Functional Analysis
Princeton University MAT520
HW8, Due Nov 10th 2023 (auto extension until Nov 12th 2023)

November 4, 2023

1. Let P,Q be two orthogonal projections onto subspaces M,N in a Hilbert space $\mathcal{H}$ such that $[P,Q]=0$.
   (a) Show $P^\perp \equiv \mathbb{1}-P$, $Q^\perp$, PQ, $P+Q-PQ$ and $P+Q-2PQ$ are orthogonal projections.
   (b) What is the relation between the projections in the previous item and M,N?

2. Let P,Q be two orthogonal projections onto subspaces M,N in a Hilbert space $\mathcal{H}$. Show that
$$\operatorname*{s-lim}_{n\to\infty} (PQ)^n$$
exists and is the orthogonal projection onto $M\cap N$.

3. Let $A\in\mathcal{B}(\mathcal{H})$. Show that the set of $\lambda\in\sigma(A)$ such that λ is not an eigenvalue of A and $\operatorname{im}(A-\lambda\mathbb{1})$ is closed but not the whole of $\mathcal{H}$ is an open subset of $\mathbb{C}$.

4. Define the numerical range $N(A)$ of $A\in\mathcal{B}(\mathcal{H})$ via
$$N(A) := \{\, \langle\psi, A\psi\rangle \mid \psi\in\mathcal{H}\wedge \|\psi\|=1 \,\}\ .$$
   (a) Show that
$$\sigma(A)\subseteq \overline{N(A)}\,.$$
   (b) Find an example where $N(A)$ is not closed and
$$\sigma(A)\nsubseteq N(A)\ .$$
   (c) Find an example where
$$\sigma(A)\neq N(A)=\overline{N(A)}\,.$$

5. Show that if $A\in\mathcal{B}(\mathcal{H})$ has $A=A^*$ then
$$\left\|(A-z\mathbb{1})^{-1}\right\| \leq \frac{1}{|\mathbb{Im}\{z\}|} \qquad (z\in\mathbb{C}:|\mathbb{Im}\{z\}|>0)\ .$$

6. Show that if $A\in\mathcal{B}(\mathcal{H})$ is an isometry then $\operatorname{im}(A)$ is closed in $\mathcal{H}$.

7. Let $V\in\mathcal{B}\left(L^2([0,1]\to\mathbb{C})\right)$ be give by
$$V(\psi) := \int_0^{\cdot}\psi \qquad \left(\psi\in L^2\right)\ .$$
   (a) Show that V is well-defined (it is a bounded linear map) with
$$V^*(\psi) = \int_{\cdot}^1 \psi \qquad \left(\psi\in L^2\right)\ .$$
   (b) Show that the spectral radius of V, $r(V)$, equals zero and that $\sigma(V)=\{\,0\,\}$.

(c) Show that $\|V\| = \frac{2}{\pi}$.

8. Let $\mathcal{F} : \ell^2(\mathbb{Z}) \to L^2(\mathbb{S}^1)$ be the Fourier series given by

$$\ell^2(\mathbb{Z}) \ni \psi \mapsto \left([0, 2\pi] \ni k \mapsto \sum_{n\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}kn} \psi_n =: \hat{\psi}(k) \right) .$$

Let $A \in \mathcal{B}\left(\ell^2(\mathbb{Z})\right)$ be the discrete Laplacian:

$$A = R + R^*$$

where R is the bilateral right shift operator

$$R\delta_n := \delta_{n+1} \qquad (n \in \mathbb{Z})$$

and $\{ \delta_n \}_{n\in\mathbb{Z}}$ the standard basis of $\ell^2(\mathbb{Z})$. Calculate

$$\mathcal{F} A \mathcal{F}^* \in \mathcal{B}\left(L^2\left(\mathbb{S}^1\right)\right) .$$

9. [extra] Call an operator $A \in \mathcal{B}\left(\ell^2\left(\mathbb{Z}^d\right)\right)$ *local* iff

$$\inf_{x,y\in\mathbb{Z}^d} -\frac{1}{\|x-y\|} \log\left(|\langle \delta_x, A\delta_y \rangle|\right) > 0 .$$

Show that if $A = A^*$ and A is local then $(A - z\mathbb{1})^{-1}$ is local too for any $z \in \mathbb{C} : |\mathbb{Im}\{z\}| > 0$.