Functional Analysis
Princeton University MAT520
HW9, Due Nov 17th 2023 (auto extension until Nov 19th 2023)

November 19, 2023

1. Prove Weyl's criterion for the spectrum of an operator. Let $A = A^* \in \mathcal{B}(\mathcal{H})$ be given. We have $\lambda \in \sigma(A)$ iff there exists some $\{ \varphi_n \}_{n\in\mathbb{N}}$ with $\|\varphi_n\| = 1$ such that
$$\lim_n \|(A - \lambda\mathbb{1})\varphi_n\| = 0\,.$$

2. Let $A \in \mathcal{B}(\mathcal{H})$ be compact and $\{ \varphi_n \}_n \subseteq \mathcal{H}$ converge *weakly* (in the sense of the Banach space weak topology on $\mathcal{H}$) to some $\varphi \in \mathcal{H}$. Show that $A\varphi_n \to A\varphi$ in norm.

3. Figure out if the following operators are compact or not (and prove what you think):
   (a) $\mathbb{1}$.
   (b) $u \otimes v^*$ for some $u, v \in \mathcal{H}$.
   (c) On *the Banach space* $X := C([0,1] \to \mathbb{C})$ with $\|\cdot\|_\infty$, let $A : X \to X$ be given
   $$(A\varphi)(x) := \int_{y=0}^{1} K(x,y)\,\varphi(y)\,\mathrm{d}y$$
   where $K : [0,1]^2 \to \mathbb{C}$ is some *continuous* function.
   (d) $A := \frac{1}{1+X^2}$ on $\ell^2(\mathbb{Z})$ where X is the position operator given by
   $$(X\psi)(n) \equiv n\psi(n) \qquad \left(n \in \mathbb{Z}; \psi \in \ell^2(\mathbb{Z})\right)$$
   and we employ the holomorphic functional calculus to define A.

4. On $\mathcal{H} \oplus \mathcal{H}$, let
$$H \quad := \quad \begin{bmatrix} 0 & S^* \\ S & 0 \end{bmatrix}$$
for some $S \in \mathcal{B}(\mathcal{H})$. Find the polar decomposition of H.

5. Show that an idempotent is compact if and only if it is of finite rank.

6. Show that no nonzero multiplication operator on $L^2([0,1])$ is compact.

7. Show that if $A \in \mathcal{B}(\mathcal{H})$ is compact and $\{ e_n \}_n$ is an ONB then $\|Ae_n\| \to 0$. Find a counter-example of the converse.

8. [extra] Let $\Omega \subseteq \mathbb{R}^3$ be a bounded region with a smooth boundary surface $\partial\Omega$. Let $f : \partial\Omega \to \mathbb{C}$ be continuous. Fix some parameter $m > 0$. Find a function $\varphi : \overline{\Omega} \to \mathbb{C}$ which is twice differentiable in Ω and continuous on $\overline{\Omega}$ such that
$$\begin{aligned} \left(-\Delta + m^2\mathbb{1}\right)\varphi &= 0 \\ \varphi|_{\partial\Omega} &= f\,. \end{aligned}$$
Find (and prove the properties of) a function $K : \overline{\Omega} \times \overline{\Omega} \to \mathbb{C}$ (called the Poisson kernel of $-\Delta + m^2\mathbb{1}$ in the interior of Ω) which allows the solution of the above Dirichlet problem be written as
$$\varphi(x) = \int_{y\in\partial\Omega} K(x,y)\,f(y)\,\mathrm{d}y\,.$$