathrm{i}} \sum_{j=1}^{m} \oint_{\gamma_j} R(z) (z\mathbb{1} - a)^{-1} \mathrm{d}z .$$

3. Let $\mathcal{A}$ be such that there exists some $a \in \mathcal{A}$ with $\sigma(a)$ not connected. Show that then $\mathcal{A}$ contains some non-trivial idempotent (an element $b \in \mathcal{A}$ with $b^2 = b \notin \{ 0, \mathbb{1} \}$).

4. Assume that $\{ a_n \}_n \subseteq \mathcal{A}$ is a sequence such that $\exists \lim_n a_n =: a \in \mathcal{A}$. Let $\Omega \in \text{Open}(\mathbb{C})$ contains a component of $\sigma(a)$. Show that $\sigma(a_n) \cap \Omega \neq \varnothing$ for all sufficiently large n. *Hint*: If $\sigma(a) \subseteq \Omega \sqcup \tilde{\Omega}$ where $\tilde{\Omega} \in \text{Open}(\mathbb{C})$ (in particular this means $\Omega \cap \tilde{\Omega} = \varnothing$), define $f : \mathbb{C} \to [0,1]$ such that $f|_{\Omega} = 1$ and $f|_{\tilde{\Omega}} = 1$.

5. Let X, Y be two Banach spaces and A, B be two bounded linear operators on X, Y respectively. Let $T \in \mathcal{B}(X \to Y)$. Show that the following two assertions are equivalent:

   (a) $TA = BT$.

   (b) $Tf(A) = f(B)T$ for any $f : \mathbb{C} \to \mathbb{C}$ holomorphic in some open set U which contains $\sigma(A) \cup \sigma(B)$.