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MAT520 — FA — HW6 Sol-ns

Q1 On $\ell^2(\mathbb{N}\to\mathbb{C}) \equiv \left\{ \psi : \mathbb{N}\to\mathbb{C} \,\middle|\, \sum_{n\in\mathbb{N}} |\psi(n)|^2 < \infty \right\}$

we define

$$\langle \varphi, \psi \rangle := \sum_{n\in\mathbb{N}} \overline{\varphi(n)}\, \psi(n) \qquad (\varphi, \psi \in \ell^2).$$

The Cauchy-Schwarz ineq. on $\mathbb{C}^N$ shows that

$$\left| \sum_{n=1}^{N} \overline{\varphi(n)}\, \psi(n) \right| \leq \sqrt{\left(\sum_{n=1}^{N} |\varphi(n)|^2\right)\left(\sum_{n=1}^{N} |\psi(n)|^2\right)}$$

for any $N \in \mathbb{N}$. Taking the limit $N\to\infty$ shows $\langle \cdot, \cdot \rangle$ is finite-valued and hence well-def. It induces the norm

$$\|\varphi\| \equiv \sqrt{\langle \varphi, \varphi \rangle} \qquad (\varphi \in \ell^2)$$

and hence the metric

$$d(\varphi, \psi) := \|\varphi - \psi\| \qquad (\varphi, \psi \in \ell^2).$$

To show $(\ell^2, \langle\cdot,\cdot\rangle)$ is a Hil. sp.

we need to show:

① ℓ^2 is a $\mathbb{C}$-v/sp.

② $\langle\cdot,\cdot\rangle$ is a sesqi-lin. form.

③ d is complete.

② is obvious from def., from which $\|\cdot\|$ is a norm so ① follows easily via the triangle ineq.

So we are left with ③:

Let $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq \ell^2(\mathbb{N})$ be Cauchy. W.T.S. it converges.

ONB of ℓ^2

$$|\varphi_n(m)-\varphi_{\tilde{n}}(m)| \equiv |\langle\delta_m, \varphi_n-\varphi_{\tilde{n}}\rangle| \overset{\text{C.S.}}{\leq} \underbrace{\|\delta_m\|}_{=1} \underbrace{\|\varphi_n-\varphi_{\tilde{m}}\|}_{\text{small}}$$

$\Rightarrow$ For fixed m, $\{\varphi_n(m)\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ is Cauchy and hence by completeness of $\mathbb{C}$

converges to some $\varphi_\infty(m)$.

W.T.S. $\{\varphi_\infty(m)\}_{m\in\mathbb{N}} \subseteq \ell^2(\mathbb{N})$.

$$\sum_{m=1}^{\infty} |\varphi_\infty(m)|^2 = \sum_{m=1}^{\infty} \left|\lim_{n\to\infty} \varphi_n(m)\right|^2$$

cont.

$$= \sum_{m=1}^{\infty} \lim_{n\to\infty} |\varphi_n(m)|^2$$

Fatou's lemma

$$\leq \liminf_{n\to\infty} \underbrace{\sum_{m=1}^{\infty} |\varphi_n(m)|^2}_{=\|\varphi_n\|_{\ell^2}^2}$$

Now, since $\big|\|\varphi_n\| - \|\varphi_m\|\big| \leq \|\varphi_n - \varphi_m\|$, $\{\|\varphi_n\|\}_n \subseteq [0,\infty)$ is Cauchy too, and so $\{\|\varphi_n\|\}_n$ converges.

$\Rightarrow$ $\varphi_\infty \in \ell^2(\mathbb{N})$.

Now, $$\|\varphi_n - \varphi_\infty\|^2 \equiv \sum_{m=1}^{\infty} |\varphi_n(m) - \varphi_\infty(m)|^2$$

$$= \sum_{m=1}^{\infty} \lim_{\tilde{n}\to\infty} |\varphi_n(m) - \varphi_{\tilde{n}}(m)|^2$$

Fatou

$$\leq \liminf_{\tilde{n}\to\infty} \sum_{m=1}^{\infty} |\varphi_n(m) - \varphi_{\tilde{n}}(m)|^2$$

$$\|\varphi_n - \varphi_{\tilde{n}}\|_{\ell^2}^2 .$$

But $\{\varphi_n\}_n$ is Cauchy, so $\|\varphi_n - \varphi_{\tilde{n}}\|$ can be made arbitrarily small if both $n, \tilde{n}$ are suff. large. Hence $\varphi_n \to \varphi_\infty$ in ℓ^2. ∎

Q2 We define $L^2(\mathbb{R} \to \mathbb{C})$ as follows:

$$f \sim g \iff \lambda(\{x \in \mathbb{R} \mid f(x) \neq g(x)\}) = 0$$

λ: Leb. msr. on $\mathbb{R}$

for any $f, g : \mathbb{R} \to \mathbb{C}$ msrble.

Then $L^2(\mathbb{R} \to \mathbb{C}) \equiv \{[f] \mid f : \mathbb{R} \to \mathbb{C} : \int |f|^2 d\lambda < \infty\}$

w/ inner prod.

$$\langle [f], [g] \rangle \equiv \int \bar{f} g \, d\lambda \in \mathbb{C}.$$

By the same arguments as above we only need to show the induced metric is complete

To that end, let $\{f_n\}_n \subseteq L^2(\mathbb{R})$ be Cauchy.

Then $\exists$ subseq $\{f_{n_j}\}_j$ s.t.

$$\|f_{n_j} - f_{n_{j+1}}\|_{L^2} \leq 2^{-j} \qquad (j \in \mathbb{N}).$$

Set

$$S_N(x) := f_{n_1}(x) + \sum_{j=1}^{N} f_{n_{j+1}}(x) - f_{n_j}(x)$$

$$A_N(x) := |f_{n_1}(x)| + \sum_{j=1}^{N} |f_{n_{j+1}}(x) - f_{n_j}(x)| \qquad (x \in \mathbb{R}, N \in \mathbb{N})$$

Then for fixed $x \in \mathbb{R}$, $\{A_N(x)\}_N$ is a non-decr. seq. and hence converges to some $A_\infty(x)$, possibly ∞. Now

$$\|A_N\|_{L^2} \leq \|f_{n_1}\|_{L^2} + \sum_{j=1}^{N} \underbrace{\|f_{n_{j+1}} - f_{n_j}\|_{L^2}}_{\leq 2^{-j}}$$

$$\leq \|f_{n_1}\|_{L^2} + 1.$$

$$\Rightarrow \quad \|A_\infty\|_{L^2} \underset{\text{Fatou}}{\leq} \liminf_{N\to\infty} \|A_N\| \leq \|f_{n_1}\|_{L^2} + 1 < \infty.$$

So $A_\infty \in L^2$. On the set where $A_\infty \neq \infty$ set $S_\infty := 0$. Otherwise for all other $x \in \mathbb{R}$, $S_\infty(x)$ converges abs. and hence has a lim.

...

$\blacksquare$

$\boxed{Q3}$ When $\mathcal{H} = \mathbb{C}^2$, $\mathcal{B}(\mathcal{H}) \cong \mathrm{Mat}_{2\times 2}(\mathbb{C})$.

Let Then
$$A := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$
$$B := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then $\|A\| = 1$, $\|B\| = 1$

$\|A+B\| = 2$, $\|A-B\| = 1$

and so $2\|A\|^2 + 2\|B\|^2 = 4$ get
$\|X+Y\|^2 + \|X-Y\|^2 = 5$!

$\boxed{Q4}$ Claim 7.9 says $(M^\perp)^\perp = \overline{M}$.

$\boxed{Q5}$ Let $\{\varphi_n\}_n \subseteq \mathcal{H}$ be pairwise $\perp$ i:

$\varphi_n \perp \varphi_m$ if $n \neq m$.

Claim: TFAE:

(1) $\sum_n \varphi_n \ \exists$ in $\|\cdot\|_{\mathcal{H}}$.

(2) $\sum_n \|\varphi_n\|_{\mathcal{H}}^2 < \infty$

(3) $\forall \psi \in \mathcal{H}, \sum_n \langle \psi, \varphi_n \rangle \ \exists$.

Proof: $\boxed{(1) \Rightarrow (2)}$

$$\left\| \sum_n \varphi_n \right\|^2 = \left\langle \sum_n \varphi_n, \sum_m \varphi_m \right\rangle$$

$$= \sum_{n,m} \langle \varphi_n, \varphi_m \rangle$$

pairwise ortho.

$$= \sum_n \|\varphi_n\|^2$$

$\boxed{(2) \Rightarrow (1)}$ $\left\| \sum_{n=1}^{N} \varphi_n - \sum_{n=1}^{M} \varphi_n \right\|^2 = \sum_{n=N+1}^{M} \|\varphi_n\|^2$ small as $\sum_n \|\varphi_n\|_{\mathcal{H}}^2 < \infty$. Hence $\left\{ \sum_{n=1}^{N} \varphi_n \right\}_N$ is Cauchy and so converges.

$\boxed{(1)\Rightarrow(3)}$ $\sum_n \langle \psi, \varphi_n \rangle = \langle \psi, \sum_n \varphi_n \rangle$ $\exists$.

$\boxed{(3)\Rightarrow(2)}$ If $\forall\, \psi$, $\sum_n \langle \psi, \varphi_n\rangle$ $\exists$,

then by Riesz, $\exists$ vector

$\Phi \in \mathcal{H}$ s.t. $\forall\, \psi \in \mathcal{H}$,

$$\sum_n \langle \psi, \varphi_n \rangle = \langle \psi, \Phi \rangle .$$

In particular

$$\langle \Phi, \varphi_n \rangle = \|\varphi_n\|^2$$

and so $\langle \Phi, \Phi \rangle = \sum_{n=1}^{\infty} \langle \Phi, \varphi_n \rangle$

$$= \sum_{n=1}^{\infty} \|\varphi_n\|^2 .$$

$\blacksquare$

$\boxed{Q6}$ In going from $(1)\Rightarrow(3)$ above we have NOT used pairwise $\perp$. But for the other direction we have!

Counterexample: On $\ell^2(\mathbb{N})$,

$\varphi_1 := \delta_1$ (Kronecker ONB).

$$\varphi_n := \delta_n - \delta_{n-1} \qquad (n \geq 2)$$

$$\sum_{n=1}^{N} \varphi_n = \delta_N .$$

$$\Longrightarrow \sum_{n=1}^{\infty} \varphi_n \text{ does NOT exist.}$$

But, $\forall\ \psi \in \ell^2(\mathbb{N})$,

$$\langle \psi, \sum_{n=1}^{N} \varphi_n \rangle = \langle \psi, \delta_N \rangle = \psi(N) \xrightarrow{N \to \infty} 0$$

as $\psi \in \ell^2$. ∎

$\boxed{Q7}$ Let $N \in \mathbb{N}$, $\alpha \in \mathbb{C}$: $\alpha^N = 1$, $\alpha^2 \neq 1$.

<u>Claim</u>: $\forall\ \varphi, \psi \in \mathcal{H}$,

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \frac{1}{N} \sum_{n=1}^{N} \alpha^n \|\alpha^n \varphi + \psi\|^2$$

<u>Proof</u>: $\|\alpha^n \varphi + \psi\|^2 = |\alpha^n|^2 \|\varphi\|^2 + \|\psi\|^2 + 2\mathrm{Re}\{\langle \alpha^n \varphi, \psi \rangle\}$

Note $\alpha^N = 1 \Rightarrow |\alpha| = 1$.

$$\Longrightarrow \alpha^n \|\alpha^n \varphi + \psi\|^2 = \alpha^n(\|\varphi\|^2 + \|\psi\|^2) + \langle \varphi, \psi \rangle + \alpha^{2n} \langle \psi, \varphi \rangle$$

<u>Claim</u>: $\displaystyle\sum_{n=1}^{N} \alpha^n = \sum_{n=1}^{N} \alpha^{2n} = 0$.

<u>Proof</u>: $\displaystyle\sum_{n=1}^{N} \alpha^n = \frac{\alpha(1-\alpha^N)}{1-\alpha} \overset{\alpha^N = 1}{=} 0$

geometric sum.

Similarly,

$$\sum_{n=1}^{N} \alpha^{2n} = \frac{\alpha^2(1-\alpha^{2N})}{1-\alpha^2} \leftarrow \neq 0!$$

But $1-\alpha^{2N} = (1-\alpha^N)(1+\alpha^N)$. $\blacksquare$

$\blacksquare$

The other claim follows using

$$\int_{\theta=0}^{2\pi} e^{i\theta}\, d\theta = 0$$

$$\int_{\theta=0}^{2\pi} e^{2i\theta}\, d\theta = 0.$$

$\boxed{Q8}$ Let $\{\varphi_n\}_n, \{\psi_n\}_n \subseteq \{\xi \in \mathcal{H} \mid \|\xi\| \leq 1\}$.

Assume $\langle \varphi_n, \psi_n \rangle \longrightarrow 1$ as $n \to \infty$.

Claim: $\lim_{n\to\infty} \|\varphi_n - \psi_n\| = 0$

Proof: $\|\varphi_n - \psi_n\|^2 = \|\varphi_n\|^2 + \|\psi_n\|^2 - 2\operatorname{Re}\{\langle \varphi_n, \psi_n \rangle\}$

$$\leq 2\left(1-\mathrm{Re}\{\langle \varphi_n, \psi_n\rangle\}\right)$$
$$= 2\,\mathrm{Re}\{\underbrace{1-\langle \varphi_n, \psi_n\rangle}_{\to 0}\}.$$

$\blacksquare$

Q9 Let $\{\varphi_n\}_n \subseteq \mathcal{H}$: $\underset{n\to\infty}{\text{w-lim}}\, \varphi_n = \varphi \ \exists \varphi \in \mathcal{H}$.

Assume $\lim_{n\to\infty} \|\varphi_n\| = \|\varphi\|$ in $\mathbb{R}$.

<u>Claim</u>: $\lim_{n\to\infty} \|\varphi_n - \varphi\| = 0$, i.e.,

$$\varphi_n \to \varphi \quad \text{in } \|\circ\|.$$

<u>Proof</u>: $\|\varphi_n - \varphi\|^2 = \|\varphi_n\|^2 + \|\varphi\|^2 - 2\,\mathrm{Re}\{\langle \varphi, \varphi_n\rangle\}$.

Since we have the weak conv., it holds in part. w/ φ itself, i.e.,

$$\lim_{n\to\infty} \langle \varphi, \varphi_n\rangle = \|\varphi\|^2.$$

Hence $\langle \varphi, \varphi_n\rangle \to \|\varphi\|^2$, whence

$$\|\varphi_n - \varphi\|^2 \to \left(\lim_{n\to\infty} \|\varphi_n\|^2\right) + \|\varphi\|^2 - 2\,\mathrm{Re}\{\|\varphi\|^2\}$$
$$= 0.$$

$\blacksquare$

Q10 Let V be an inner prod. sp. and $\{\varphi_n\}_{n=1}^N \subseteq V$ an orthonormal set.

For fixed $\psi \in V$, define

$$F_\psi : \mathbb{C}^N \to [0,\infty)$$

$$\alpha \mapsto \left\| \psi - \sum_{n=1}^N \alpha_n \varphi_n \right\| .$$

Claim: F_ψ is minimized on

$\alpha_{min} \in \mathbb{C}^N$ w/

$$(\alpha_{min})_n := \langle \varphi_n, \psi \rangle .$$

Proof: $$F(\alpha_{min}+\beta)^2 = \left\| \psi - \sum_{n=1}^N (\langle \varphi_n, \psi\rangle + \beta_n)\varphi_n \right\|^2$$

$$= \left\| \psi - \sum_{n=1}^N \langle \varphi_n, \psi\rangle \varphi_n \right\|^2 + \sum_{n=1}^N |\beta_n|^2 \|\varphi_n\|^2 -$$

$$- 2\operatorname{Re}\left\{ \left\langle \psi - \sum_{n=1}^N \langle \varphi_n, \psi\rangle \varphi_n , \sum_{n=1}^N \beta_n \varphi_n \right\rangle \right\}$$

But $\left(\psi - \sum_{n=1}^N \langle \varphi_n, \psi\rangle \varphi_n\right) \perp \varphi_m \quad \forall\, m$:

$$\left\langle \varphi_m, \psi - \sum_{n=1}^N \langle \varphi_n, \psi\rangle \varphi_n \right\rangle \overset{\text{orthonormality}}{=} \langle \varphi_m, \psi\rangle - \langle \varphi_m, \psi\rangle$$

$$= 0.$$

$$\Rightarrow F(\alpha_{min}+\beta) = \| \psi - \sum_{n=1}^{N} \langle \varphi_n, \psi \rangle \varphi_n \|^2 + \sum_{n=1}^{N} |\beta_n|^2 \|\varphi_n\|^2 .$$

Clearly min when $\beta = 0$.

$\blacksquare$