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MAT 520 — FA — Midterm Sol-ns

Q1 X, Y two Banach sp.

$\forall\, p \geq 1$, define on $X \times Y$

$$\|(x,y)\|_p := \begin{cases} \max(\{\|x\|_X, \|y\|_Y\}) & p=\infty \\ (\|x\|_X^p + \|y\|_Y^p)^{1/p} & p<\infty \end{cases}$$

(a) Claim: $\|\cdot\|_\infty$ is a norm.

Proof: ⊛ Abs. homogen. clear. ✓

⊛ If $\|(x,y)\|_p = 0$, both $\|x\|=0$ and $\|y\|=0$, so $x=0$ and $y=0$ so $(x,y)=(0,0)$. ✓

⊛ $\boxed{\Delta \neq}$:

$$\|\underbrace{(x,y)+(\tilde{x},\tilde{y})}_{(x+\tilde{x},\, y+\tilde{y})}\|_\infty = \max(\{\underbrace{\|x+\tilde{x}\|}_{\leq \|x\|+\|\tilde{x}\|}, \underbrace{\|y+\tilde{y}\|}_{\leq \|y\|+\|\tilde{y}\|}\})$$

$$\leq \max(\{\|x\|+\|\tilde{x}\|, \|y\|+\|\tilde{y}\|\})$$

$$\leq \max(\{\|x\|, \|y\|\}) + \max(\{\|\tilde{x}\|, \|\tilde{y}\|\})$$

Where in $\boxed{0}$ we have used

$$\underbrace{\max(\{a+b, c+d\})}_{\mathrm{I}} \leq \underbrace{\max(\{a,c\})}_{\mathrm{II}} + \underbrace{\max(\{b,d\})}_{\mathrm{III}}.$$

If $a+b \geq c+d$, $\mathrm{I} = a+b$

$a \geq c$ $\quad$ $\mathrm{II} = a$

$b \geq d$ $\quad$ $\mathrm{III} = b$

and then $a+b \overset{?}{\leq} a+b$ ✓

$a \leq c$ $\quad$ $\mathrm{II} = c$

$b \geq d$ $\quad$ $\mathrm{III} = b$

$a+b \overset{?}{\leq} c+b$ ✓

etc. . .

<u>Claim</u>: $\|\circ\|_p$ is a norm $\forall\ p \in (0,1)$

<u>Proof</u>: ⊛ Abs. homogen. is clear. ✓

⊛ If $\|(x,y)\|_p = 0$, $\|x\|^p + \|y\|^p = 0$

$\Longrightarrow \|x\| = 0$ and $\|y\| = 0$

$\Longrightarrow x = 0$ and $y = 0$.

⊛ $\boxed{\Delta \neq}$

$$\|(x,y) + (\tilde{x}, \tilde{y})\|_p \equiv \|(x+\tilde{x}, y+\tilde{y})\|_p$$

$$= \left(\|x+\tilde{x}\|^p + \|y+\tilde{y}\|^p\right)^{1/p}$$

$\Delta \neq$ in X, Y

and $\alpha \mapsto \alpha^p$ incr.

$$\leq \left((\|x\|+\|\tilde{x}\|)^p + (\|y\|+\|\tilde{y}\|)^p \right)^{1/p}$$

Minkowski $\neq$

$$\leq \left(\|x\|^p + \|y\|^p \right)^{1/p} + \left(\|\tilde{x}\|^p + \|\tilde{y}\|^p \right)^{1/p}$$

$$\equiv \|(x,y)\|_p + \|(\tilde{x},\tilde{y})\|_p .$$

<u>Claim</u>: $\|\circ\|_\infty$ is complete.

<u>Proof</u>: Say $\{(x_n,y_n)\}_n$ is Cauchy in $(X \times Y, \|\circ\|_\infty)$.

Then $\forall\ \varepsilon > 0\ \exists\ N_\varepsilon \in \mathbb{N}$: if $n,m \geq N_\varepsilon$ then $\|(x_n,y_n)-(x_m,y_m)\|_\infty < \varepsilon$

$$\Updownarrow$$

$$\max(\{\|x_n - x_m\|, \|y_n - y_m\|\}) < \varepsilon$$

So $\{x_n\}_n$ and $\{y_n\}_n$ are both separately Cauchy, and hence $\exists\ (x,y) \in X \times Y$: $x_n \to x$, $y_n \to y$.

But then

$$\|(x_n,y_n)-(x,y)\|_\infty = \max(\{\|x_n - x\|, \|y_n - y\|\})$$

$\longrightarrow 0 \quad \text{as} \quad n \to \infty.$

Claim: $\|\cdot\|_p$ is complete for $p \in [1, \infty)$.

Proof: Say $\{(x_n, y_n)\}_n$ is Cauchy in $(X \times Y, \|\cdot\|_p)$.

Then $\forall \varepsilon > 0\ \exists N_\varepsilon \in \mathbb{N}: \forall n, m \geq N_\varepsilon,$

$$\|(x_n, y_n) - (x_m, y_m)\|_p < \varepsilon$$
$$\Updownarrow$$
$$\|x_n - x_m\|^p + \|y_n - y_m\|^p < \varepsilon^p$$
$$\Downarrow$$
$$\|x_n - x_m\| < \varepsilon \quad \wedge \quad \|y_n - y_m\| < \varepsilon$$
$$\Downarrow$$
$$\exists (x, y) \in X \times Y : x_n \to x \text{ and } y_n \to y.$$

Then

$$\|(x_n, y_n) - (x, y)\|_p \equiv \left(\|x_n - x\|^p + \|y_n - y\|^p\right)^{1/p}.$$

Pick $N \in \mathbb{N}$: if $n \geq N$ then $\begin{cases} \|x_n - x\| \leq 2^{-\frac{1}{p}} \varepsilon \\ \|y_n - y\| \leq 2^{-\frac{1}{p}} \varepsilon. \end{cases}$

(b) Claim: All p - norms are equiv. on $X \times Y$.

Proof: Since norm equiv. is an equiv. rel., it suffices to show $\|\cdot\|_p$ and $\|\cdot\|_\infty$

are equiv. $\forall \quad p \in [1,\infty)$.

To that end,

$$\|(x,y)\|_p \equiv (\|x\|^p + \|y\|^p)^{1/p}$$

$$\leq \left(2\max(\{\|x\|^p, \|y\|^p\})\right)^{1/p}$$

$\alpha \mapsto \alpha^p$ incr.

$$\overset{?}{=} \left(2\max(\{\|x\|, \|y\|\})^p\right)^{1/p}$$

$$= 2^{1/p} \|(x,y)\|_\infty .$$

The reverse direction follows by HW2Q4 (inverse mapping).

$\boxed{Q2}$ $M: X \to Y$ where X, Y are normed $\mathbb{C}$-v/sp.

Assume ① $M(0) = 0$

② $M(\frac{1}{2}(x+\tilde{x})) = \frac{1}{2}M(x) + \frac{1}{2}M(\tilde{x}) \qquad (x, \tilde{x} \in X)$.

③ M is cont.

Claim: M is $\mathbb{R}$-linear.

Proof: ⊛ ② w/ $\tilde{x} = 0 \Rightarrow M(\frac{1}{2}x) = \frac{1}{2}M(x)$.

⊛ $M(x+\tilde{x}) = M(\frac{1}{2}(2x+2\tilde{x}))$

$= \frac{1}{2}M(2x) + \frac{1}{2}M(2\tilde{x})$ ②

$= M(x) + M(\tilde{x})$.

⊛ $M(nx) = n\,M(x)$ by the above $\forall \; n \in \mathbb{N}_{\geq 0}$.

⊛ $M(\frac{1}{n}x) = \frac{1}{n}\cdot n\,M(\frac{1}{n}x) = \frac{1}{n}M(x) \quad \forall \; n \in \mathbb{N}_{\geq 1}$.

(*) Let $\alpha \in \mathbb{R}$. Then $\exists \{\alpha_n\}_n \subseteq \mathbb{Q}$: $\alpha_n \to \alpha$.

Since we have $M(\alpha_n x) = \alpha_n M(x) \;\forall x \in X$, by cont. of M we get

$$M(\alpha x) = \alpha M(x).$$

Note M may fail to be $\mathbb{C}$-lin. since we know nothing about $M(ix)$...

$\boxed{Q3}$ (a) A normed v/sp. which is NOT a Banach sp:

Example 3.10 in LN.

(b) A linear f^n-al which is NOT cont.:

On $X := \{ \psi : \mathbb{N} \to \mathbb{C} \mid |\psi^{-1}(\{0\}^c)| < \infty \}$

w/ $\|\cdot\|_\infty$ norm. This is not a Banach sp.

Def. $\lambda : X \to \mathbb{C}$

$$\psi \mapsto \sum_{n \in \mathbb{N}} n \psi_n$$

(c) A TVS which is NOT locally convex:

$L^p([0,1])$ w/ $p \in (0,1)$ is a TVS

but not locally convex: any open convex set about the origin is the entire sp.

(d) $X = \mathbb{C}^n$ is a Banach sp. whose closed unit ball is cpt.

(e) A Banach sp. which is NOT reflexive:

$\ell^1(\mathbb{N})$ has $\left(\ell^1(\mathbb{N})\right)^* = \ell^\infty(\mathbb{N})$

but $\left(\ell^\infty(\mathbb{N})\right)^* \neq \ell^1(\mathbb{N})$.

$\boxed{Q4}$ Claim: If X, Y are two Banach sp. and $A \in \mathcal{B}(X \to Y)$ then

$$x_n \xrightarrow{w} x \quad \Longrightarrow \quad Ax_n \xrightarrow{w} Ax$$
in X in Y.

Proof: Weak conv. $\overset{\text{Lemma 5.11 in LN}}{\Longleftrightarrow}$ $\lambda(x_n) \to \lambda(x) \ \forall \lambda \in X^*$.

Let $\lambda \in Y^*$. Want $\lambda A x_n \to \lambda A x$.

But $\lambda \circ A \in X^*$ as A is cont. ∎

$\boxed{Q5}$ See Rudin $\boxed{11.23}$.

For counter examples, use $\mathcal{A} = \mathrm{Mat}_{2\times 2}(\mathbb{C})$.