Functional Analysis
Princeton University MAT520
October 2023 Take-home Midterm Exam

October 11, 2023

Instructions: This is a take-home exam, meaning you are expected to do it "at home" by yourself in a quiet environment with no other person aiding you (this is not a group assignment). You may very well use any source material which is "not interactive" such as books / notes / lecture notes / etc. But you may *not* communicate with other people, nor post any questions online. Pretend you were in a classroom and could bring with you whichever *non-electronic offline* reference material you needed. Obviously I have no way of policing that, so we shall rely on Princeton's honor code.

- The exam has five questions, each worth 25 points, to yield a maximum of 125 points (which will then be truncated to a maximal 100).
- You are expected to complete the exam in four hours, starting from whenever you open it on Gradescope. You may take breaks as needed but not communicate with others about the exam during that time.
- If you somehow can't submit the assignment through Gradescope just send it to my email (shapiro@math.princeton.edu). I trust you that you time yourself correctly for the four hours.
- During your exam it is possible to email us (shapiro@math.princeton.edu or juihui@princeton.edu) if there is any doubt about the phrasing of the questions (but don't expect to get help or hints, in the interest of fairness).
- The exam is meant to be easy and straightforward and there are no trick questions. If something seems too easy maybe it's just easy.

1. Let X, Y be two Banach spaces. Define on the Cartesian product

$$X \times Y$$

coordinate-wise addition and scalar multiplication. For $p \in [1, \infty]$, define

$$\|(x,y)\|_p := \begin{cases} \max\left(\{\ \|x\|_X, \|y\|_Y\ \}\right) & p = \infty \\ \left(\|x\|_X^p + \|y\|_Y^p\right)^{\frac{1}{p}} & p \in [1, \infty) \end{cases}.$$

   (a) Show that with these definitions, $X \times Y$ is a Banach space (i.e., show it is a complete normed vector space).

   (b) Show that all p-norms are equivalent on $X \times Y$.

2. Let $M : X \to Y$ be a continuous map between normed spaces X, Y such that

$$M(0) = 0$$

and

$$M\left(\frac{1}{2}(x + \tilde{x})\right) = \frac{1}{2}M(x) + \frac{1}{2}M(\tilde{x}) \qquad (x, \tilde{x} \in X).$$

Show that M is linear.

3. Provide an example (no further explanation or proof is necessary) for each of the following:

   (a) A normed vector space which is not a Banach space.

   (b) A linear functional that is not continuous.

(c) A topological vector space which is not locally convex.

(d) A Banach space whose closed unit ball is compact.

(e) A Banach space which is not reflexive.

4. Show that if X, Y are Banach spaces and $A \in \mathcal{B}(X \to Y)$ then if $x_n \to x$ weakly in X then $Ax_n \to Ax$ weakly in Y.

5. In a Banach algebra $\mathcal{A}$, let $a, b \in \mathcal{A}$. Show that if $ab = ba$ then

$$\sigma(a+b) \subseteq \sigma(a) + \sigma(b)$$

and

$$\sigma(ab) \subseteq \sigma(a)\,\sigma(b) \; .$$

Find examples where these containments are strict, and find examples when these containments are false if $ab \neq ba$.