Functional Analysis
Princeton University MAT520
HW7, Due Nov 3rd 2023 (auto extension until Nov 5th 2023)

November 1, 2023

1 C-star algebras

In the following exercises, $\mathscr{A}$ is a C-star algebra with involution $* : \mathscr{A} \to \mathscr{A}$ and norm $\|\cdot\|$. $a, b, \dots \in \mathscr{A}$.

1. Show that if a is a partial isometry (i.e. $|a|^2$ is an idempotent) then $a = aa^*a = aa^*aa^*a$.
2. Show that a is a partial isometry iff a^* is a partial isometry.
3. Show that if p, q are self-adjoint projections then $\|p - q\| \leq 1$.
4. Show that if u, v are unitary then $\|u - v\| \leq 2$.
5. Show that if a is self-adjoint with $\|a\| \leq 1$ then

$$a + \mathrm{i}\sqrt{1 - a^2}, \qquad a - \mathrm{i}\sqrt{1 - a^2}$$

   are unitary. Conclude that any $b \in \mathscr{A}$ is the linear combination of four unitaries.

6. Two self-adjoint projections p, q are said to be orthogonal (written $p \perp q$) iff $pq = 0$. Show that the following are equivalent:
   (a) $p \perp q$.
   (b) $p + q$ is a self-adjoint projection.
   (c) $p + q \leq \mathbb{1}$.
7. Let $v_1, \dots, v_n$ be partial isometries and suppose that

$$\sum_{j=1}^{n} |v_j|^2 = \sum_{j=1}^{n} |v_j^*|^2 = \mathbb{1}\,.$$

   Show that $\sum_{j=1}^{n} v_j$ is unitary.

8. [extra] Show that for any $\varepsilon > 0$ there exists a $\delta_\varepsilon > 0$ such that if a obeys

$$\max\left(\{\, \|a - a^*\|, \left\|a^2 - a\right\| \,\}\right) \leq \delta_\varepsilon$$

   then there exists a self-adjoint projection p with $\|a - p\| \leq \varepsilon$.

9. [extra] Show that for any $\varepsilon > 0$ there exists a $\delta_\varepsilon > 0$ such that if a obeys

$$\max\left(\left\{\, \left\||a|^2 - \mathbb{1}\right\|, \left\||a^*|^2 - \mathbb{1}\right\| \,\right\}\right) \leq \delta_\varepsilon$$

   then there exists a unitary u with $\|a - u\| \leq \varepsilon$.

10. [extra] Show that $\sigma(p) \subseteq \{\, 0, 1 \,\}$ for an idempotent p.
11. [extra] Show that $\|p\| = 1$ for a non-zero self-adjoint projection p.

12. [extra] Show that the spectral radius $r(a)$ of a self-adjoint a equals its norm $\|a\|$.

13. [extra] Show that $\sigma(u) \subseteq \mathbb{S}^1$ if u is unitary (i.e. $|u|^2 = |u^*|^2 = \mathbb{1}$).

14. [extra] Show that $\sigma(a) \subseteq [0, \infty)$ if a is positive (i.e. $a = |b|^2 \exists b$).

15. [extra] Show that $\sigma(a) \subseteq \mathbb{R}$ if $a = a^*$.

16. [extra] Show that a is invertible if $|a|^2 \geq \varepsilon \mathbb{1}$ for some $\varepsilon > 0$; (recall $a \geq b$ iff $a - b \geq 0$ iff $a - b = |c|^2$ for some c).

2 Hilbert spaces

In this section $\mathcal{H}$ is a Hilbert space.

17. Show that

$$\mathcal{H} := \ell^2(\mathbb{R}) \equiv \left\{ f : \mathbb{R} \to \mathbb{C} \;\middle|\; f^{-1}(\mathbb{C} \setminus \{0\}) \text{ is a countable set and } \sum_{x \in \mathbb{R}} |f(x)|^2 < \infty \right\}$$

is *not* a separable Hilbert space.

18. Let R be the unilateral right shift operator on $\ell^2(\mathbb{N})$:

$$Re_j := e_{j+1} \qquad (j \in \mathbb{N})$$

where $\{ e_j \}_{j \in \mathbb{N}}$ is the standard basis of $\ell^2(\mathbb{N})$ and extend linearly.

    (a) Calculate R^*.
    (b) Calculate $|R|^2$ and $|R^*|^2$.
    (c) Show that R is a partial isometry.
    (d) Calculate $\sigma(R), \sigma(R^*), \sigma\left(|R|^2\right)$ and $\sigma\left(|R^*|^2\right)$.

19. Let $\hat{R}$ be the bilateral right shift operator on $\ell^2(\mathbb{Z})$:

$$\hat{R}e_j := e_{j+1} \qquad (j \in \mathbb{Z})$$

where $\{ e_j \}_{j \in \mathbb{Z}}$ is the standard basis of $\ell^2(\mathbb{Z})$ and extend linearly.

    (a) Calculate $\hat{R}^*$.
    (b) Calculate $\left|\hat{R}\right|^2$ and $\left|\hat{R}^*\right|$.
    (c) Show that $\hat{R}$ is a unitary.
    (d) Calculate $\sigma\left(\hat{R}\right), \sigma\left(\hat{R}^*\right), \sigma\left(\left|\hat{R}\right|^2\right)$ and $\sigma\left(\left|\hat{R}^*\right|^2\right)$.

20. Let $\frac{1}{X} \in \mathcal{B}\left(\ell^2(\mathbb{N})\right)$ be given by

$$\frac{1}{X} e_j := \frac{1}{j} e_j \qquad (j \in \mathbb{N})$$

and extend linearly.

    (a) Calculate $\left(\frac{1}{X}\right)^*$.
    (b) Calculate $\sigma\left(\frac{1}{X}\right)$.
    (c) Show that $\frac{1}{X}$ does *not* have closed range.

21. Show that if M is a closed linear subspace and $P_M : \mathcal{H} \to \mathcal{H}$ is given by

$$P_M \psi := a$$

where $\psi = a + b$ in the unique decomposition $\mathcal{H} = M \oplus M^\perp$, then P_M is a *self-adjoint projection*, i.e., show that $P_M = P_M^* = P_M^2$. Conversely, given any self-adjoint projection $P \in \mathcal{B}(\mathcal{H})$, find a closed linear subspace M such that $P = P_M$.

22. [extra] Let $\{ A_n \}_n \subseteq \mathcal{B}(\mathcal{H})$ such that for any $\varphi, \psi \in \mathcal{H}$,

$$\exists \lim_n \langle \varphi, A_n \psi \rangle \ .$$

Show there exists $A \in \mathcal{B}(\mathcal{H})$ such that $A_n \to A$ weakly.

23. For any $t > 0$, let $T_t \in \mathcal{B}\left(L^2(\mathbb{R})\right)$ be given by

$$T_t \varphi \quad := \quad \varphi(\cdot + t) \qquad \left(\varphi \in L^2\right) \ .$$

   (a) Calculate $\|T_t\|$.
   (b) Find a limit to which T_t converges as $t \to \infty$ (in which operator topology?).

24. [extra] Show that multiplication is not jointly continuous as a map

$$\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$$

if $\mathcal{B}(\mathcal{H})$ is given the strong operator topology.

25. Let $A_n \to A$, $B_n \to B$ in the strong operator topology. Show that $A_n B_n \to AB$ in the strong operator topology.

26. Let $A_n \to A$, $B_n \to B$ in the weak operator topology. Find a counter example for $A_n B_n \to AB$ in the weak operator topology.

27. Show that for $A \in \mathcal{B}(\mathcal{H})$,

$$\|A\|_{\text{op}} = \sup\left(\{ |\langle \varphi, A\psi \rangle| \mid \|\varphi\| = \|\psi\| = 1 \}\right)$$

and if $A = A^*$ then

$$\|A\|_{\text{op}} = \sup\left(\{ |\langle \varphi, A\varphi \rangle| \mid \|\varphi\| = 1 \}\right) \ .$$

28. Show that if $A_n \geq 0$, $A_n \to A$ in norm (resp. strongly) then $\sqrt{A_n} \to \sqrt{A}$ in norm (resp. strongly).

29. Show that if $A_n \to A$ in norm then $|A_n| \to |A|$ in norm.

30. [extra] Show that if $A_n \to A$ and $A_n^* \to A^*$ strongly then $|A_n| \to |A|$ strongly.

31. [extra] Find a counter example to

$$\| |A| - |B| \| \leq \|A - B\| \, .$$