Topics in Mathematical Physics: Mathematical Aspects of Condensed Matter Physics Princeton University MAT 595 / PHY 508 Lecture Notes

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Created: Jan 27 2024, Last Typeset: April 15, 2024

Abstract

These lecture notes correspond to a course given in the Spring semester of 2024 in the math and physics departments of Princeton University.

Contents

Syllabus

- The main source of material for the lectures: this very document (to be published and weekly updated on the course website–please do not print before the course is finished and the label "final version" appears at the top).
- Official course textbook: No one, main official text will be used but in preparing these notes; I will probably make heavy use of [Sha16], [Rud91] as well as [RS80] and [BB89]. In particular for the part about random operators I will use the textbook by [AW15].
- Other sources one may consult are [Sto11, Kir07].
- Two lectures per week: Tue and Thur, 1:30 pm – 2:50 pm in Jadwin Hall 343.
- People involved:
  - Instructor: Jacob Shapiro shapiro@math.princeton.edu
    Office hours: by appointment.
  - Assistant: ???
- HW will be periodically posted on the course website but is not meant to be submitted.
- Grade: this is an auditing class.

- Anonymous Ed discussion enabled. Use it to ask questions or to raise issues (technical or academic) with the course.
- If you alert me about typos and mistakes in this manuscript (unrelated to the sections marked [todo]) I'll be most grateful.
  - Thanks goes to: David Shustin, Grace Sommers ($\times 3$).

Semester plan

List of (big) theorems and topics aimed at being included:

- Quantum dynamics.
- Quantum transport and linear response theory.
- Effects of disorder on quantum dynamics: Anderson localization and delocalization.
- The quantum Hall effect.
- Topological phases of matter and the classification of insulators.

Semester plan by date:

1. Jan 30th: Quantum dynamics
2. Feb 1st: Quantum dynamics
3. Feb 6th: Connection between spectral types and quantum dynamics: RAGE
4. Feb 8th: Connection between spectral types and quantum dynamics: Mourre
5. Feb 13th: Linear response theory and the Kubo formula
6. Feb 15th: Linear response theory and the Kubo formula: DC conductivity as a function of the Greens function
7. Feb 20th: Anderson localization: introduction to random operators
8. Feb 22nd: Anderson localization: introduction to random operators
9. Feb 27th: Anderson localization: proof of Anderson localization at high disorder
10. Feb 29th: Anderson localization: proof of Anderson localization at weak disorder at spectral edges
11. Mar 5th: Anderson localization: proof of complete Anderson localization in 1D
12. Mar 7th:Anderson localization: physics "proof" of Anderson de-localization
13. Mar 12th: SPRING RECESS
14. Mar 14th: SPRING RECESS
15. Mar 19th: Introduction to the integer quantum Hall effect, Landau Hamiltonian, etc.
16. Mar 21st: Proof that Chern number is well-defined, connection between trace formula, index formula and periodic k-space formula
17. Mar 26th: Some elements of Fredholm theory
18. Mar 28th: Proof that the edge Hall conductivity is an integer and the bulk-edge correspondence for the IQHE
19. Apr 2nd: The $\mathbb{Z}_2$-index and the Fu-Kane-Mele index
20. Apr 4th: The 1D classification of topological insulators
21. Apr 9th: Some elements of K-theory
22. Apr 11th: K-theoretic classification and the Kitaev periodic table
23. Apr 16th: Introduction to many-body quantum mechanics.
24. Apr 18th: Introduction to many-body quantum mechanics.
25. Apr 23rd: The Fraas et al proof that the Hall conductivity of an interacting system is quantized.
26. Apr 25th: The Kitaev-Fidkowski $\mathbb{Z}_8$ index for interacting systems.

1 Quantum dynamics

Our goal is to understand the dynamics of electrons in solids. To that end, we will (mostly) make the following assumptions:

- Electrons do *not* interacts with each other.
- Real space is a discrete lattice.
- Quantum mechanics is applicable.

As a result of these assumptions, the appropriate setting to explore models of electrons in thus in single-particle Hilbert space

$$\mathcal{H} = \ell^2\left(\mathbb{Z}^d\right)$$

where d is the space dimension. Sometimes we will consider other lattices besides $\mathbb{Z}^d$, but in principle this generalization is not very important right now. At other times it will also be useful to allow internal degrees of freedom on each lattice site, i.e., that the wave function is a map

$$\psi : \mathbb{Z}^d \to \mathbb{C}^N$$

and one way to write that Hilbert space is as

$$\mathcal{H} = \ell^2\left(\mathbb{Z}^d \to \mathbb{C}^N\right) \cong \ell^2\left(\mathbb{Z}^d \to \mathbb{C}\right) \otimes \mathbb{C}^N \,.$$

To specify a model for the dynamics of electrons on this Hilbert space, we must pick a Hamiltonian, which for us will be a bounded linear operator $H \in \mathcal{B}(\mathcal{H})$ which is furthermore *self-adjoint*.

1.1 Locality

Beyond being *any* bounded linear operator which is self-adjoint, a Hamiltonian better be *local*. To discuss locality we need to single out a basis on Hilbert space, which is the main reason why it is important to stipulate that we are working with $\ell^2\left(\mathbb{Z}^d\right)$: because otherwise, all separable Hilbert spaces are isomorphic, so in principle we have the isomorphism

$$\ell^2\left(\mathbb{Z}^d\right) \cong \ell^2\left(\mathbb{Z}^{\tilde{d}}\right) \,.$$

Hence, let us choose the position basis as

$$\{\, \delta_x \,\}_{x\in\mathbb{Z}^d} \subseteq \ell^2\left(\mathbb{Z}^d\right)$$

defined as

$$\left(\delta_x\right)_y \equiv \delta_{xy} \equiv \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases} \qquad \left(x, y \in \mathbb{Z}^d\right) \,.$$

With this basis, we may form matrix elements of the Hamiltonian H as

$$H_{xy} \equiv \langle \delta_x, H\delta_y \rangle \qquad \left(x, y \in \mathbb{Z}^d\right) \,.$$

Note that if $\mathcal{H} = \ell^2\left(\mathbb{Z}^d\right) \otimes \mathbb{C}^N$ then H_{xy} is actually an $N \times N$ matrix, whose matrix elements are

$$\left(H_{xy}\right)_{ij} \equiv \langle \delta_x \otimes e_i, H\delta_y \otimes e_j \rangle \qquad (i, j = 1, \dots, N)$$

where $\{\, e_i \,\}_{i=1}^N$ is the standard basis of $\mathbb{C}^N$. With this definition we may finally make the

Definition 1.1 (Local operator)**.** The operator $A \in \mathcal{B}\left(\ell^2\left(\mathbb{Z}^d\right) \otimes \mathbb{C}^N\right)$ is called *local* (or *exponentially-local*) iff there exists some $R < \infty$ and $\mu > 0$ such that

$$-\frac{1}{\|x-y\|} \log\left(\|A_{x,y}\|\right) \geq \mu \qquad \left(x, y \in \mathbb{Z}^d : \|x-y\| \geq R\right) \,. \tag{1.1}$$

Here by $\|A_{xy}\|$ we either mean the absolute value (if $N = 1$) or any matrix norm (they are all equivalent) if $N > 1$.

Claim 1.2. (1.1) is equivalent to: there exist $C, \mu \in (0, \infty)$ such that

$$\|A_{xy}\| \leq C\mathrm{e}^{-\mu\|x-y\|} \qquad \left(x, y \in \mathbb{Z}^d\right) \,. \tag{1.2}$$

Proof. Let us assume (1.1). Taking its exponential we find

$$\|A_{xy}\| \leq \mathrm{e}^{-\mu\|x-y\|} \qquad \left(x,y\in\mathbb{Z}^d : \|x-y\|\geq R\right) .$$

Since A is bounded, we of course also have

$$\|A_{xy}\| \leq \|A\| \qquad \left(x,y\in\mathbb{Z}^d\right) .$$

At those $x,y\in\mathbb{Z}^d$ for which $\|x-y\|<R$ we have

$$\begin{aligned}\|A_{xy}\| &\leq \|A\|\mathrm{e}^{\mu\|x-y\|}\mathrm{e}^{-\mu\|x-y\|} \\ &\leq \|A\|\mathrm{e}^{\mu R}\mathrm{e}^{-\mu\|x-y\|}\end{aligned}$$

so if we define

$$C \quad := \quad \max\left(\left\{\ \|A\|\mathrm{e}^{\mu R}, 1\ \right\}\right)$$

then we find (1.2). Conversely, assuming (1.2), we have

$$-\frac{1}{\|x-y\|}\log\left(\|A_{xy}\|\right) \geq \mu - \frac{\log\left(C\right)}{\|x-y\|} \qquad \left(x,y\in\mathbb{Z}^d\right) .$$

Take now $R := \frac{\log(C)}{\frac{1}{2}\mu}$ to get that if $\|x-y\|\geq R$ then

$$\frac{\log\left(C\right)}{\|x-y\|} \quad \leq \quad \frac{1}{2}\mu\,.$$

□

Sometimes it is useful to also have other modes of locality. We shall introduce them as we go along. One can generalize in two possible directions:

- replace exponential decay with various other rates of decay in the off-diagonal direction
- allow the rate of exponential (or any other) decay to depend on the diagonal position. One way to do so will be called *weakly-local*: there exists some $\mu>0$ such that for any $\varepsilon>0$ there exists some $C_\varepsilon<\infty$ with which

$$\|A_{xy}\| \leq C_\varepsilon \mathrm{e}^{-\mu\|x-y\|+\varepsilon\|x\|} \qquad \left(x,y\in\mathbb{Z}^d\right) .$$

Why do we call such operators *local*? Because we interpret the number (or matrix) H_{xy} as the *transition amplitude* to go between the state (or space) δ_x and δ_y. Sometimes these terms are also called *hopping terms.* We may think of them as the transition amplitude of infinitesimal time, since

$$\left\langle \delta_x, \mathrm{e}^{-\mathrm{i}tH}\delta_y\right\rangle \approx \underbrace{\left\langle \delta_x,\delta_y\right\rangle}_{=0} - \mathrm{i}t\left\langle \delta_x, H\delta_y\right\rangle + \mathcal{O}\left(t^2\right) .$$

Example 1.3 (Kinetic energy)**.** We present the *discrete Laplacian* on $\ell^2\left(\mathbb{Z}^d\right)\ni\psi$ as

$$\left(-\Delta\psi\right)_x := \sum_{y\sim x}\psi_x-\psi_y \qquad \left(x\in\mathbb{Z}^d\right) .$$

Here $y\sim x$ means all vertices $y\in\mathbb{Z}^d$ which share an edge with x, i.e., nearest neighbors of x. There are different ways to denote the discrete Laplacian, as well as normalize it. First, consider $\left\{\ R_j\ \right\}_{j=1}^d$ as the right-shift operators on $\ell^2\left(\mathbb{Z}^d\right)$. In particular, they are defined as

$$\left(R_j\psi\right)_x := \psi_{x-e_j} \qquad \left(x\in\mathbb{Z}^d\right)$$

where $\{ e_j \}_{j=1}^d$ is the standard basis for $\mathbb{R}^d$. Then

$$-\Delta = 2d\mathbb{1} - \sum_{j=1}^d R_j + R_j^* \,.$$

We shall shortly see that with this normalization,

$$\sigma(-\Delta) = \sigma_{\text{ac}}(-\Delta) = [0, 4d] \,.$$

Example 1.4 (A potential)**.** Let $V : \mathbb{Z}^d \to \mathbb{R}$ be any sequence (with fine print on it later). Then we define on $\ell^2(\mathbb{Z}^d) \ni \psi$, the operator $V(X)$ as

$$(V(X)\psi)_x := V(x)\psi_x \qquad (x \in \mathbb{Z}^d) \,.$$

This may be denoted also using *the position operator* as follows: Define the (unbounded, vector-valued) position operator X on $\ell^2(\mathbb{Z}^d) \ni \psi$ as:

$$(X\psi)_x := x\psi_x \qquad (x \in \mathbb{Z}^d) \,.$$

Then we proceed to interpret $V(X)$ via the measurable functional calculus (using the fact that $[X_i, X_j] = 0$).

Example 1.5 (Non-local operator)**.** It is instructive to consider an example of a *non-local* operator. To that end, consider the operator A on $\ell^2(\mathbb{Z})$ given by the matrix elements

$$A_{xy} := \frac{2(-1 + \cos(\pi(x-y)) + \pi(x-y)\sin(\pi(x-y)))}{(x-y)^2} \qquad (x, y \in \mathbb{Z}^d) \,.$$

We claim that this definition yields a bounded operator. But from this equation it is clear that its decay is merely like $n \mapsto \frac{1}{n}$ which is very slow, not even summable. We shall never call operators whose integral kernel does not even decay in a summable way local.

1.2 Bloch decomposition–the Fourier series

A basic tool for us to understand and diagonalize certain operators will be the Fourier series. In physics language, this is "going to momentum space" by way of a Fourier transform. Concretely, we define

$$\mathcal{F} : \ell^2(\mathbb{Z}^d) \to L^2(\mathbb{T}^d)$$

where $\mathbb{T}^d \equiv [-\pi, \pi)^d$ is the d-dimensional torus, and we take the L^2 space on it with the Lebesgue measure.

$$L^2(\mathbb{T}^d) \equiv \left\{ \hat{\psi} : \mathbb{T}^d \to \mathbb{C} \,\middle|\, \int \left|\hat{\psi}(k)\right| \mathrm{d}k < \infty \right\} \,.$$

The definition is then

$$(\mathcal{F}\psi)(k) := \sum_{x \in \mathbb{Z}^d} \mathrm{e}^{-\mathrm{i}k\cdot x}\psi_x \qquad (k \in \mathbb{T}^d) \,.$$

To make sense initially we would define $\mathcal{F}$ on $\ell^1 \cap \ell^2$ and then extend. Its inverse is given by

$$\left(\mathcal{F}^{-1}\hat{\psi}\right)_x := (2\pi)^{-d} \int_{k \in \mathbb{T}^d} \mathrm{e}^{\mathrm{i}k\cdot x}\hat{\psi}(k)\,\mathrm{d}k \,.$$

With this definition, we have

Claim 1.6 (Parseval)*.* $\mathcal{F}$ is a unitary operator (up to a constant)

Proof. We calculate

$$\begin{aligned}\langle \mathcal{F}\psi,\mathcal{F}\varphi\rangle_{L^2} &\equiv \int_{k\in\mathbb{T}^d}\overline{(\mathcal{F}\psi)(k)}(\mathcal{F}\varphi)(k)\,\mathrm{d}k\\ &\overset{\star}{=} \int_{k\in\mathbb{T}^d}\overline{\sum_{x\in\mathbb{Z}^d}\mathrm{e}^{-\mathrm{i}k\cdot x}\psi_x}\sum_{y\in\mathbb{Z}^d}\mathrm{e}^{-\mathrm{i}k\cdot x}\varphi_y\mathrm{d}k\\ &= \sum_{x,y\in\mathbb{Z}^d}\overline{\psi_x}\varphi_y\underbrace{\int_{k\in\mathbb{T}^d}\mathrm{e}^{\mathrm{i}k\cdot(x-y)}\mathrm{d}k}_{=(2\pi)^d\delta_{xy}}\\ &= (2\pi)^d\sum_{x\in\mathbb{Z}^d}\overline{\psi_x}\varphi_x\\ &\equiv (2\pi)^d\langle\psi,\varphi\rangle_{\ell^2}\,.\end{aligned}$$

To complete the proof, in $\star$ we should use Abel summation (see e.g. my Complex Analysis lecture notes [Sha23], the proof of Theorem 8.5; we avoid these details here). □

The big advantage of the Fourier series is in the fact that certain operators are easy to diagonalize using it. These operators are the *periodic* or *translation invariant* operators

Definition 1.7 (Periodic operator)**.** An operator $A\in\mathcal{B}\left(\ell^2\left(\mathbb{Z}^d\right)\right)$ is called periodic iff

$$A_{x,y}=A_{x+z,y+z}\qquad\left(x,y,z\in\mathbb{Z}^d\right)\,. \tag{1.3}$$

Lemma 1.8. *If $A\in\mathcal{B}\left(\ell^2\left(\mathbb{Z}^d\right)\right)$ is periodic then there exists some $a:\mathbb{T}^d\to\mathbb{C}$ such that, if M_a is the diagonal multiplication operator on $L^2\left(\mathbb{T}^d\right)$ by the function a, i.e.,*

$$\left(M_a\hat{\psi}\right)(k)\equiv a(k)\,\hat{\psi}(k)\qquad\left(k\in\mathbb{T}^d;\psi\in L^2\left(\mathbb{T}^d\right)\right)$$

then

$$\mathcal{F}A\mathcal{F}^*=M_a\,.$$

In fact,

$$a(k):=\sum_{x\in\mathbb{Z}^d}\mathrm{e}^{\mathrm{i}\langle k,x\rangle}A_{0,x}\qquad\left(k\in\mathbb{T}^d\right)\,.$$

a is called the symbol *associated to A.*

Moreover,

$$\sigma(A)=\sigma_{ac}(A)=\mathrm{im}(a)\equiv\left\{\,a(k)\;\middle|\;k\in\mathbb{T}^d\,\right\}\,.$$

Proof. We calculate

$$\begin{aligned}\left(\mathcal{F}A\mathcal{F}^*\hat{\psi}\right)(k) &\equiv \sum_{x\in\mathbb{Z}^d}\mathrm{e}^{-\mathrm{i}\langle k,x\rangle}\sum_{y\in\mathbb{Z}^d}A_{xy}\int_{p\in\mathbb{T}^d}\mathrm{e}^{+\mathrm{i}\langle p,y\rangle}\hat{\psi}(p)\,\mathrm{d}p\\ &= \sum_{x\in\mathbb{Z}^d}\mathrm{e}^{-\mathrm{i}\langle k,x\rangle}\sum_{y\in\mathbb{Z}^d}A_{0,y-x}\int_{p\in\mathbb{T}^d}\mathrm{e}^{+\mathrm{i}\langle p,y\rangle}\hat{\psi}(p)\,\mathrm{d}p\\ &= \sum_{x\in\mathbb{Z}^d}\mathrm{e}^{-\mathrm{i}\langle k,x\rangle}\sum_{z\in\mathbb{Z}^d}A_{0,z}\int_{p\in\mathbb{T}^d}\mathrm{e}^{+\mathrm{i}\langle p,z+x\rangle}\hat{\psi}(p)\,\mathrm{d}p\\ &= \sum_{z\in\mathbb{Z}^d}A_{0,z}\int_{p\in\mathbb{T}^d}\mathrm{e}^{+\mathrm{i}\langle p,z\rangle}\delta(k-p)\,\hat{\psi}(p)\,\mathrm{d}p\\ &= \sum_{z\in\mathbb{Z}^d}\mathrm{e}^{\mathrm{i}\langle k,z\rangle}A_{0,z}\hat{\psi}(k)\,.\end{aligned}$$

Let us thus define

$$a(k) := \sum_{z\in\mathbb{Z}^d} \mathrm{e}^{\mathrm{i}\langle k,z\rangle} A_{0,z} \qquad \left(k\in\mathbb{T}^d\right) .$$

The claim about the spectrum follows via the functional calculus of diagonal operators. We leave the part about the spectrum being purely absolutely continuous as an exercise to the reader. □

Example 1.9. The right-shift in direction $j = 1, \dots, d$ operator R_j is defined as

$$(R_j\psi)_y \quad\equiv\quad \psi_{y-e_j} \qquad \left(y\in\mathbb{Z}^d, \psi\in\ell^2\left(\mathbb{Z}^d\right)\right)$$

It is a periodic operator and and hence its Fourier representation is a diagonal multiplication operator:

$$\mathcal{F}R_j\mathcal{F}^* \quad=\quad M_{r_j}$$

with

$$\begin{aligned} r_j(k) &= \sum_{x\in\mathbb{Z}^d} \mathrm{e}^{-\mathrm{i}k\cdot x} (R_j)_{0,x} \\ &= \sum_{x\in\mathbb{Z}^d} \mathrm{e}^{-\mathrm{i}k\cdot x} \langle\delta_0, R_j\delta_x\rangle \\ &= \sum_{x\in\mathbb{Z}^d} \mathrm{e}^{-\mathrm{i}k\cdot x} \left\langle\delta_0, \delta_{x-e_j}\right\rangle \\ &= \sum_{x\in\mathbb{Z}^d} \mathrm{e}^{-\mathrm{i}k\cdot x} \delta_{x-e_j,0} \\ &= \sum_{x\in\mathbb{Z}^d} \mathrm{e}^{-\mathrm{i}k\cdot x} \delta_{x,e_j} \\ &= \mathrm{e}^{-\mathrm{i}k_j} . \end{aligned}$$

for all $k\in\mathbb{T}^d$.

Example 1.10. The discrete Laplacian $-\Delta$ defined via

$$(-\Delta\psi)_x \quad:=\quad \sum_{y\sim x}\psi_x-\psi_y \qquad \left(\psi\in\ell^2\left(\mathbb{Z}^d\right), x\in\mathbb{Z}^d\right)$$

is periodic. It may be re-written using the right-shift operator as

$$-\Delta \quad=\quad 2\sum_{j=1}^d \mathbb{1}-\mathbb{Re}\{R_j\}$$

with $\mathbb{Re}\{A\}\equiv\frac{1}{2}(A+A^*)$. In momentum (i.e., Fourier) space, it is given as multiplication by the function

$$\mathcal{E}(k) \quad:=\quad 2\sum_{j=1}^d 1-\cos(k_j) \qquad \left(k\in\mathbb{T}^d\right) .$$

We note that

$$2(1-\cos(k_j)) = 4\left[\sin\left(\frac{1}{2}k_j\right)\right]^2$$

so that for infinitesimal k,

$$\mathcal{E}(k)\approx\|k\|^2$$

which resembles the dispersion relation of the Laplacian on $L^2\left(\mathbb{R}^d\right)$. Hence, the discrete Laplacian is “accurate” for small momenta and “distorted” for large momenta (small distances). But in condensed matter physics we are mainly interested in large scales, i.e., small momenta, so that this distortion is not something we care about: it is just making our lives easier mathematically speaking.

Example 1.11. The position operator in the jth direction $(j = 1, \dots, d)$

$$(X_j\psi)_y \equiv y_j\psi_y \qquad \left(y \in \mathbb{Z}^d, \psi \in \ell^2\left(\mathbb{Z}^d\right)\right) \tag{1.4}$$

gets mapped to to derivative with respect to momentum, i.e.,

$$\mathcal{F} X_j \mathcal{F}^* = \mathrm{i}\partial_{k_j} \qquad (j = 1, \dots, d) \; .$$

Proof. We calculate

$$\begin{aligned}
\left(\mathcal{F} X_j \mathcal{F}^* \hat{\psi}\right)(k) &= \sum_{x\in\mathbb{Z}^d} \mathrm{e}^{-\mathrm{i}k\cdot x} \left(X_j \mathcal{F}^* \hat{\psi}\right)_x \\
&= \sum_{x\in\mathbb{Z}^d} \underbrace{\mathrm{e}^{-\mathrm{i}k\cdot x} x_j}_{=\mathrm{i}\partial_{k_j}} \left(\mathcal{F}^* \hat{\psi}\right)_x \\
&= \mathrm{i}\partial_{k_j} \left(\mathcal{F}\mathcal{F}^* \hat{\psi}\right)(k) \; .
\end{aligned}$$

□

Example 1.12. If A is periodic then the commutator $[X_j, A]$ is mapped to the derivative:

$$[X_j, A] \mapsto \mathrm{i} M_{\partial_j a} \, .$$

Proof. We have

$$\begin{aligned}
\left(\mathcal{F}\left[X_j, A\right]\mathcal{F}^* \hat{\psi}\right)(k) &= \left(\mathcal{F}\left[X_j, A\right]\mathcal{F}^* \hat{\psi}\right)(k) \\
&= \left(\mathcal{F} X_j \mathcal{F}^* \mathcal{F} A \mathcal{F}^* \hat{\psi}\right)(k) - \left(\mathcal{F} A \mathcal{F}^* \mathcal{F} X_j \mathcal{F}^* \hat{\psi}\right)(k) \\
&= \left(\mathrm{i}\partial_j M_a \hat{\psi}\right)(k) - \left(M_a \mathrm{i}\partial_j \hat{\psi}\right)(k) \\
&\overset{\text{Leibniz}}{=} M_{\mathrm{i}\partial_{k_j} a} \hat{\psi} \, .
\end{aligned}$$

□

Example 1.13. A multiplication operator in real space M_v by the function $v : \mathbb{Z}^d \to \mathbb{R}$ is mapped onto the convolution operator $C_{\hat{v}}$ in momentum space.

Proof. Use the convolution theorem for Fourier series:

$$\begin{aligned}
\left(\mathcal{F} v\left(X\right) \mathcal{F}^* \hat{\psi}\right)(k) &= \sum_{x\in\mathbb{Z}^d} \mathrm{e}^{-\mathrm{i}k\cdot x} \left(v\left(X\right) \mathcal{F}^* \hat{\psi}\right)_x \\
&= \sum_{x\in\mathbb{Z}^d} \mathrm{e}^{-\mathrm{i}k\cdot x} v\left(x\right) \left(\mathcal{F}^* \hat{\psi}\right)_x \, .
\end{aligned}$$

If we identify

$$\left(\mathcal{F} v\right)(k) \equiv \hat{v}\left(k\right) \equiv \sum_{x\in\mathbb{Z}^d} \mathrm{e}^{-\mathrm{i}k\cdot x} v\left(x\right)$$

then

$$\begin{aligned}\left(\mathcal{F}v\left(X\right)\mathcal{F}^{*}\hat{\psi}\right)\left(k\right) & = \sum_{x\in\mathbb{Z}^{d}}\mathrm{e}^{-\mathrm{i}k\cdot x}\left(2\pi\right)^{-d}\int_{p\in\mathbb{T}^{d}}\mathrm{e}^{\mathrm{i}p\cdot x}\hat{v}\left(p\right)\mathrm{d}p\left(\mathcal{F}^{*}\hat{\psi}\right)_{x}\\ & = \left(2\pi\right)^{-d}\int_{p\in\mathbb{T}^{d}}\hat{v}\left(p\right)\mathrm{d}p\sum_{x\in\mathbb{Z}^{d}}\mathrm{e}^{-\mathrm{i}\left(k-p\right)\cdot x}\left(\mathcal{F}^{*}\hat{\psi}\right)_{x}\\ & = \int_{p\in\mathbb{T}^{d}}\hat{v}\left(p\right)\hat{\psi}\left(k-p\right)\mathrm{d}p\\ & \equiv \left(C_{\hat{v}}\hat{\psi}\right)\left(k\right)\,.\end{aligned}$$

We thus recognize that

$$\mathcal{F}M_{v}\mathcal{F}^{*}=C_{\mathcal{F}v}\,.$$

□

The following theorem from classical harmonic analysis [Kat04, pp. 27] associates locality in real space to regularity in momentum space:

Theorem 1.14. *(Riemann-Lebesgue) If A is local as in (1.1) and periodic as in (1.3), so that $\mathcal{F}A\mathcal{F}^{*}=M_{a}$, then $a:\mathbb{T}^{d}\to\mathbb{C}$ is analytic in an annulus.*

Proof. We have from Lemma 1.8 that

$$a\left(z\right) = \sum_{x\in\mathbb{Z}^{d}}\prod_{j=1}^{d}z_{j}^{x_{j}}A_{0,x}\qquad\left(z\in\left(\mathbb{S}^{1}\right)^{d}\right)\,.$$

Now deforming z, we write it instead as of $\tilde{z}=rz$ where $r\in\left(0,\infty\right)^{d}$ so that

$$\begin{aligned}\left|a\left(\tilde{z}\right)\right| & \leq\sum_{x\in\mathbb{Z}^{d}}\left(\prod_{j=1}^{d}r_{j}^{x_{j}}\right)\left\Vert A_{0,x}\right\Vert \\ & \leq C_{A}\sum_{x\in\mathbb{Z}^{d}}\left(\prod_{j=1}^{d}r_{j}^{x_{j}}\right)\mathrm{e}^{-\mu\left\Vert x\right\Vert }\,.\,. & \text{(Using ??)}\end{aligned}$$

Now we use

$$\left\Vert x\right\Vert \geq\sum_{j=1}^{d}\frac{\left|x_{j}\right|}{\sqrt{d}}$$

to get

$$\left|a\left(\tilde{z}\right)\right|\leq C_{A}\prod_{j=1}^{d}\sum_{x_{j}\in\mathbb{Z}}r_{j}^{x_{j}}\mathrm{e}^{-\frac{\mu}{\sqrt{d}}\left|x_{j}\right|}$$

Clearly this is finite if $\mathrm{e}^{-\frac{\mu_{A}}{\sqrt{d}}}<\max\left(\left\{ r_{j},\frac{1}{r_{j}}\right\} \right)$. Hence we get a convergent power series in an annulus about the torus which is equivalent to analyticity on that annulus [TODO: cite this equivalence]. □

Remark 1.15. We also have the converse statement: if $a:\mathbb{T}^{d}\to\mathbb{C}$ is analytic in an annulus then $\mathcal{F}^{*}M_{a}\mathcal{F}$ is exponentially local. We leave this as an exercise to the reader (see e.g. [Sha23] Lemma 8.4).

Remark 1.16. More generally, any-rate polynomial decay will be mapped to smooth “symbols”, and ℓ^{p} locality will be mapped to C^{p} reguliarty of the symbol.

1.3 Consequences of locality

The significance of locality is clear from the following Lieb-Robinson theorem. It is usually discussed in the context of many-body quantum mechanics [LR72], but here in the single-particle setting, obtains a particularly simple guise, which we take from [AW15, Exercises 2.2 (a)]:

Theorem 1.17. *(Single-particle Lieb-Robinson) If $H = H^* \in \mathcal{B}\left(\ell^2\left(\mathbb{Z}^d\right) \otimes \mathbb{C}^N\right)$ is local as in (1.1), i.e., that there exist $C_H, \mu_H \in (0, \infty)$ such that*

$$\|H_{xy}\| \leq C_H \mathrm{e}^{-\mu_H \|x-y\|} \qquad \left(x, y \in \mathbb{Z}^d\right) .$$

Then there is some velocity $v_H \in (0, \infty)$ and some $D < \infty$ such that for any $v > v_H$,

$$\mathbb{P}\left[\{ \text{ a particle starting at the origin is outside } B_{vt}(0) \text{ after time } t \}\right] \leq D\mathrm{e}^{-\frac{\mu_H}{2}(v-v_H)t} \qquad (t \geq 0) . \tag{1.5}$$

Here we mean $B_{vt}(0) \equiv \left\{ x \in \mathbb{Z}^d \mid \|x\| < vt \right\}$.

Proof. First we interpret the LHS probability. We know from quantum mechanics that the state of a particle starting in the origin is δ_0. Since we have internal degrees of freedom we allow for an arbitrary state φ in $\mathbb{C}^N$ so we take the initial state of the particle as $\delta_0 \otimes \varphi$. We know that after time t, its state, according to quantum mechanics, is

$$\mathrm{e}^{-\mathrm{i}tH}\delta_0 \otimes \varphi$$

and finally, the probability to measure its position at some $y \in \mathbb{Z}^d$ (in some internal state $\psi \in \mathbb{C}^N$) is

$$\left|\left\langle \delta_y \otimes \psi, \mathrm{e}^{-\mathrm{i}tH}\delta_0 \otimes \varphi\right\rangle\right|^2 .$$

We thus bound the LHS of (1.5) by

$$N \times \sup_{\varphi,\psi \in \mathbb{C}^N : \|\varphi\|=\|\psi\|=1} \sum_{y \in B_{vt}(0)^c} \left|\left\langle \delta_y \otimes \psi, \mathrm{e}^{-\mathrm{i}tH}\delta_0 \otimes \varphi\right\rangle\right|^2 = \sum_{y \in B_{vt}(0)^c} \left\|\left\langle \delta_y, \mathrm{e}^{-\mathrm{i}tH}\delta_0\right\rangle\right\|^2 . \tag{1.6}$$

Now, we begin with a few preliminary estimates: For any $n \in \mathbb{N}$,

$$\begin{aligned}
\left\|\left(H^n\right)_{xy}\right\| &\equiv \left\| \sum_{z_1,\dots,z_{n-1} \in \mathbb{Z}^d} H_{x,z_1} \dots H_{z_{n-1},y} \right\| \\
&\leq \sum_{z_1,\dots,z_{n-1} \in \mathbb{Z}^d} \|H_{x,z_1}\| \dots \|H_{z_{n-1},y}\| \\
&\leq \sum_{z_1,\dots,z_{n-1} \in \mathbb{Z}^d} C_H^n \mathrm{e}^{-\mu_H(\|x-z_1\|+\dots+\|z_{n-1}-y\|)} && \text{(Locality of } H\text{)} \\
&\leq C_H^n \mathrm{e}^{-\frac{\mu_H}{2}\|x-y\|} \sum_{z_1,\dots,z_{n-1} \in \mathbb{Z}^d} \mathrm{e}^{-\frac{\mu_H}{2}(\|z_2-z_1\|+\dots+\|z_{n-1}-y\|)} .
\end{aligned}$$

In this last step, we have used the triangle inequality:

$$\|x - z_1\| + \dots + \|z_{n-1} - y\| \quad \geq \quad \|x - y\|$$

as well as dropping the first term since it is clearly positive. Since $\mathbb{Z}^d$ is invariant under translations, we find

$$\sum_{z_1,\dots,z_{n-1} \in \mathbb{Z}^d} \mathrm{e}^{-\nu(\|z_2-z_1\|+\dots+\|z_{n-1}-y\|)} \quad = \quad \left(\sum_{z \in \mathbb{Z}^d} \mathrm{e}^{-\nu\|z\|} \right)^{n-1}$$

but the inner sum is clearly finite. E.g., $\|z\| \geq \frac{1}{\sqrt{d}}\|z\|_1$ with $\|z\|_1 \equiv \sum_{j=1}^d |z_j|$ which then factorizes:

$$\begin{aligned} D_{\nu,d} &:= \sum_{z\in\mathbb{Z}^d} \mathrm{e}^{-\nu\|z\|} \\ &\leq \left(\sum_{z\in\mathbb{Z}} \mathrm{e}^{-\frac{\nu}{\sqrt{d}}|z|}\right)^d \\ &= \left[\coth\left(\frac{\nu}{2\sqrt{d}}\right)\right]^d \\ &< \infty\,. \end{aligned} \tag{1.7}$$

Combining everything together we have the estimate for any $n \in \mathbb{N}_{\geq 1}$,

$$\begin{aligned} \left\|(H^n)_{xy}\right\| &\leq C_H^n \mathrm{e}^{-\frac{\mu_H}{2}\|x-y\|}\left(D_{\frac{\mu_H}{2},d}\right)^{n-1} \\ &= \frac{1}{D_{\frac{\mu_H}{2},d}}\left(C_H D_{\frac{\mu_H}{2},d}\right)^n \mathrm{e}^{-\frac{\mu_H}{2}\|x-y\|}\,. \end{aligned}$$

Next, we have

$$\begin{aligned} \left\|\left\langle\delta_y, \mathrm{e}^{-\mathrm{i}tH}\delta_0\right\rangle\right\| &= \left\|\sum_{n=0}^\infty \frac{(-\mathrm{i}t)^n}{n!}\left\langle\delta_y, H^n\delta_0\right\rangle\right\| \\ &\leq \sum_{n=0}^\infty \frac{t^n}{n!}\left\|(H^n)_{y,0}\right\| \\ &\leq \sum_{n=0}^\infty \frac{t^n}{n!}\frac{1}{D_{\frac{\mu_H}{2},d}}\left(C_H D_{\frac{\mu_H}{2},d}\right)^n \mathrm{e}^{-\frac{\mu_H}{2}\|y\|} && \text{(Previous estimate)} \\ &= \frac{1}{D_{\frac{\mu_H}{2},d}}\mathrm{e}^{tC_H D_{\frac{\mu_H}{2},d}-\frac{\mu_H}{2}\|y\|} \end{aligned}$$

and hence the RHS of (1.6) is bounded by (using $y \in B_{vt}(0)^c$ implies $\|y\| \geq vt$):

$$\begin{aligned} \sum_{y\in B_{vt}(0)^c}\left(D_{\frac{\mu_H}{2},d}\right)^{-2}\mathrm{e}^{2tC_H D_{\frac{\mu_H}{2},d}-\mu_H\|y\|} &\leq \left(D_{\frac{\mu_H}{2},d}\right)^{-2}\mathrm{e}^{2tC_H D_{\frac{\mu_H}{2},d}-\frac{\mu_H}{2}vt}\sum_{y\in\mathbb{Z}^d}\mathrm{e}^{-\frac{\mu_H}{2}\|y\|} \\ &\leq \left(D_{\frac{\mu_H}{2},d}\right)^{-2}\mathrm{e}^{2tC_H D_{\frac{\mu_H}{2},d}-\frac{\mu_H}{2}vt}D_{\frac{\mu_H}{2},d} \\ &= \frac{1}{D_{\frac{\mu_H}{2},d}}\mathrm{e}^{-\frac{\mu_H}{2}\left(v-4\frac{C_H D_{\frac{\mu_H}{2},d}}{\mu_H}\right)t} \end{aligned}$$

and so we identify $v_H := 4\frac{C_H D_{\frac{\mu_H}{2},d}}{\mu_H}$ and $D := N\frac{1}{D_{\frac{\mu_H}{2},d}}$. □

While the Lieb-Robinson bound gives an intuitive sense for what locality implies for quantum dynamics, we will find more for the Combes-Thomas estimate. Again, originally presented in the context of many-body quantum mechanics [CT73], the single-particle version ([AW15, Chapter 10.3]), presented here roughly speaking says that the analytic functional calculus of Hamiltonians preserves locality:

Theorem 1.18. *(The Combes-Thomas estimate) If $H = H^* \in \mathcal{B}\left(\ell^2\left(\mathbb{Z}^d\right)\otimes\mathbb{C}^N\right)$ is local as in (1.1) with decay estimate μ_H, and $z\in\mathbb{C}$ with*

$$\delta := \operatorname{dist}\left(z,\sigma\left(H\right)\right) > 0 \tag{1.8}$$

then there is some $\tilde{\mu}_H > 0$ (which remains finite as $\delta\to 0$) such that

$$\left\|R\left(z\right)_{xy}\right\| \leq \frac{2}{\delta}\mathrm{e}^{-\tilde{\mu}_H\delta\|x-y\|} \qquad \left(x,y\in\mathbb{Z}^d\right)$$

with $R\left(z\right)\equiv\left(H-z\mathbb{1}\right)^{-1}$ being the resolvent operator and $\sigma\left(H\right)\subseteq\mathbb{R}$ the spectrum of H.

The constant $\tilde{\mu}_H$ may be expressed in terms of μ_H as

$$\frac{1}{\delta}\min\left(\left\{\frac{\delta}{4C_H D_{\frac{1}{2}\mu_H,d}},\frac{1}{4}\mu_H\right\}\right).$$

Corollary 1.19. *(The analytic functional calculus of a local self-adjoint operator is local) Assume that $f:\mathbb{R}\to\mathbb{C}$ is real-analytic, i.e., that,*

$$f\left(\lambda\right) = \frac{1}{2\pi\mathrm{i}}\oint_\Gamma\frac{1}{z-\lambda}f\left(z\right)\mathrm{d}z \qquad \left(\lambda\in\mathbb{R}\right)$$

for some closed CCW contour Γ which encloses $\sigma\left(H\right)$. Then if $H=H^*\in\mathcal{B}\left(\ell^2\left(\mathbb{Z}^d\right)\otimes\mathbb{C}^N\right)$ is local as in (1.1) then $f\left(H\right)$ is local also as in (1.1).

Proof of Corollary 1.19. Write

$$f\left(H\right) \quad = \quad \frac{\mathrm{i}}{2\pi}\oint_\Gamma R\left(z\right)f\left(z\right)\mathrm{d}z$$

where Γ is a closed CCW contour which encloses $\sigma\left(H\right)$. Since H is bounded, $\sigma\left(H\right)$ has a finite diameter. Let Γ be a contour which always stays distance 1 away from $\sigma\left(H\right)$, so that, say,

$$\oint_\Gamma|\mathrm{d}z|\leq 2\left(\|H\|+1\right)+2\,.$$

Then

$$\begin{aligned}\left\|f\left(H\right)_{xy}\right\| &\leq \frac{1}{2\pi}\sup_{z\in\Gamma}\left\|R\left(z\right)_{xy}\right\|\sup_{z\in\Gamma}|f\left(z\right)|\oint_\Gamma|\mathrm{d}z|\\ &\leq \frac{1}{2\pi}\left(\frac{2}{1}\mathrm{e}^{-\tilde{\mu}_H 1\|x-y\|}\right)\|f\|_{L^\infty(\Gamma)}2\left(\|H\|+2\right)\\ &= \frac{2}{\pi}\|f\|_{L^\infty(\Gamma)}\left(\|H\|+2\right)\mathrm{e}^{-\tilde{\mu}_H 1\|x-y\|}\,.\end{aligned}$$

Note we could indeed make the contour bigger so as to make δ bigger (and get better exponential decay) but that would worsen the constants outside the exponential. □

Proof of Theorem 1.18 . Let $f:\mathbb{Z}^d\to\mathbb{C}$ be a *bounded* sequence function such that there is some $\nu\in(0,\infty)$ with

$$|f\left(x\right)-f\left(y\right)| \quad\leq\quad \nu\|x-y\| \qquad \left(x,y\in\mathbb{Z}^d\right).$$

Define then

$$H_f \quad := \quad \mathrm{e}^{f(X)}H\mathrm{e}^{-f(X)}$$

which is clearly also bounded. A short calculation yields

$$\begin{aligned}\left[(H_f - z\mathbb{1})^{-1}\right]_{xy} &= \left[\left(\mathrm{e}^{f(X)}H\mathrm{e}^{-f(X)} - z\mathbb{1}\right)^{-1}\right]_{xy}\\ &= \left[\left(\mathrm{e}^{f(X)}H\mathrm{e}^{-f(X)} - z\mathrm{e}^{f(X)}\mathrm{e}^{-f(X)}\right)^{-1}\right]_{xy}\\ &= \left[\mathrm{e}^{-f(X)}R(z)\,\mathrm{e}^{f(X)}\right]_{xy}\\ &= \mathrm{e}^{-f(x)}R(z)_{xy}\,\mathrm{e}^{f(y)}\end{aligned}$$

Hence,

$$\begin{aligned}\left\|R(z)_{xy}\right\| &= \mathrm{e}^{f(x)-f(y)}\left\|R_f(z)_{xy}\right\|\\ &\leq \mathrm{e}^{f(x)-f(y)}\|R_f(z)\|\,.\end{aligned}$$

But for any $\varphi \in \mathcal{H}$,

$$\begin{aligned}\|(H_f - z\mathbb{1})\varphi\| &= \|(H - z\mathbb{1})\varphi\| - \|(H_f - H)\varphi\|\\ &\geq \delta\|\varphi\| - \|H_f - H\|\|\varphi\|\end{aligned}$$

where we have used (1.8) in the last step. Let us remark that

$$\begin{aligned}(H_f - H)_{xy} &\equiv (H_f)_{xy} - H_{xy}\\ &= \mathrm{e}^{f(x)}H_{xy}\mathrm{e}^{-f(y)} - H_{xy}\\ &= \left(\mathrm{e}^{f(x)-f(y)} - 1\right)H_{xy}\,.\end{aligned}$$

Hence, using Holmgren's bound (see Lemma 1.20 just below) and the fact that

$$|\alpha| \leq \beta \Longrightarrow |\mathrm{e}^{\alpha} - 1| \leq \mathrm{e}^{\beta} - 1$$

we have then

$$\begin{aligned}\|H_f - H\| &\leq \max_{x\leftrightarrow y}\sup_x\sum_y\left\|(H_f - H)_{xy}\right\|\\ &= \max_{x\leftrightarrow y}\sup_x\sum_y\left|\mathrm{e}^{f(x)-f(y)} - 1\right|\|H_{xy}\|\\ &\leq \max_{x\leftrightarrow y}\sup_x\sum_y\left(\mathrm{e}^{\nu\|x-y\|} - 1\right)C_H\mathrm{e}^{-\mu_H\|x-y\|}\\ &= \sum_y\left(\mathrm{e}^{\nu\|y\|} - 1\right)C_H\mathrm{e}^{-\mu_H\|y\|}\\ &\leq 2C_H\nu\sum_y\mathrm{e}^{-(\mu_H-2\nu)\|y\|} \qquad \left(\text{Use } \mathrm{e}^{\nu\|y\|} - 1 \leq 2\nu\mathrm{e}^{2\nu\|y\|}\right)\\ &= 2C_H\nu D_{\mu_H-2\nu,d}\,.\end{aligned}$$

where $D_{\alpha,d} \equiv \sum_{x\in\mathbb{Z}^d}\mathrm{e}^{-\alpha\|x\|}$. Assuming that $\nu \leq \frac{1}{4}\mu_H$ we have

$$D_{\mu_H-2\nu,d} \leq D_{\frac{1}{2}\mu_H,d}$$

so we pick ν as

$$\nu := \min\left(\left\{\frac{\delta}{4C_H D_{\frac{1}{2}\mu_H,d}}, \frac{1}{4}\mu_H\right\}\right)$$

we find $\|R_f(z)\| \leq \frac{\delta}{2}$. Now thanks to the freedom $f \mapsto -f$ we have

$$\left\|R(z)_{xy}\right\| \leq \frac{2}{\delta}\min\left(\left\{ \mathrm{e}^{f(x)-f(y)}, \mathrm{e}^{-f(x)+f(y)} \right\}\right) = \frac{2}{\delta}\mathrm{e}^{-|f(x)-f(y)|}\,.$$

If we now take, for any $L \geq 0$,

$$f_L(x) := \nu \min(\{ L, \|\cdot - y\| \})$$

then clearly f_L is bounded by νL and

$$f_L(x) - f_L(y) = 0 - \nu\min(\{ L, \|x-y\| \})$$

since L is arbitrary here, we may take the limit $L \to \infty$ to obtain

$$\left\|R(z)_{xy}\right\| \leq \frac{2}{\delta}\mathrm{e}^{-\nu\|x-y\|} \qquad \left(x, y \in \mathbb{Z}^d\right)\,.$$

We recognize that

$$\tilde{\mu}_H := \frac{1}{\delta}\min\left(\left\{ \frac{\delta}{4C_H D_{\frac{1}{2}\mu_H,d}}, \frac{1}{4}\mu_H \right\}\right)\,.$$

□

Above we have used the following basic

Lemma 1.20. *(Holmgren's bound) For any operator A on a Hilbert space with an ONB $\{ \psi_j \}_j$ we have*

$$\|A\| \leq \sqrt{\sup_j \sum_k |\langle\psi_j, A\psi_k\rangle|}\sqrt{\sup_k \sum_j |\langle\psi_j, A\psi_k\rangle|}$$

Proof. Start by the characterization $\|A\| = \sup(\{ |\langle\varphi, A\psi\rangle| \mid \|\varphi\| = \|\psi\| = 1 \})$, and use

$$\begin{aligned} |\langle\varphi, A\psi\rangle| &\leq \sum_{i,j} |\varphi_i|\,|A_{ij}|\,|\psi_j| \\ &= \sum_{i,j}\left(|\varphi_i|\sqrt{|A_{ij}|}\right)\left(\sqrt{|A_{ij}|}\,|\psi_j|\right) \\ &\leq \sqrt{\sum_{i,j}|\varphi_i|^2|A_{ij}|}\sqrt{\sum_{i,j}|A_{ij}|\,|\psi_j|^2} && \text{(Cauchy-Schwarz)} \\ &\leq \sqrt{\left(\sup_i\sum_j|A_{ij}|\right)\left(\sum_i|\varphi_i|^2\right)}\sqrt{\left(\sup_j\sum_i|A_{ij}|\right)\left(\sum_j|\psi_j|^2\right)} \\ &= \sqrt{\sup_i\sum_j|A_{ij}|}\sqrt{\sup_j\sum_i|A_{ij}|}\,. \end{aligned}$$

□

For the sake of concreteness, let us get an estimate on the decay rate for operators which are nearest-neighbors:

Claim 1.21. If H is nearest-neighbor, in the sense that

$$H_{xy} = H_{xy}\chi_{\{0,1\}}(\|x-y\|) \qquad \left(x,y\in\mathbb{Z}^d\right)$$

and $z\in\mathbb{C}$ is such that

$$0<\delta:=\operatorname{dist}(z,\sigma(H))<\mathrm{e}\|H\|\coth\left(\frac{1}{4\sqrt{d}}\right)$$

then

$$\left|\left\langle\delta_x,(H-z\mathbb{1})^{-1}\delta_y\right\rangle\right| \leq \frac{2}{\delta}\exp\left(-\frac{C_d}{\|H\|}\delta\|x-y\|\right) \qquad \left(x,y\in\mathbb{Z}^d\right)$$

with

$$C_d:=\left(4\mathrm{e}\coth\left(\frac{1}{4\sqrt{d}}\right)\right)^{-1}.$$

In particular, for the discrete Laplacian on $\ell^2\left(\mathbb{Z}^d\right)$ normalized to have spectrum in $[0,4d]$ we have $\|H\|=4d$ and so

$$\left|\left\langle\delta_x,(-\Delta-z\mathbb{1})^{-1}\delta_y\right\rangle\right| \leq \frac{2}{\delta}\exp\left(-\frac{1}{16\mathrm{e}d\coth\left(\frac{1}{4\sqrt{d}}\right)}\delta\|x-y\|\right) \qquad \left(x,y\in\mathbb{Z}^d\right).$$

Proof. The locality estimate is obeyed with

$$\begin{aligned}|H_{xy}| &\leq \|H\|\chi_{\{0,1\}}(\|x-y\|)\\ &\leq \mathrm{e}\|H\|\mathrm{e}^{-\|x-y\|}.\end{aligned}$$

We thus recognize

$$\begin{aligned}C_H &:= \mathrm{e}\|H\|\\ \mu_H &:= 1\end{aligned}$$

for the locality of H and from this we conclude

$$\tilde{\mu}_H=\frac{1}{4\delta}\min\left(\left\{\frac{\delta}{\mathrm{e}\|H\|D_{\frac{1}{2},d}},1\right\}\right).$$

Now recall that

$$\begin{aligned}D_{\frac{1}{2},d} &\equiv \sum_{z\in\mathbb{Z}^d}\mathrm{e}^{-\frac{1}{2}\|z\|}\\ &\leq \sum_{z\in\mathbb{Z}^d}\mathrm{e}^{-\frac{1}{2}\frac{1}{\sqrt{d}}\|z\|_1}\\ &= \left(\sum_{z\in\mathbb{Z}}\mathrm{e}^{-\frac{1}{2\sqrt{d}}|z|}\right)^d\\ &= \coth\left(\frac{1}{4\sqrt{d}}\right).\end{aligned}$$

Assuming that $\delta<\mathrm{e}\|H\|\coth\left(\frac{1}{4\sqrt{d}}\right)$ then, we find

$$\tilde{\mu}_H \geq \frac{1}{4\mathrm{e}\|H\|\coth\left(\frac{1}{4\sqrt{d}}\right)}$$

□

1.3.1 Other regularity classes of functional calculus and their induced locality

We have seen that the *holomorphic* functional calculus preserves locality in Corollary 1.19. If we don't pay attention to regularity we could quickly lose locality, as the following example illustrates

Example 1.22 (Discontinuous function of local is *not* local)**.** Let $-\Delta$ be the discrete Laplacian on $\ell^2(\mathbb{Z})$ normalized to have spectrum on $[0,4]$. Clearly it is local as in (1.2) since it is nearest neighbor. Consider the spectral projection

$$P := \chi_{[0,2]}(-\Delta)$$

which is merely a bounded measurable function of $-\Delta$. It corresponds to the Fermi projection with Fermi energy $E_F = 2$. We can calculate P explicitly to see it does *not* decay quickly:

$$\begin{aligned} P_{xy} &= \frac{1}{2\pi}\int_{k=0}^{2\pi} \mathrm{e}^{\mathrm{i}k(x-y)}\chi_{[0,2]}(2-2\cos(k))\,\mathrm{d}k \\ &= \frac{1}{2\pi}\int_{k=0}^{2\pi} \mathrm{e}^{\mathrm{i}k(x-y)}\chi_{[0,2]}\left(4\left[\sin\left(\frac{k}{2}\right)\right]^2\right)\mathrm{d}k \\ &= \frac{1}{2\pi}\int_{k=-k_0}^{k_0} \mathrm{e}^{\mathrm{i}k(x-y)}\mathrm{d}k \\ &= \frac{1}{2\pi}\left[\frac{\mathrm{e}^{\mathrm{i}k_0(x-y)}}{\mathrm{i}k_0(x-y)} - \frac{\mathrm{e}^{-\mathrm{i}k_0(x-y)}}{-\mathrm{i}k_0(x-y)}\right] \\ &= \frac{1}{\pi}\frac{\cos(k_0(x-y))}{k_0(x-y)} \end{aligned}$$

where k_0 is the solution to

$$4\left[\sin\left(\frac{k_0}{2}\right)\right]^2 \equiv 2\,.$$

It is thus clear that

$$|P_{xy}| \sim \frac{1}{\pi k_0}\frac{1}{|x-y|}$$

and it hence does *not* exhibit rapid off-diagonal decay.

What about continuous functions? We shall use the following basic

Claim 1.23. If $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is a C-star algebra then the *continuous* functional calculus on normal operators within $\mathcal{A}$ lands within $\mathcal{A}$.

Proof. For some $A \in \mathcal{A}$ normal and $f : \mathbb{C} \to \mathbb{C}$ continuous, since $\sigma(A)$ is bounded, let $\{ p_k \}_{k\in\mathbb{N}}$ be a sequence of polynomials converging *uniformly* to f on $\sigma(A)$. Then $p_k(A) \in \mathcal{A}$ by the algebraic properties of a C-star algebra and hence $f(A) \in \mathcal{A}$ since $\|p_k(A) - f(A)\| \to 0$ as $k \to \infty$. □

Hence, if we can exhibit our locality as a C-star sub-algebra of $\mathcal{B}(\mathcal{H})$ we'll get locality of the continuous functional calculus. Is it so? It seems unlikely that one could prove that the condition of (1.2) forms a C-star algebra, essentially because the norm limit of exponentially decaying elements need not be exponentially decaying. Sometimes [TODO: cite 1D classification chapter] our locality will be given in terms a C-star algebraic condition which makes the continuous functional calculus very convenient.

However, it is also useful for us sometimes to consider the *smooth* functional calculus, which more easily preserves (1.2).

1.3.2 The smooth functional calculus

Let $f : \mathbb{R} \to \mathbb{C}$ be a smooth function, let $A : \mathcal{B}(\mathcal{H})$ be a bounded *self-adjoint* operator. Our goal is to define the operator $f(A)$.

We define the Wirthinger derivative

$$\partial_{\overline{z}} := \partial_x + \mathrm{i}\partial_y$$

as an operator on any function of a complex variable.

Let $\chi : \mathbb{R} \to \mathbb{R}$ be an *even* smooth function of compact support such that $\chi|_{B_\delta(0)} = 1$ for some $\delta > 1$. Let $N \in \mathbb{N}_{\geq 1}$. We define $\tilde{f} : \mathbb{C} \to \mathbb{C}$ as a *quasi-analytic extension* of f (which a-priori depends on χ, δ, N, but we shall see soon that these choices do not matter):

$$\tilde{f}(x + \mathrm{i}y) \quad := \quad \chi(y) \sum_{k=0}^{N} f^{(k)}(x) \frac{(\mathrm{i}y)^k}{k!} \qquad (x \in \mathbb{R}, y \in \mathbb{R}) \ .$$

First of all we note this is indeed an extension of f, since for $y = 0$ we obtain that only the $k = 0$ term survives, in which case $\tilde{f}(x) = f(x)$ as desired. Secondly, it is *quasi-analytic* in the sense that $\partial_{\overline{z}} \tilde{f}\Big|_{\mathbb{R}} = 0$ (recall that an analytic function $g : \mathbb{C} \to \mathbb{C}$ is one which obeys the Cauchy-Riemann equations, i.e., $\partial_{\overline{z}} g = 0$, so $\tilde{f}$ is *quasi*-analytic in the sense that it obeys the Cauchy-Riemann equations only when restricted to the real line).

Now we calculate

$$\begin{aligned}
\left(\partial_{\overline{z}} \tilde{f}\right)(x + \mathrm{i}y) &= \sum_{k=0}^{N} (\partial_x + \mathrm{i}\partial_y) f^{(k)}(x) \frac{(\mathrm{i}y)^k}{k!} \chi(y) \\
&= \sum_{k=0}^{N} f^{(k+1)}(x) \frac{(\mathrm{i}y)^k}{k!} \chi(y) - \sum_{k=1}^{N} f^{(k)}(x) \frac{(\mathrm{i}y)^{k-1}}{(k-1)!} \chi(y) + \mathrm{i} \sum_{k=0}^{N} f^{(k)}(x) \frac{(\mathrm{i}y)^k}{k!} \chi'(y) \\
&= \sum_{k=1}^{N+1} f^{(k)}(x) \frac{(\mathrm{i}y)^{k-1}}{(k-1)!} \chi(y) - \sum_{k=1}^{N} f^{(k)}(x) \frac{(\mathrm{i}y)^{k-1}}{(k-1)!} \chi(y) + \mathrm{i} \sum_{k=0}^{N} f^{(k)}(x) \frac{(\mathrm{i}y)^k}{k!} \chi'(y) \\
&= f^{(N+1)}(x) \frac{(\mathrm{i}y)^N}{N!} \chi(y) + \mathrm{i} \sum_{k=0}^{N} f^{(k)}(x) \frac{(\mathrm{i}y)^k}{k!} \chi'(y)
\end{aligned}$$

So that

$$\left(\partial_{\overline{z}} \tilde{f}\right)(x) \quad = \quad \mathrm{i}\chi'(0) f(x)$$

but since χ is a constant function (by *choice*) about zero, $\chi'(0) = 0$ and so $\left(\partial_{\overline{z}} \tilde{f}\right)(x) = 0$ so that $\tilde{f}$ is indeed quasi-analytic.

Fact 1.24. *We assume that f has compact support, which implies now that $f^{(k)}$ has compact support for all $k \geq 0$.*

Now we claim that, in analogy with the Cauchy integral formula, one has the following *two-dimensional integral* identity

$$f(a) \quad = \quad \frac{1}{2\pi} \int_{z \in \mathbb{C}} \left(\partial_{\overline{z}} \tilde{f}\right)(z) (a - z)^{-1} \mathrm{d}z \qquad (a \in \mathbb{R}) \ .$$

where the integral converges *absolutely*. In fact, this integral must be understood in the improper sense because for $z \in \mathbb{R}$, $a - z$ may be zero, so by writing this integral we *really* mean the limit

$$\frac{1}{2\pi} \int_{z \in \mathbb{C}} \left(\partial_{\overline{z}} \tilde{f}\right)(z) (a - z)^{-1} \mathrm{d}z \quad \equiv \quad \lim_{\varepsilon \to 0} \int_{z \in \{ z' \in \mathbb{C} \mid |\mathbb{Im}\{z\}| > \varepsilon \}} \left(\partial_{\overline{z}} \tilde{f}\right)(z) (a - z)^{-1} \mathrm{d}z \, .$$

By the formula

$$\left(\partial_{\overline{z}} \tilde{f}\right)(x + \mathrm{i}y) \quad = \quad f^{(N+1)}(x) \frac{(\mathrm{i}y)^N}{N!} \chi(y) + \mathrm{i} \sum_{k=0}^{N} f^{(k)}(x) \frac{(\mathrm{i}y)^k}{k!} \chi'(y)$$

we see that the first term is of compact support in $\mathbb{C}$ as $f^{(k)}$ has compact support and so does χ and χ'. Hence there is no question of integrability at infinity. Let us call the collective support of all terms involved K. To study integrability near the real axis, we study

$$\left|\left(\partial_{\overline{z}} \tilde{f}\right)(x + \mathrm{i}y)\right| \quad \leq \quad \frac{1}{N!} \left|f^{(N+1)}(x)\right| |y|^N |\chi(y)| + \sum_{k=0}^{N} \frac{1}{k!} \left|f^{(k)}(x)\right| |y|^k |\chi'(y)|$$

by assumption we have $|\chi'(y)| = |\chi'(y)| \chi_{\mathbb{R} \setminus B_1(0)}(y)$ where the second χ is the characteristic function. Hence

$$\left|\left(\partial_{\overline{z}} \tilde{f}\right)(x + \mathrm{i}y)\right| \quad \leq \quad C_N |y|^N$$

But $|a-z|^{-1} \leq |y|^{-1}$ so that

$$\begin{aligned}\left|\frac{1}{2\pi}\int_{z\in\mathbb{C}}\left(\partial_{\overline{z}}\tilde{f}\right)(z)(a-z)^{-1}\mathrm{d}z\right| &= \left|\frac{1}{2\pi}\int_{z\in K}\left(\partial_{\overline{z}}\tilde{f}\right)(z)(a-z)^{-1}\mathrm{d}z\right| \\ &\leq \frac{1}{2\pi}\int_{z\in K} C_N |y|^N |y|^{-1}\mathrm{d}z \\ &< \infty\end{aligned}$$

where the last bound is because $|K| < \infty$. So the integral converges absolutely.

One verifies that the limit

$$\lim_{\varepsilon\to 0}\underbrace{\frac{1}{2\pi}\int_{z\in\{\, z'\in\mathbb{C} \mid \mathbb{Im}\{z'\}>\varepsilon\,\}}\left(\partial_{\overline{z}}\tilde{f}\right)(z)(a-z)^{-1}\mathrm{d}z}_{=:f_\varepsilon(a)} = f(a)$$

converges pointwise: Indeed, $\mathbb{C} \ni z \mapsto (a-z)^{-1}$ is actually quasi-analytic, so that $\partial_{\overline{z}}(a-z)^{-1} = 0$. Hence partial integration (with respect to both x and y, which is tantamount to an application of Stokes theorem) yields only a boundary term,

$$f_\varepsilon(a) = \frac{1}{2\pi\mathrm{i}}\int_{x\in\mathbb{R}}\tilde{f}(x+\mathrm{i}y)(a-x-\mathrm{i}y)^{-1}\Big|_{y=-\varepsilon}^{y=\varepsilon}\mathrm{d}x\,.$$

Now

$$\begin{aligned}\tilde{f}(x\pm\mathrm{i}\varepsilon) &= \chi(\pm\varepsilon)\sum_{k=0}^{N} f^{(k)}(x)\frac{(\pm\mathrm{i}\varepsilon)^k}{k!} \\ &= \chi(0)f(x)+\varepsilon(\chi'(0)f(x)\pm\mathrm{i}\chi(0)f'(x))+\mathcal{O}\left(\varepsilon^2\right) \\ &= f(x)\pm\mathrm{i}\varepsilon f'(x)+\mathcal{O}\left(\varepsilon^2\right)\end{aligned}$$

so that

$$\begin{aligned}f_\varepsilon(a) &= \int_{x\in\mathbb{R}} f(x)\frac{1}{\pi}\mathbb{Im}\left\{(a-x-\mathrm{i}\varepsilon)^{-1}\right\}\mathrm{d}x + \\ &\quad +\int_{x\in\mathbb{R}} f'(x)\frac{1}{2\pi}\varepsilon\left((a-x-\mathrm{i}\varepsilon)^{-1}+(a-x+\mathrm{i}\varepsilon)^{-1}\right)\mathrm{d}x\end{aligned}$$

The first term is an approximate delta function (in the distribution sense, as $\varepsilon \to 0^+$). For the second term we use

$$\begin{aligned}\frac{1}{w}+\frac{1}{\overline{w}} &= \frac{\overline{w}+w}{|w|^2} \\ &= \frac{2\,\mathbb{Re}\{w\}}{|w|^2}\end{aligned}$$

to get

$$\begin{aligned}\frac{1}{2\pi}\varepsilon\left((a-x-\mathrm{i}\varepsilon)^{-1}+(a-x+\mathrm{i}\varepsilon)^{-1}\right) &= \frac{\varepsilon}{\pi}\frac{(a-x)}{(a-x)^2+\varepsilon^2} \\ &\overset{\varepsilon\to 0^+}{\longrightarrow} \begin{cases} 0 & x\neq a \\ 0 & x=a \end{cases} \\ &= 0\end{aligned}$$

Hence we learn that

$$\lim_{\varepsilon\to 0^+} f_\varepsilon(a) = f(a)\,,$$

as desired.

Now due to the functional calculus, which says that if $\lim_{\varepsilon\to 0} f_\varepsilon = f$ pointwise then $\text{s-}\lim_{\varepsilon\to 0} f_\varepsilon(A) = f(A)$ for any self-adjoint operator A, we learn that we may write $f(A)$ as

$$f(A) = \frac{1}{2\pi}\int_{z\in\mathbb{C}}\left(\partial_{\overline{z}}\tilde{f}\right)(z)(A-z\mathbb{1})^{-1}\mathrm{d}z \tag{1.9}$$

and since the integral converges absolutely, the convergence of the integral on the operator-valued function is in operator norm.

1.3.3 The smooth functional calculus preserves locality

Theorem 1.25 (The smooth functional calculus preserves locality)**.** *Let $f:\mathbb{R}\to\mathbb{C}$ be smooth of compact support and $A=A^*\in\mathcal{B}(\mathcal{H})$ be given which obeys (1.2). Then there is some $\mu>0$ such that for any $N\in\mathbb{N}$ there exists some $C_N<\infty$ such that*

$$\left\|f(A)_{xy}\right\| \leq C_N\left(1+\mu\|x-y\|\right)^{-N}. \tag{1.10}$$

In particular we are not *showing that $f(A)$ obeys (1.2) but in most applications Theorem 1.25 is certainly good enough.*

Proof. Using the smooth functional calculus, assuming $N\geq 2$,

$$\begin{aligned}
\left\|f(A)_{xy}\right\| &= \left\|\frac{1}{2\pi}\int_{z\in\mathbb{C}}\left(\partial_{\overline{z}}\tilde{f}\right)(z)\left(A-z\mathbb{1}\right)_{xy}^{-1}\mathrm{d}z\right\| \\
&\leq \frac{1}{2\pi}\int_{z\in\mathbb{C}}\left|\left(\partial_{\overline{z}}\tilde{f}\right)(z)\right|\left\|\left(A-z\mathbb{1}\right)_{xy}^{-1}\right\|\mathrm{d}z \\
&\quad \left(\text{Apply Combes-Thomas and estimates on }\left|\left(\partial_{\overline{z}}\tilde{f}\right)(z)\right|\right) \\
&\leq \frac{1}{2\pi}\int_{\mathbb{Re}\{z\},\mathbb{Im}\{z\}\in\mathbb{R}} C_N\left|\mathbb{Im}\{z\}\right|^N\frac{2}{\left|\mathbb{Im}\{z\}\right|}\mathrm{e}^{-\mu\left|\mathbb{Im}\{z\}\right|\|x-y\|}\mathrm{d}\mathbb{Re}\{z\}\mathrm{d}\mathbb{Im}\{z\} \\
&\quad \left(\text{The integral on the real part cannot be larger than }2\|A\|\right) \\
&\leq \frac{1}{\pi}C_N 2\|A\|\int_{\mathbb{Im}\{z\}\in\mathbb{R}}\left|\mathbb{Im}\{z\}\right|^{N-1}\mathrm{e}^{-\mu\left|\mathbb{Im}\{z\}\right|\|x-y\|}\mathrm{d}\mathbb{Im}\{z\} \\
&= \frac{1}{\pi}C_N 2\|A\| 2\int_{\eta=0}^{\infty}\eta^{N-1}\mathrm{e}^{-\mu\eta\|x-y\|}\mathrm{d}\eta =: \star \\
&= \frac{4}{\pi}C_N\|A\|(N-1)!\left(\frac{1}{\mu\|x-y\|}\right)^N.
\end{aligned}$$

This whole derivation assumed that $x\neq y$. If that's not the case, we replace the Combes-Thomas estimate with the trivial bound

$$\left\|\left(A-z\mathbb{1}\right)_{xx}^{-1}\right\|\leq\frac{1}{\left|\mathbb{Im}\{z\}\right|}\qquad (z\in\mathbb{C}\setminus\mathbb{R}).$$

Hence we can combine these two estimates together to conclude that

$$\left\|f(A)_{xy}\right\| \leq C\left(1+\mu\|x-y\|\right)^{-N}$$

for N arbitrarily large, for some $C<\infty$, $\mu>0$ which depend on A, f and N. □

1.3.4 Decay of difference of smooth functional calculus

Next we turn to the question of what can we say about $f(A)-f(B)$ if f is smooth of compact support and A,B are both self-adjoint, where we assume that we know something a-priori about $\left\|(A-B)_{xy}\right\|$. Let us further assume that both A,B are local so that we can apply a Combes-Thomas estimate on them.

We have

$$\begin{aligned}
\left\|\left(f\left(A\right)-f\left(B\right)\right)_{xy}\right\| &\leq \left\|\frac{1}{2\pi}\int_{z\in\mathbb{C}}\left(\partial_{\overline{z}}\tilde{f}\right)\left(z\right)\left(\left(A-z\mathbb{1}\right)_{xy}^{-1}-\left(B-z\mathbb{1}\right)_{xy}^{-1}\right)\mathrm{d}z\right\| \\
&\quad \text{(Use the resolvent identity)} \\
&\leq \left\|\frac{1}{2\pi}\int_{z\in\mathbb{C}}\left(\partial_{\overline{z}}\tilde{f}\right)\left(z\right)\left(\sum_{x',x''}\left(A-z\mathbb{1}\right)_{xx'}^{-1}\left(B-A\right)_{x'x''}\left(B-z\mathbb{1}\right)_{x''y}^{-1}\right)\mathrm{d}z\right\| \\
&\leq \frac{1}{2\pi}\sum_{x',x''}\|(B-A)_{x'x''}\|\int_{z\in\mathbb{C}}\left|\left(\partial_{\overline{z}}\tilde{f}\right)\left(z\right)\right|\left\|\left(A-z\mathbb{1}\right)_{xx'}^{-1}\right\|\left\|\left(B-z\mathbb{1}\right)_{x''y}^{-1}\right\|\mathrm{d}z \\
&\leq \frac{1}{2\pi}\sum_{x',x''}\|(B-A)_{x'x''}\|\frac{1}{\pi}C_N 2\max\left(\{\,\|A\|,\|B\|\,\}\right)\int_{\mathbb{Im}\{z\}\in\mathbb{R}}|\mathbb{Im}\,\{z\}|^{N-2}\,\mathrm{e}^{-\mu|\mathbb{Im}\{z\}|\left(\|x-x'\|+\|x''-y\|\right)}\mathrm{d}\mathbb{Im}\,\{z\}
\end{aligned}$$

Since N may be chosen arbitrarily large, for the last factor we again get polynomial decay (at rate $N-1$) so that

$$\left\|\left(f\left(A\right)-f\left(B\right)\right)_{xy}\right\| \leq C\sum_{x',x''}\|(B-A)_{x'x''}\|\left(1+\mu\left(\|x-x'\|+\|x''-y\|\right)\right)^{-N+1}\,. \tag{1.11}$$

As an application of the above, consider for instance the case when we assume that $\|(B-A)_{x'x''}\|\leq C\mathrm{e}^{-\mu\|x'-x''\|}\mathrm{e}^{-\nu\left(|x'_j|+|x''_j|\right)}$ for some $j=1,\dots,d$. Then

$$\left\|\left(f\left(A\right)-f\left(B\right)\right)_{xy}\right\| \leq C\sum_{x',x''}C\mathrm{e}^{-\mu\|x'-x''\|}\mathrm{e}^{-\nu\left(|x'_j|+|x''_j|\right)}\left(1+\mu\left(\|x-x'\|+\|x''-y\|\right)\right)^{-N+1}$$

Now using (many applications of) the triangle inequality (prove this!) we can conclude that

$$\left\|\left(f\left(A\right)-f\left(B\right)\right)_{xy}\right\| \leq C\left(1+\mu\|x-y\|\right)^{-N+1}\left(1+\mu'\left(|x_j|+|y_j|\right)\right)^{-N+1}\,. \tag{1.12}$$

1.4 Types of quantum motion

An important quantity in the study of quantum dynamics is the second moment of the position operator:

$$\mathcal{M}_{ij}\left(t\right):=\left\langle\delta_0,\mathrm{e}^{\mathrm{i}tH}X_iX_j\mathrm{e}^{-\mathrm{i}tH}\delta_0\right\rangle\qquad\left(t>0,\,i,j=1,\dots,d\right)\,.$$

It represents the expectation value of X_iX_j evolved to time t on a state δ_0.

If H is reflection invariant, i.e., $H_{xy}=H_{-x,-y}$ then the off-diagonal elements are zero:

$$\begin{aligned}
\mathcal{M}_{ij}\left(t\right) &= \sum_{x\in\mathbb{Z}^d}x_ix_j\left\langle\delta_0,\mathrm{e}^{\mathrm{i}tH}\delta_x\right\rangle\left\langle\delta_x,\mathrm{e}^{-\mathrm{i}tH}\delta_0\right\rangle \\
&= \sum_{x\in\mathbb{Z}^d}x_ix_j\left|\mathrm{e}^{-\mathrm{i}tH}\left(x,0\right)\right|^2 \\
&= 0\,.
\end{aligned}$$

For this reason it is mainly the diagonal (and if H is isotropic, then all of them are the same) that are interesting, so we focus on

$$\mathcal{M}\left(t\right):=\sum_{x\in\mathbb{Z}^d}\|x\|^2\left|\mathrm{e}^{-\mathrm{i}tH}\left(x,0\right)\right|^2\,.$$

This has the probabilistic interpretation of the *variance* of the position at time t of a particle starting at the origin at time zero.

Definition 1.26 (Types of motion)**.** We say that the particle exhibits *ballistic motion* iff

$$\mathcal{M}(t) \sim t^2 \qquad (t \to \infty) \,.$$

This is because in classical ballistic motion,

$$x = vt$$

or

$$x^2 = v^2 t^2 \,.$$

Conversely, if

$$\mathcal{M}(t) \sim t \qquad (t \to \infty)$$

then the motion is called *diffusive*. Finally, if

$$\mathcal{M}(t) \sim \mathcal{O}(1) \qquad (t \to \infty)$$

then we say the motion is *localized*.

Proposition 1.27. *If H is local and periodic, then the motion is ballistic.*

Proof. Since H is periodic, it is judicious to write $\mathcal{M}$ in momentum space:

$$\begin{aligned}
\mathcal{M}_{ij}(t) &\equiv \left\langle \delta_0, \mathrm{e}^{\mathrm{i}tH} X_i X_j \mathrm{e}^{-\mathrm{i}tH} \delta_0 \right\rangle_{\ell^2} \\
&= (2\pi)^{-d} \left\langle \mathcal{F}\delta_0, \mathcal{F}\mathrm{e}^{\mathrm{i}tH} X_i X_j \mathrm{e}^{-\mathrm{i}tH} \delta_0 \right\rangle_{L^2} \\
&= (2\pi)^{-d} \left\langle \mathcal{F}\delta_0, \mathcal{F}\mathrm{e}^{\mathrm{i}tH} \mathcal{F}^* \mathcal{F} X_i \mathcal{F}^* \mathcal{F} X_j \mathcal{F}^* \mathcal{F} \mathrm{e}^{-\mathrm{i}tH} \mathcal{F}^* \mathcal{F} \delta_0 \right\rangle_{L^2} \,.
\end{aligned}$$

Now,

$$\begin{aligned}
(\mathcal{F}\delta_0)(k) &= \sum_{x \in \mathbb{Z}^d} \mathrm{e}^{-\mathrm{i}k \cdot x} (\delta_0)_x \\
&= 1
\end{aligned}$$

so we get

$$\begin{aligned}
\mathcal{M}_{ij}(t) &= (2\pi)^{-d} \int_{k \in \mathbb{T}^d} \mathrm{e}^{\mathrm{i}th(k)} \mathrm{i}\partial_i \mathrm{i}\partial_j \mathrm{e}^{-\mathrm{i}th(k)} \mathrm{d}k \\
&= -(2\pi)^{-d} \int_{k \in \mathbb{T}^d} \mathrm{e}^{\mathrm{i}th(k)} \partial_i \mathrm{e}^{-\mathrm{i}th(k)} \left(-\mathrm{i}t \left(\partial_j h\right)(k)\right) \mathrm{d}k \\
&= -(2\pi)^{-d} \int_{k \in \mathbb{T}^d} \left(-\mathrm{i}t \left(\partial_i \partial_j h\right)(k) - t^2 \left(\partial_i h\right)(k) \left(\partial_j h\right)(k)\right) \mathrm{d}k \,.
\end{aligned}$$

We can proceed in various ways. For example, due to reflection symmetry we could say that $\int \partial_i \partial_j h = 0$. Another possibility is to say that since in real space $\mathcal{M}_{ij}(t)$ is clearly real, and h is real-valued as H is self-adjoint, it must be the case that $\int \partial_i \partial_j h = 0$. It any event, we find that

$$\mathcal{M}_{ij}(t) = t^2 (2\pi)^{-d} \int_{\mathbb{T}^d} (\partial_i h) \partial_j h$$

which is indeed ballistic. We interpret

$$\sqrt{(2\pi)^{-d} \int_{\mathbb{T}^d} (\partial_i h) \partial_j h}$$

as *the velocity*. □

Later on we will see non-trivial examples of localized motion, associated with *Anderson localization*. Here is a trivial example:

Example 1.28 (Localized motion)**.** Assume that H is diagonal in space, i.e., $H_{xy} \sim \delta_{xy}$. Then

$$\begin{aligned} \mathcal{M}_{ij}(t) &= \sum_{x\in\mathbb{Z}^d} x_i x_j \left|\mathrm{e}^{-\mathrm{i}tH}(x,0)\right|^2 \\ &= \sum_{x\in\mathbb{Z}^d} x_i x_j \left|\mathrm{e}^{-\mathrm{i}tH}(x,0)\right|^2 \delta_{x0} \\ &= 0\,. \end{aligned}$$

1.4.1 Relation to the diffusion equation

The diffusion equation is given by

$$\partial_t n(t,x) = -D\Delta n(t,x) \qquad \left(t>0, x\in\mathbb{Z}^d, i,j=1,\dots,d\right)$$

where n is the density of particles (as a function of time t and space x; perhaps one interprets $-\Delta$ as a discrete Laplacian). Suppose for a moment that this relationship indeed holds. Then, using the n-expectation value

$$\left\langle X_i X_j\right\rangle_n \equiv \frac{\sum_x x_i x_j n(x,t)}{\sum_x n(x,t)}$$

we find

$$\begin{aligned} \partial_t \sum_x x_i x_j n(x,t) &= \sum_x x_i x_j \partial_t n(x,t) \\ &\overset{\text{Diffusion equation}}{=} \sum_x x_i x_j D(-\Delta n)(t,x) \\ &\overset{\text{I.B.P}}{=} D\sum_x \left[-\Delta(x_i x_j)\right] n(t,x) \\ &= 2D\delta_{ij}\sum_x n(t,x)\,. \end{aligned}$$

As a result we find the equation

$$\partial_t \left\langle X_i X_j\right\rangle_n = 2D\delta_{ij}\,.$$

In deriving this equation we have assumed that D is isotropic and homogeneous (with obvious generalization otherwise). Integrating this equation we find

$$\left\langle X_i X_j\right\rangle_n = 2tD\delta_{ij} + C$$

so that

$$\lim_{t\to\infty}\frac{\left\langle X_i X_j\right\rangle_n}{t} = 2D\delta_{ij}\,.$$

For this reason, we make the following

Definition 1.29 (Diffusion coefficient)**.** Let $\psi\in\mathcal{H}$ with $\|\psi\|=1$. Then if the motion of H and ψ is diffusion, in the sense that

$$\left\langle\psi, \mathrm{e}^{\mathrm{i}tH}X_i X_j \mathrm{e}^{-\mathrm{i}tH}\psi\right\rangle \sim \delta_{ij}t \qquad (t\to\infty)$$

then we define the diffusion coefficient associated with ψ as

$$D(\psi) := \frac{1}{2}\lim_{t\to\infty}\frac{\left\langle\psi, \mathrm{e}^{\mathrm{i}tH}X_1^2 \mathrm{e}^{-\mathrm{i}tH}\psi\right\rangle}{t}\,.$$

With this we see the relationship between D and $\mathcal{M}$, whenever the motion *is* diffusive.

Remark 1.30. It could very well be that for different initial states ψ the motion has different behavior. The correct thing to imagine is that ψ is a wave packet concentrated in energy in a part of the spectrum that is associated with different kinds of motion (which may well happen).

1.5 The relationship between dynamics and spectral *type*

We now turn to an interesting relationship between spectral *type* (i.e., eigenvalues versus continuous spectrum) and dynamics, i.e., bound states versus scattering states.

We first remark that if ψ is an *eigenstate* of the Hamiltonian, in the sense that $\psi \in \ell^2$ and

$$H\psi = \lambda\psi$$

for some $\lambda \in \mathbb{R}$, then just by being in ℓ^2 we have some form of spatial decay for ψ. However, as we apply time evolution on ψ, we merely get a phase

$$t \mapsto \mathrm{e}^{-\mathrm{i}tH}\psi = \mathrm{e}^{-\mathrm{i}t\lambda}\psi$$

so that

$$\left|\left\langle \varphi, \mathrm{e}^{-\mathrm{i}tH}\psi\right\rangle\right|^2 = \left|\langle \varphi, \psi\rangle\right|^2$$

is constant in time, regardless of φ. This is in stark difference to any states in the *continuous* part of the Hilbert space, as the following theorem shows. The material in this section is taken from [Tes09, AW15].

We first start by a measure-theoretic result (see [Tes09] Theorem 5.5):

Theorem 1.31 (Wiener)**.** *Let μ be a finite complex Borel measure on $\mathbb{R}$ and*

$$\hat{\mu}(t) := \int_{E\in\mathbb{R}} \mathrm{e}^{-\mathrm{i}tE}\mathrm{d}\mu(E) \qquad (t \geq 0)$$

is its Fourier transform. Then the Cesàro time average of $\hat{\mu}$ has the following limit

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T |\hat{\mu}(t)|^2\,\mathrm{d}t = \sum_{E\in\mathbb{R}} |\mu(\{\,E\,\})|^2$$

where the sum on the right-hand side is finite.

Proof. We write

$$\begin{aligned}\frac{1}{T}\int_0^T |\hat{\mu}(t)|^2\,\mathrm{d}t &= \frac{1}{T}\int_0^T \overline{\left(\int_{E\in\mathbb{R}} \mathrm{e}^{-\mathrm{i}tE}\mathrm{d}\mu(E)\right)}\left(\int_{\tilde{E}\in\mathbb{R}} \mathrm{e}^{-\mathrm{i}t\tilde{E}}\mathrm{d}\mu\left(\tilde{E}\right)\right)\mathrm{d}t \\ &\overset{\text{Fubini}}{=} \int_{E\in\mathbb{R}}\int_{\tilde{E}\in\mathbb{R}}\left(\frac{1}{T}\int_0^T \mathrm{e}^{-\mathrm{i}t\left(\tilde{E}-E\right)}\mathrm{d}t\right)\overline{\mathrm{d}\mu(E)}\mathrm{d}\mu\left(\tilde{E}\right).\end{aligned}$$

Now, the function in parenthesis is bounded by 1 and converges pointwise to

$$\chi_{\{\,0\,\}}\left(\tilde{E}-E\right)$$

so the domincated convergence theorem of the limit $T \to \infty$ yields the result

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{T}\int_0^T |\hat{\mu}(t)|^2\,\mathrm{d}t &= \int_{E\in\mathbb{R}}\int_{\tilde{E}\in\mathbb{R}}\chi_{\{\,0\,\}}\left(\tilde{E}-E\right)\overline{\mathrm{d}\mu(E)}\mathrm{d}\mu\left(\tilde{E}\right) \\ &= \int_{E\in\mathbb{R}}\mu\left(\left\{\,\tilde{E}\,\right\}\right)\overline{\mathrm{d}\mu\left(\tilde{E}\right)} \\ &= \sum_{E\in\mathbb{R}}|\mu(\{\,E\,\})|^2\,.\end{aligned}$$

□

Remark 1.32. If $\mu_{H,\psi,\varphi}$ is the spectral projection associated to the triplet H, ψ, φ as

$$\mu_{H,\psi,\varphi}(S) \equiv \langle \psi, \chi_S(H)\,\varphi\rangle$$

then: if $\mu_{H,\psi,\psi}$ is continuous, or absolutely continuous (w.r.t. the Lebesgue measure), then so is $\mu_{H,\varphi,\psi}$.

Proof. Let $P_\sharp$ be the projection onto the continuous or absolutely continuous part of the Hilbert space, depending on $\sharp$. Then, by definition, $\mu_{H,\psi,\psi}$ is $\sharp$ iff $\psi \in \operatorname{im}(P_\sharp)$. Moreover, $P_\sharp$ commutes with the functional calculus of H. Hence

$$\begin{aligned}\langle\varphi, \chi_S(H)\psi\rangle &= \langle\varphi, \chi_S(H)P_\sharp\psi\rangle \\ &= \langle\varphi, P_\sharp\chi_S(H)\psi\rangle \\ &= \langle P_\sharp\varphi, \chi_S(H)\psi\rangle\,.\end{aligned}$$

But since the off-diagonal measure is defined via the polarization identity, we find that $\langle\varphi, \chi_\cdot(H)\psi\rangle$ is also $\sharp$. ☐

Conclusions for us:

1. If $\psi \in \operatorname{im}(P_c)$ then for any $\varphi \in \mathcal{H}$,
$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \left|\widehat{\mu_{H,\varphi,\psi}}(t)\right|^2 \mathrm{d}t = 0\,.$$
2. If $\psi \in \operatorname{im}(P_{ac})$ then for any $\varphi \in \mathcal{H}$, then already using the Riemann-Lebesgue lemma we know that
$$t \mapsto \widehat{\mu_{H,\varphi,\psi}}(t)$$
is continuous and decays to zero at infinity.

We sharpen this statement with aid of the following intermediate abstract result

Theorem 1.33. *Let A be a bounded self-adjoint operator and assume that K is bounded and compact, i.e., that $K(A - z\mathbb{1})^{-1}$ is compact for some (and hence all) $z \in \sigma(A)^c$. Then*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \left\|K\mathrm{e}^{-\mathrm{i}tA}P_c\psi\right\|^2 \mathrm{d}t = 0$$

and

$$\lim_{t\to\infty}\left\|K\mathrm{e}^{-\mathrm{i}tA}P_{ac}\psi\right\| = 0$$

for all $\psi \in \mathcal{H}$.

Note: this theorem extends to unbounded operators, see [Tes09] for the details.

Proof. Let $\psi \in \mathcal{H}_\sharp$ with $\sharp \in \{\,\mathrm{c}, \mathrm{ac}\,\}$. Assume that F is finite rank, with $\{\,\varphi_j\,\}_{j=1}^n$ an ONB for $\operatorname{im}(F)$. Then $F = F\sum_{j=1}^n \varphi_j \otimes \varphi_j^*$ and hence

$$F\eta \;=\; \sum_{j=1}^n \langle\varphi_j, F\eta\rangle\,\varphi_j = \sum_{j=1}^n \langle F^*\varphi_j, \eta\rangle\,\varphi_j \qquad (\eta \in \mathcal{H})$$

i.e., any finite rank operator F may be written as

$$F = \sum_{j=1}^n \varphi_j \otimes \psi_j^*$$

for some ONB $\{\,\varphi_j\,\}_{j=1}^n$ and $\psi_j = F^*\varphi_j$. Then

$$\begin{aligned}\left\|F\mathrm{e}^{-\mathrm{i}tH}\psi\right\|^2 &= \left\|\sum_{j=1}^n \varphi_j\left\langle\psi_j, \mathrm{e}^{-\mathrm{i}tH}\psi\right\rangle\right\|^2 \\ &= \sum_{j=1}^n \left|\left\langle\psi_j, \mathrm{e}^{-\mathrm{i}tH}\psi\right\rangle\right|^2 && \text{(ONB property)}\end{aligned}$$

We recognize that

$$\langle \psi_j, \mathrm{e}^{-\mathrm{i}tH}\psi\rangle \equiv \int_{\lambda\in\mathbb{R}} \mathrm{e}^{-\mathrm{i}t\lambda}\mathrm{d}\mu_{H,\psi_j,\psi}(\lambda) = \widehat{\mu_{H,\psi_j,\psi}}(t)$$

with $\mu_{H,\psi_j,\psi}$ the spectral projection associated to the triplet H, ψ_j, ψ. Hence the result follows thanks to the Wiener theorem above.

Now assume that K is compact and let $F_n \to K$ be a sequence of finite rank operators such that

$$\|K - F_n\| \leq \frac{1}{n} \qquad (n \in \mathbb{N})$$

so that

$$\begin{aligned} \left\|K\mathrm{e}^{-\mathrm{i}tH}\psi\right\|^2 &\leq \left(\left\|F_n\mathrm{e}^{-\mathrm{i}tH}\psi\right\| + \frac{1}{n}\left\|\mathrm{e}^{-\mathrm{i}tH}\psi\right\|\right)^2 \\ &\leq 2\left\|F_n\mathrm{e}^{-\mathrm{i}tH}\psi\right\|^2 + \frac{2}{n^2}\|\psi\|^2 \qquad (\text{Using } (a+b)^2 \leq 2a^2+2b^2) \end{aligned}$$

Take now the limit $t \to \infty$ and then $n \to \infty$ to obtain the result. □

One then has the following precise statement due to Ruelle, Amrein, Georgescu and Enss [Rue69, AG74, Ens78]:

Theorem 1.34 (RAGE)**.** *Let H be a self-adjoint operator and K_n a sequence of compact operators such that*

$$\operatorname*{s-lim}_{n\to\infty} K_n = \mathbb{1}\,.$$

Then

$$\mathcal{H}_c(H) = \left\{ \psi \in \mathcal{H} \;\middle|\; \lim_{n\to\infty}\lim_{T\to\infty}\frac{1}{T}\int_0^T \left\|K_n\mathrm{e}^{-\mathrm{i}tH}\psi\right\|\mathrm{d}t = 0 \right\}$$

and

$$\mathcal{H}_{pp}(H) = \left\{ \psi \in \mathcal{H} \;\middle|\; \lim_{n\to\infty}\sup_{t\geq 0}\left\|(\mathbb{1} - K_n)\,\mathrm{e}^{-\mathrm{i}tH}\psi\right\| = 0 \right\}.$$

In particular, it is useful to think of

$$K_n = \chi_{B_n(0)}(X)$$

i.e., the projection onto a ball of size n about the origin in position space. This is finite rank (and hence compact) and indeed converges strongly to the identity. Then the statement is saying that $\psi \in \mathcal{H}_{\mathrm{pp}}(H)$ iff

$$\lim_{n\to\infty}\sup_{t\geq 0}\left\|\chi_{B_n(0)^c}(X)\,\mathrm{e}^{-\mathrm{i}tH}\psi\right\| = 0$$

i.e., a particle evolved to arbitrary time, will eventually escape a ball of arbitrary size.

Proof. Assume first that $\psi \in \mathcal{H}_{\mathrm{c}}(H)$. Then by Cauchy-Schwarz

$$\begin{aligned} \frac{1}{T}\int_0^T \left\|K_n\mathrm{e}^{-\mathrm{i}tH}\psi\right\|\mathrm{d}t &\leq \frac{1}{T}\left(\int_0^T \left\|K_n\mathrm{e}^{-\mathrm{i}tH}\psi\right\|^2\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_0^T 1\mathrm{d}t\right)^{\frac{1}{2}}\mathrm{d}t \\ &= \left(\frac{1}{T}\int_0^T \left\|K_n\mathrm{e}^{-\mathrm{i}tH}\psi\right\|^2\mathrm{d}t\right)^{\frac{1}{2}} \\ &\overset{T\to\infty}{\to} 0 \end{aligned}$$

by the previous theorem. Conversely, if $\psi \notin \mathcal{H}_{\mathrm{c}}(H)$, we may write $\psi = \psi^{\mathrm{c}} + \psi^{\mathrm{pp}}$. By the previous estimate it merely suffices to estimate $\left\|K_n\mathrm{e}^{-\mathrm{i}tH}\psi^{pp}\right\|$ from below. Let us write

$$\psi^{\mathrm{pp}} = \sum_j \alpha_j\psi_j$$

where $\{ \psi_j \}_j$ are the eigenfunctions of H with eigenvalues λ_j. Then

$$\mathrm{e}^{-\mathrm{i}tH}\psi^{\mathrm{pp}} = \sum_j \mathrm{e}^{-\mathrm{i}t\lambda_j}\alpha_j\psi_j \,.$$

Truncating this expansion after N terms, we find that this part converges uniformly by the strong convergences of $K_n \to \mathbb{1}$:

$$\begin{aligned}\lim_{n\to\infty}\sup_{t\geq 0}\left\|(\mathbb{1}-K_n)\sum_{j=1}^{N}\mathrm{e}^{-\mathrm{i}t\lambda_j}\alpha_j\psi_j\right\| &\leq \sum_{j=1}^{N}|\alpha_j|\lim_{n\to\infty}\sup_{t\geq 0}\|(\mathbb{1}-K_n)\psi_j\| \\ &\overset{K_n\to\mathbb{1}\text{ strongly}}{=} 0\,.\end{aligned}$$

By the uniform boundedness principle, we have $\|K_n\| \leq M$ so that the error can be made arbitrarily small by taking N sufficiently large.

If $\psi \in \mathcal{H}_{\mathrm{pp}}$, then the claim follows by the estimate we have just proven. Conversely, if $\psi \notin \mathcal{H}_{\mathrm{pp}}$, write again $\psi = \psi^{\mathrm{c}} + \psi^{\mathrm{pp}}$ and it suffices to show that

$$\left\|(\mathbb{1}-K_n)\,\mathrm{e}^{-\mathrm{i}tH}\psi^{\mathrm{c}}\right\|$$

does not tend to zero as $n \to \infty$. Assume otherwise. Then

$$\begin{aligned}0 &= \lim_{T\to\infty}\frac{1}{T}\int_0^T\left\|(\mathbb{1}-K_n)\,\mathrm{e}^{-\mathrm{i}tH}\psi^{\mathrm{c}}\right\|\mathrm{d}t \\ &\geq \|\psi^{\mathrm{c}}\| - \lim_{T\to\infty}\frac{1}{T}\int_0^T\left\|K_n\mathrm{e}^{-\mathrm{i}tH}\psi^{\mathrm{c}}\right\|\mathrm{d}t \\ &= \|\psi^{\mathrm{c}}\|\end{aligned}$$

which is a contradiction. □

1.6 AC Spectrum–A vague form of delocalization

As we have seen above, there is great interest in establishing that operators actually have purely absolutely continuous spectrum, since this is an indication of either ballistic or diffusive motion (and also has far reaching consequences for scattering theory). In this section we explore various different ways to establish that the spectrum of an operator (on an interval) is absolutely continuous.

1.6.1 Stability of AC spectrum

We begin with a basic observation:

Theorem 1.35. *The essential spectrum of an operator is stable against compact perturbations.*

Proof. In a sense this statement is trivial, if we define the essential spectrum appropriately (see [Sha24]). One reasonable definition is

$$\sigma_{\mathrm{ess}}(A) \equiv \{ z \in \mathbb{C} \mid (A - z\mathbb{1}) \notin \mathcal{F}(\mathcal{H}) \} \tag{1.13}$$

where $\mathcal{F}(\mathcal{H})$ is the set of Fredholm operators on a Hilbert space (the space $\mathcal{F}(\mathcal{H})$ of those operators F on $\mathcal{H}$ such that $\dim(\ker(F)), \dim(\ker(F^*))$ are finite and such that $\mathrm{im}(F)$ is closed). One basic fact about Fredholm operators is that they are stable under compact perturbations.

To reiterate, we are trying to prove that if K is compact, then

$$\sigma_{\mathrm{ess}}(A) = \sigma_{\mathrm{ess}}(A+K)\,.$$

Using (1.13) we find that

$$\begin{aligned} z \notin \sigma_{\text{ess}}(A+K) &\iff (A+K-z\mathbb{1}) \in \mathcal{F}(\mathcal{H}) \\ &\iff (A-z\mathbb{1}) \in \mathcal{F}(\mathcal{H}) \\ &\iff z \notin \sigma_{\text{ess}}(A) \end{aligned}$$

so we find the result. □

We ask whether there is an analogous statement for the absolutely continuous spectrum. It turns out that this is indeed the case, if we replace compactness with the trace class property.

Theorem 1.36. *Let A be a normal operator and T be a trace-class operator so that $A+T$ is also normal. Then*

$$\sigma_{ac}(A) \quad = \quad \sigma_{ac}(A+T)\,.$$

Proof. We postpone the proof of this fact until we can prove the existence of wave operators implies ac spectrum (the proof may be found in [Kat84], pp. 542 Theorem 4.4. □

1.6.2 The limiting absorption principle

The limiting absorption principle is the statement that in some sense if one goes into the *absolutely continuous spectrum*, the resolvent still has a bounded limit (though not in ℓ^2). To warm up, we start with the following characterization of the spectral measure (see more details in [Sha24]):

Lemma 1.37 (Characterization of measure type via the Borel transform)**.** *Let μ be a finite Borel measure and f its Borel transform, given by*

$$f(\lambda) \quad \equiv \quad \int_{\lambda\in\mathbb{R}} \frac{1}{E-\lambda}\mathrm{d}\mu(E)\,.$$

Then

$$\lim_{\varepsilon\to 0^+} \frac{1}{\pi}\mathbb{Im}\{f(\lambda+\mathrm{i}\varepsilon)\}$$

exists a.e. w.r.t. both μ and $\mathcal{L}$ (the Lebesgue measure). Moreover,

$$\left\{\, \lambda\in\mathbb{R} \,\middle|\, \mathbb{Im}\left\{f\left(\lambda+\mathrm{i}0^+\right)\right\} = \infty \,\right\}$$

and

$$\left\{\, \lambda\in\mathbb{R} \,\middle|\, 0 < \mathbb{Im}\left\{f\left(\lambda+\mathrm{i}0^+\right)\right\} < \infty \,\right\}$$

are the support of the singular and absolutely continuous parts of μ respectively. Moreover, the set of point masses of μ is given by

$$\left\{\lambda \,\middle|\, \lim_{\varepsilon\to 0^+} \varepsilon\,\mathbb{Im}\{f(\lambda+\mathrm{i}\varepsilon)\} > 0 \right\}.$$

Proof. See [Jak06]. □

This we calibrate with the following basic statement about the resolvent implying ac spectrum:

Proposition 1.38. *Let H be a self-adjoint operator on a separable Hilbert space $\mathcal{H}$. Assume that for any φ in a dense subset of $\mathcal{H}$, either*

$$\sup_{E\in[a,b],\varepsilon\in(0,1)} \left|\left\langle \varphi, (H-(E+\mathrm{i}\varepsilon)\mathbb{1})^{-1}\varphi\right\rangle\right| < \infty\,. \tag{1.14}$$

or there exists some $p>1$ such that

$$\sup_{\varepsilon\in(0,1)} \int_a^b \frac{1}{\pi}\left[\mathbb{Im}\{\langle\varphi, R(E+\mathrm{i}\varepsilon)\varphi\rangle\}\right]^p \mathrm{d}E < \infty \tag{1.15}$$

With $R(z)\equiv(H-z\mathbb{1})^{-1}$. Then H has purely absolutely continuous spectrum on $[a,b]$.

Proof. We claim first that (1.14) implies (1.15). Actually it is possible to show that $\mathbb{Im}\left\{\langle\varphi,R\left(E+\mathrm{i}\varepsilon\right)\varphi\rangle\right\}\geq0$ if $\varepsilon>0$ thanks to the Herglotz property, so we can safely take the p power with no absolute value. To show (1.15), assuming (1.14) holds, we use

$$\mathbb{Im}\left\{\langle\varphi,R\left(E+\mathrm{i}\varepsilon\right)\varphi\rangle\right\}\leq\left|\langle\varphi,R\left(E+\mathrm{i}\varepsilon\right)\varphi\rangle\right|$$

as well as

$$\left|\langle\varphi,R\left(E+\mathrm{i}\varepsilon\right)\varphi\rangle\right|\leq\frac{1}{\varepsilon}\|\varphi\|^{2}\,.$$

Hence we assume (1.15) and work towards showing that

$$\langle\varphi,\chi_{\cdot}\left(H\right)\varphi\rangle$$

is ac on $[a,b]$. We have for any interval I, by Stone's formula

$$\langle\varphi,\chi_{I}\left(H\right)\varphi\rangle\quad\leq\quad\lim_{\varepsilon\to0^{+}}\int_{I}\frac{1}{\pi}\mathbb{Im}\left\{\langle\varphi,R\left(E+\mathrm{i}\varepsilon\right)\varphi\rangle\right\}\mathrm{d}E\,.$$

B Hoelder's inequality, the RHS integral is estimated by

$$\int_{I}\frac{1}{\pi}\mathbb{Im}\left\{\langle\varphi,R\left(E+\mathrm{i}\varepsilon\right)\varphi\rangle\right\}\mathrm{d}E\quad\leq\quad\left(\int_{I}\left[\frac{1}{\pi}\mathbb{Im}\left\{\langle\varphi,R\left(E+\mathrm{i}\varepsilon\right)\varphi\rangle\right\}\right]^{p}\mathrm{d}E\right)^{\frac{1}{p}}|I|^{\frac{1}{q}}$$

with $q=\left(1-\frac{1}{p}\right)^{-1}$. Since we know (1.15) we conclude that

$$\langle\varphi,\chi_{I}\left(H\right)\varphi\rangle\leq C\left|I\right|^{s}$$

and hence the measure is absolutely continuous. □

The following material is taken from [Tao11]:

Definition 1.39 (Limiting absorption principle)**.** H is said to have the limiting absorption principle at $E\in\mathbb{R}$ iff for any $\psi\in\ell^{2}$ sufficiently "nice" (on the lattice, with finite support is enough), and for any $\sigma>0$ there exists some $C_{\sigma}\in(0,\infty)$ (only depending on σ) such that

$$\sup_{\varepsilon\neq0}\left\|\left(H-\left(E+\mathrm{i}\varepsilon\right)\mathbb{1}\right)^{-1}\psi\right\|_{H^{-\frac{1}{2}-\sigma}}\leq C_{\sigma}\frac{1}{\sqrt{|E|}}\|\psi\|_{H^{\frac{1}{2}+\sigma}}$$

where

$$\|\psi\|_{H^{s}}:=\left\|\langle X\rangle^{s}\psi\right\|_{\ell^{2}}$$

and

$$\langle x\rangle:=\left(1+\|x\|^{2}\right)^{\frac{1}{2}}\,.$$

Claim 1.40*.* The limiting absorption principle holds for $H=-\Delta$ on $\ell^{2}\left(\mathbb{Z}^{d}\right)$.

Proposition 1.41. *Any operator H admitting the limiting absorption principle at E has purely absolutely continuous spectrum in a small interval about E.*

Proof. Show that

$$\lim_{\varepsilon\to0^{+}}\frac{1}{\pi}\mathbb{Im}\left\{\langle\varphi,R\left(\cdot+\mathrm{i}\varepsilon\right)\varphi\rangle\right\}$$

exists and is finite. □

1.6.3 Existence of wave operators

The material in this section is taken from [RS79] Chapter XI.3. We start with a pedestrian criterion for existence of ac spectrum.

Theorem 1.42. *Let A, B be two bounded self-adjoint operators on a separable Hilbert space $\mathcal{H}$ and assume that*

$$\operatorname*{s-lim}_{t\to\infty} e^{-itA}e^{itB}$$

exists. Then $\sigma_{ac}(B) \subseteq \sigma_{ac}(A)$.

Let us motivate this statement a bit with scattering theory. We first note that if φ is an eigenvector of B with eigenvalue λ, then

$$e^{itA}e^{-itB}\varphi = e^{itA}e^{-it\lambda}\varphi$$

and we know that that limit exists only if φ is also an eigenvector of A (which is generically false). Hence we should really define the wave operator

$$\Omega^{\pm}(A,B) := \operatorname*{s-lim}_{t\to\pm\infty} e^{-itA}e^{itB}P_{\mathrm{ac}}(B)$$

if the limit exists. We also define

$$\mathcal{H}_{\pm} := \operatorname{im}\left(\Omega^{\pm}\right) .$$

Proposition 1.43. *If $\Omega^{\pm}(A,B)$ both exist, then*

1. *$\Omega^{\pm}$ are partial isometries with initial subspaces $P_{\mathrm{ac}}(B)\mathcal{H}$ and final subspaces $\mathcal{H}_{\pm}$.*
2. *$\mathcal{H}_{\pm}$ are invariant subspaces for A and*
$$\Omega^{\pm}(\mathcal{D}(B)) \subseteq \mathcal{D}(A)$$
$$A\Omega^{\pm} = \Omega^{\pm}B$$
3. *$\mathcal{H}_{\pm} \subseteq \operatorname{im}(P_{ac}(A))$.*

Recall that for a partial isometry $I \in \mathcal{B}(\mathcal{H})$, $\ker(I)^{\perp}$ is called the initial subspace and $\operatorname{im}(I)$ is called the final subspace. Hence $\Omega^{\pm}$ define unitary equivalences between

$$P_{\mathrm{ac}}(B)\mathcal{H} \quad \to \quad \mathcal{H}_{\pm}$$

and so the first and third point together imply that

$$\operatorname{im}(P_{\mathrm{ac}}(B)) \subseteq \operatorname{im}(P_{\mathrm{ac}}(A)) .$$

Proof. For the first item, let us take $\varphi \in (P_{\mathrm{ac}}(B)\mathcal{H})^{\perp}$. Then by definition of the wave operator itself, $\Omega^{\pm}\varphi = 0$. Conversely, if $\varphi \in P_{\mathrm{ac}}(B)\mathcal{H}$, then by unitarity,

$$\left\|e^{-itA}e^{itB}P_{\mathrm{ac}}(B)\varphi\right\| = \|\varphi\|$$

so $\Omega^{\pm}$are indeed partial isometries are claimed.

For the second item, note that for any fixed s, we have

$$\begin{aligned} \Omega^{\pm} &= e^{-isA}\Omega^{\pm}e^{isB} \\ &\Updownarrow \\ e^{-isA}\Omega^{\pm} &= \Omega^{\pm}e^{isB} \end{aligned}$$

since this holds for any s, Stone's theorem for unitary groups implies that $A\Omega^{\pm} = \Omega^{\pm}B$. Next, if $\varphi \in \mathcal{H}_{\pm}$, then

$$\varphi = \Omega^{\pm}\psi$$

for some ψ, and hence,

$$A\varphi = A\Omega^{\pm}\psi = \Omega^{\pm}B\psi \in \operatorname{im}\left(\Omega^{\pm}\right) \equiv \mathcal{H}_{\pm}$$

so that these are indeed invariant subspaces for A. This also shows the statement about the domains.

Finally, by the previous arguments, $A|_{\mathcal{H}_{\pm}}$ is unitarily equivalent via $\Omega^{\pm}$ to $B|_{P_{\mathrm{ac}}(B)\mathcal{H}}$. Hence $A|_{\mathcal{H}_{\pm}}$ has purely absolutely continuous spectrum. ☐

Proposition 1.44 (Chain rule)**.** *If $\Omega^{\pm}(A,B)$ and $\Omega^{\pm}(B,C)$ exist, then $\Omega^{\pm}(A,C)$ exist, and*

$$\Omega^{\pm}(A,C)=\Omega^{\pm}(A,B)\,\Omega^{\pm}(B,C)\ .$$

Proof. By the third item in the proposition above,

$$\operatorname{im}\left(\Omega^{\pm}(B,C)\right)\quad\subseteq\quad\operatorname{im}\left(P_{\text{ac}}(B)\right)$$

so

$$\underset{t\to\pm\infty}{\text{s-lim}}\,P_{\text{ac}}(B)^{\perp}\,\mathrm{e}^{-\mathrm{i}tB}\mathrm{e}^{\mathrm{i}tC}P_{\text{ac}}(C)=0\,.$$

Hence

$$\begin{aligned}\mathrm{e}^{-\mathrm{i}tA}\mathrm{e}^{\mathrm{i}tC}P_{\text{ac}}(C)\,\varphi &= \mathrm{e}^{-\mathrm{i}tA}\mathrm{e}^{\mathrm{i}tB}P_{\text{ac}}(B)\,\mathrm{e}^{-\mathrm{i}tB}\mathrm{e}^{\mathrm{i}tC}P_{\text{ac}}(C)\,\varphi+\\ &\quad +\mathrm{e}^{-\mathrm{i}tA}\mathrm{e}^{\mathrm{i}tB}P_{\text{ac}}(B)^{\perp}\,\mathrm{e}^{-\mathrm{i}tB}\mathrm{e}^{\mathrm{i}tC}P_{\text{ac}}(C)\,\varphi\\ &\to \Omega^{\pm}(A,B)\,\Omega^{\pm}(B,C)\,\varphi\end{aligned}$$

since the strong limit of a product is the product of the strong limits. □

Definition 1.45. We say that we have *asymptotic completeness* if

$$\mathcal{H}_{+}=\mathcal{H}_{-}=\left(P_{\text{pp}}(A)\,\mathcal{H}\right)^{\perp}\ .$$

Definition 1.46. We say that that the wave operators $\Omega^{\pm}$ are *complete* iff

$$\mathcal{H}_{+}=\mathcal{H}_{-}=\operatorname{im}\left(P_{\text{ac}}(A)\right)\ .$$

Hence the distinction between these two is that asymptotic completeness further requires that $\sigma_{\text{sc}}(A)=\varnothing$.

Proposition 1.47. *Assume that $\Omega^{\pm}(A,B)$ exist. Then they are complete iff $\Omega^{\pm}(B,A)$ exist.*

Proof. Assume that both $\Omega^{\pm}(A,B)$ and $\Omega^{\pm}(B,A)$ exist. By the chain rule,

$$P_{\text{ac}}(A)=\Omega^{\pm}(A,A)=\Omega^{\pm}(A,B)\,\Omega^{\pm}(B,A)$$

so that

$$\operatorname{im}\left(P_{\text{ac}}(A)\right)\subseteq\operatorname{im}\left(\Omega^{\pm}(A,B)\right)\ .$$

But we also know that $\operatorname{im}\left(\Omega^{\pm}(A,B)\right)\subseteq\operatorname{im}\left(P_{\text{ac}}(A)\right)$, we have completeness.

Conversely, if $\Omega^{\pm}(A,B)$ exist and are complete, let $\varphi\in\operatorname{im}\left(P_{\text{ac}}(A)\right)$, Then $\varphi=\Omega^{\pm}(A,B)\,\psi$ for some ψ. This implies that

$$\left\|\mathrm{e}^{-\mathrm{i}tA}\varphi-\mathrm{e}^{-\mathrm{i}tB}P_{\text{ac}}(B)\,\psi\right\|\to 0$$

as $t\to\infty$. But $\mathrm{e}^{-\mathrm{i}tB}$ is unitary, so

$$\lim_{t\to\infty}\mathrm{e}^{\mathrm{i}tB}\mathrm{e}^{-\mathrm{i}tA}\varphi$$

exists and equals $P_{\text{ac}}(B)\,\psi$. □

Theorem 1.48 (Cook's method)**.** *Assume that A,B are self-adjoint operators and that there exists some set*

$$\mathcal{D}\subseteq\mathcal{D}(B)\cap\operatorname{im}\left(P_{ac}(B)\right)$$

which is dense in $\operatorname{im}\left(P_{ac}(B)\right)$ so that for any $\varphi\in\mathcal{D}$, there is some T_0 satisfying

1. *For any $|t|>T_0$, $\mathrm{e}^{-\mathrm{i}tB}\varphi\in\mathcal{D}(A)$.*
2. *$\int_{T_0}^{\infty}\left(\left\|(B-A)\,\mathrm{e}^{-\mathrm{i}tB}\varphi\right\|+\left\|(B-A)\,\mathrm{e}^{\mathrm{i}tB}\varphi\right\|\right)\mathrm{d}t<\infty$.*

Then $\Omega^{\pm}(A,B)$ exist.

Proof. Define $\eta(t) := e^{itA}e^{-itB}\varphi$ for some $\varphi \in \mathcal{D}$. For $t > T_0$, $e^{-itB}\varphi \in \mathcal{D}(A) \cap \mathcal{D}(B)$, so η is strongly differentiable on (T_0, ∞) and

$$\eta'(t) = -ie^{itA}(B-A)e^{-itB}\varphi\,.$$

Hence for $t > s > T_0$ we find that

$$\|\eta(t) - \eta(s)\| \le \int_s^t \|\eta'(u)\| du \le \int_s^t \left\|(B-A)e^{-iuB}\varphi\right\| du$$

goes to zero as $s \to \infty$ by the assumption above. Hence η is Cauchy as $t \to \infty$, so

$$\lim_{t\to\infty} e^{itA}e^{-itB}P_{ac}(B)\psi$$

exists for all $\psi \in \mathcal{D}$. It also exists trivially for all $\psi \in \operatorname{im}\left(P_{ac}(B)^{\perp}\right)$, so, for all ψ lying in a dense set. The existence in a dense implies the existence for all ψ by a $\frac{\varepsilon}{3}$ argument, which shows that Ω^- exists. The proof for Ω^+ is identical. $\square$

Example 1.49. Imagine that $B = -\Delta$ (the discrete Laplacian) and $A = -\Delta + V(X)$ for some V. Then apparently to guarantee that A has ac spectrum in $[0, \infty)$ we need to verify that

$$\int_1^{\infty} \left(\left\|V(X)e^{-it(-\Delta)}\varphi\right\| + \left\|V(X)e^{it(-\Delta)}\varphi\right\|\right) dt < \infty$$

for all φ in a dense subset. For instance, the dense subset could be φ with compact support. At this moment it is probably good to mention that the propagator for the continuum Laplacian has time dependence like

$$e^{-it(-\Delta)}(x,y) \sim t^{-\frac{d}{2}}$$

so this is going to be integrable if $d \ge 3$. In fact, [Krishna 1992] has shown that for the discrete Laplacian it is sufficient to show

$$\sum_{x\in\mathbb{Z}^d} |V(x)|^2 \left|e^{-it(-\Delta)}(x,0)\right|^2 \sim |t|^{-2-\varepsilon}$$

as $t \to \infty$.

1.6.4 Mourre theory

In this section we want to establish an additional criterion for pure ac spectrum, which, as we saw above, has consequences for quantum dynamics. The material in this section is mostly taken from [CyconKirschFroeseSimon].

Proposition 1.50. *Let H be self-adjoint and assume that for each φ in some dense set there exists some $C_\varphi < \infty$ such that*

$$\limsup_{\varepsilon\to0^+} \sup_{\mu\in(a,b)} \left\langle \varphi, \operatorname{Im}\left\{(H - (\mu + i\varepsilon)\mathbb{1})^{-1}\right\}\varphi\right\rangle \le C_\varphi\,.$$

Then H has purely absolutely continuous spectrum in (a, b).

Proof. Stone's formula says that

$$\frac{1}{2}\left\langle\varphi, \left(\chi_{(a',b')}(H) + \chi_{[a',b']}(H)\right)\varphi\right\rangle = \lim_{\varepsilon\to0^+}\frac{1}{\pi}\int_{a'}^{b'}\left\langle\varphi, \operatorname{Im}\left\{(H-(\mu+i\varepsilon)\mathbb{1})^{-1}\right\}\varphi\right\rangle d\mu$$

and using the fact that $\chi_{(a',b')} \le \chi_{[a',b']}$, if $(a', b') \subseteq (a, b)$,

$$\left\langle\varphi, \chi_{(a',b')}(H)\varphi\right\rangle \le \frac{1}{\pi}\int_{a'}^{b'} C_\varphi = \frac{1}{\pi}C_\varphi(b'-a')$$

for a dense subset of φ's. But this implies that

$$\langle\varphi,\chi_S(H)\varphi\rangle\leq\frac{1}{\pi}C_\varphi|\Omega|$$

for any measurable $\Omega\subseteq(a,b)$ which implies in turn that the spectral measures $\mu_{H,\varphi,\varphi}$ are absolutely continuous. But since we assume this for a dense subset of φ's, the spectrum itself is absolutely continuous. □

Theorem 1.51 (Putnam)**.** *Let H and A be bounded and self-adjoint and furthermore assume that*

$$\mathrm{i}[H,A]=|C|^2\equiv C^*C$$

for some operator C for which $\ker(C)=\{0\}$. *Then H has purely absolutely continuous spectrum.*

Proof. We have, with $R(z)\equiv(H-z\mathbb{1})^{-1}$, we get via the C-star identity,

$$\begin{aligned}\|CR(z)\|^2 &= \|R(\overline{z})C^*CR(z)\|\\ &= \|R(\overline{z})\mathrm{i}[H,A]R(z)\|\\ &= \|R(\overline{z})\mathrm{i}[H-z\mathbb{1},A]R(z)\|\\ &\leq \|R(\overline{z})(H-z\mathbb{1})AR(z)\|+\|R(\overline{z})A(H-z\mathbb{1})R(z)\|\\ &= \|R(\overline{z})(H-(\overline{z}-\overline{z}-z)\mathbb{1})AR(z)\|+\|R(\overline{z})A\|\\ &\leq \|AR(z)\|+\|R(\overline{z})A\|+2|\mathbb{Im}\{z\}|\|R(\overline{z})AR(z)\|\\ &\leq 4\frac{1}{|\mathbb{Im}\{z\}|}\|A\|.\end{aligned}$$

As a result,

$$\begin{aligned}2\|C\,\mathbb{Im}\{R(z)\}C^*\| &= 2\|CR(z)2\mathrm{i}\,\mathbb{Im}\{z\}R(\overline{z})C^*\|\\ &\leq 8\|A\|.\end{aligned}$$

Now, $\mathrm{im}(C^*)$ is dense since we have $\mathrm{im}(C^*)^\perp=\ker(C)=\{0\}$, so using Proposition 1.50 we find the result. □

Imagine we could show that

$$\mathrm{i}[H,A]\geq\alpha\mathbb{1}\tag{1.16}$$

for some $\alpha>0$. That is, that not only does $\ker(C)=\{0\}$ but also that it is invertible. Then the previous estimates show us that

$$\begin{aligned}\|R(z)\| &= \left\|C^{-1}CR(z)\right\|\\ &\leq \left\|C^{-1}\right\|\|CR(z)\|\\ &\leq \left\|C^{-1}\right\|2\frac{1}{\sqrt{|\mathbb{Im}\{z\}|}}\sqrt{\|A\|}.\end{aligned}$$

But we also know that for self-adjoint operators,

$$\|R(z)\| = \frac{1}{\mathrm{dist}(z,\sigma(H))}$$

so that

$$\frac{1}{\mathrm{dist}(z,\sigma(H))} \leq \left\|C^{-1}\right\|2\frac{1}{\sqrt{|\mathbb{Im}\{z\}|}}\sqrt{\|A\|}\qquad(z\in\mathbb{C}).$$

If $E\in\sigma(H)$ and we take $z=E+\mathrm{i}\varepsilon$ for $\varepsilon>0$ we get

$$\frac{1}{\varepsilon}\leq2\left\|C^{-1}\right\|\sqrt{\|A\|}\frac{1}{\sqrt{\varepsilon}}\qquad(\varepsilon>0)$$

which leads to a contradiction, i.e., $\sigma(H)=\varnothing$. Hence (1.16) is impossible for bounded H,A.

The Mourre estimate is a weak form of this hypothesis for unbounded operators.

Definition 1.52 (Mourre estimate)**.** Let H, A be two self-adjoint operators (possibly unbounded) on a separable Hilbert space such that:

1. $\mathcal{D}(A) \cap \mathcal{D}(H)$ is dense in $\mathcal{D}(H)$.

2. The form of $\mathrm{i}[H, A]$ defined on $\mathcal{D}(A) \cap \mathcal{D}(H)$ extends to a bounded operator from $\mathcal{D}(H)$ to $\mathcal{D}(H)^*$.

We say that H obeys the Mourre estimate on Δ (with respect to A) iff there exists a positive number $\alpha > 0$ and a compact operator K such that

$$\chi_\Delta(H)\,\mathrm{i}[H, A]\,\chi_\Delta(H) \quad \geq \quad \alpha\chi_\Delta(H) + K\,.$$

Example 1.53. Let

$$H = -\Delta + V(X)$$

on $L^2(\mathbb{R}^d)$ where $V : \mathbb{R}^d \to \mathbb{R}$ is some function such that

1. $V(X)(-\Delta + \mathbb{1})^{-1}$ is compact.

2. $(-\Delta + \mathbb{1})^{-1} X \cdot (\nabla V)(X)(-\Delta + \mathbb{1})^{-1}$ is compact.

Define $A := \frac{1}{2}(X \cdot P + P \cdot X)$, which is the generator of dilations. Then

$$\begin{aligned}
\mathrm{i}\left[P^2, A\right] &= \mathrm{i}\left[P^2, \frac{1}{2}(X \cdot P + P \cdot X)\right] \\
&= \frac{\mathrm{i}}{2}\left([P_jP_j, X_iP_i] + [P_jP_j, P_iX_i]\right) \\
&= \frac{\mathrm{i}}{2}\left([P_jP_j, X_i]P_i + P_i[P_jP_j, X_i]\right) \\
&= \frac{\mathrm{i}}{2}\left(P_j[P_j, X_i]P_i + [P_j, X_i]P_jP_i + P_iP_j[P_j, X_i] + P_i[P_j, X_i]P_j\right)
\end{aligned}$$

and now using the fact that $[P_i, X_j] = \mathrm{i}\delta_{ij}$ we get that

$$\begin{aligned}
\mathrm{i}\left[P^2, A\right] &= \frac{\mathrm{i}}{2}\left(-P_j\mathrm{i}\delta_{ji}P_i - \mathrm{i}\delta_{ji}P_jP_i - P_iP_j\mathrm{i}\delta_{ji} - P_i\mathrm{i}\delta_{ij}P_j\right) \\
&= 2P^2
\end{aligned}$$

so that $-\Delta \equiv P^2$ itself easily satisfies a Mourre estimate on any interval not containing zero. Using our assumptions on V we find furthermore the same is also true for $H \equiv P^2 + V(X)$:

$$\begin{aligned}
\mathrm{i}[H, A] &= 2P^2 + \mathrm{i}\left[V(X), \frac{1}{2}(X \cdot P + P \cdot X)\right] \\
&= 2P^2 + \frac{\mathrm{i}}{2}\left(X_i[V(X), P_i] + [V(X), P_i]X_i\right) \\
&= 2P^2 + \frac{\mathrm{i}}{2}\left(X_i(\mathrm{i}(\partial_iV)(X)) + (\mathrm{i}(\partial_iV)(X))X_i\right) \\
&= 2P^2 - X \cdot (\nabla V)(X) \\
&= 2H - 2V(X) - X \cdot (\nabla V)(X)
\end{aligned}$$

Now by our assumptions: $2V(X) + X \cdot (\nabla V)(X)$ is compact when sandwiched with $\chi_{(a,b)}(H)$. If furthermore, $a > 0$, we find

$$\chi_{(a,b)}(H)\,2H\chi_{(a,b)}(H) \geq 2a\chi_{(a,b)}(H)$$

so that the Mourre estimate is satisfied.

The Virial Theorem Let $\psi \in \ell^2$ be an eigenfunction of H, in the sense that $H\psi = \lambda\psi$. Then

$$\begin{aligned} \langle \psi, \mathrm{i}\,[H, \mathrm{i}A]\,\psi \rangle &= \mathrm{i}\left(\langle \psi, HA\psi \rangle - \langle \psi, AH\psi \rangle\right) \\ &= \mathrm{i}\left(\langle H\psi, A\psi \rangle - \lambda\,\langle \psi, AH\psi \rangle\right) \\ &= \mathrm{i}\lambda\left(\langle \psi, A\psi \rangle - \langle \psi, A\psi \rangle\right) \\ &= 0\,. \end{aligned}$$

Note that when H, A are unbouned some care must be taken to handle the fact this form is well-defined. We avoid these subtleties here and refer the reader to [CyconKirschFroeseSimon].

Control of embedded eigenvalues

Theorem 1.54. *Assume that H satisfies the Mourre estimate on the interval Δ (with respect to A). Assume moreover that there exists some self-adjoint operator H_0 (in applications this is usually $H_0 = -\Delta$) such that:*

1. $(H_0 - z\mathbb{1})^{-1}\,\mathcal{D}(A) \subseteq \mathcal{D}(A)$ *for some* $z \in \rho(H_0)$.
2. $\mathcal{D}(H_0) \cap \mathcal{D}(H_0 A)$ *is dense in* $\mathcal{D}(H)$ *and*
3. *The form* $\mathrm{i}\,[H_0, A]$ *defined on* $\mathcal{D}(A) \cap \mathcal{D}(H)$ *extends to a bounded operator from* $\mathcal{D}(H)$ *to* $\mathcal{H}$. *Then H has at most finitely many eigenvalues in Δ and each eigenvalue has finite multiplicity.*

Then H has at most finitely many eigenvalues in Δ and each eigenvalue has finite multiplicity.

Proof. Suppose there are infinitely many eigenvalues of H in Δ, or that some eigenvalue has infinite multiplicity. Let $\{\,\psi_n\,\}_n$ be the corresponding orthonormal eigenfunctions of this space. By the Virial theorem and the Mourre estimate we get

$$\begin{aligned} 0 &= \langle \psi_n, \mathrm{i}\,[H, A]\,\psi_n \rangle \\ &= \langle \psi_n, \chi_\Delta(H)\,\mathrm{i}\,[H, A]\,\chi_\Delta(H)\,\psi_n \rangle \\ &\geq \alpha \|\psi_n\|^2 + \langle \psi_n, K\psi_n \rangle\ . \end{aligned}$$

Since $\|\psi_n\| = 1$ and $\psi_n \to 0$ weakly, and K is compact, we have $\langle \psi_n, K\psi_n \rangle \to 0$ as $n \to \infty$. But this is impossible as $\alpha > 0$. □

Absence of singular continuous spectrum

Lemma 1.55. *Assume that H satisfies the Mourre estimate on the open interval Δ (with respect to A). Then actually the Mourre estimate is obeyed with $K = 0$ away from eigenvalues of H.*

Proof. Let $\Delta' \subseteq \Delta$ be any interval which does *not* contain an eigenvalue. Then

$$\chi_{\Delta'}(H)\,\mathrm{i}\,[H, A]\,\chi_{\Delta'}(H) \geq \alpha\chi_{\Delta'}(H) + \chi_{\Delta'}(H)\,K\chi_{\Delta'}(H)\ .$$

Now since Δ' does not contain any eigenvalues, $\chi_{\Delta'}(H)\,K\chi_{\Delta'}(H)$ tends to zero in norm as Δ' shrinks to zero width about any point. Hence let us pick Δ' such that

$$\chi_{\Delta'}(H)\,\mathrm{i}\,[H, A]\,\chi_{\Delta'}(H) \geq \alpha\chi_{\Delta'}(H) - \frac{1}{2}\alpha\mathbb{1}\,.$$

Now multiply both sides again by $\chi_{\Delta'}(H)$ to get the result. □

Theorem 1.56. *Assume that H satisfies the Mourre estimate on the interval Δ (with respect to A). Assume moreover that there exists some self-adjoint operator H_0 (in applications this is usually $H_0 = -\Delta$) such that:*

1. $(H_0 - z\mathbb{1})^{-1}\mathcal{D}(A) \subseteq \mathcal{D}(A)$ *for some* $z \in \rho(H_0)$.
2. $\mathcal{D}(H_0) \cap \mathcal{D}(H_0 A)$ *is dense in* $\mathcal{D}(H)$ *and*
3. *The form* $\mathrm{i}[H_0, A]$ *defined on* $\mathcal{D}(A) \cap \mathcal{D}(H)$ *extends to a bounded operator from* $\mathcal{D}(H)$ *to* $\mathcal{H}$. *Then* H *has at most finitely many eigenvalues in* Δ *and each eigenvalue has finite multiplicity.*
4. *The form* $\mathrm{i}[\mathrm{i}[H, A], A]$ *extends from* $\mathcal{D}(A) \cap \mathcal{D}(H)$ *to a bounded map from* $\mathcal{D}(H)$ *to* $\mathcal{D}(H)^*$.

Then if the Mourre estimate actually holds with $K = 0$ then

$$\limsup_{\delta\to 0^+}\sup_{\mu\in\Delta}\left\|\left(|A|+\mathbb{1}\right)^{-1}\left(H-(\mu+\mathrm{i}\delta)\mathbb{1}\right)^{-1}\left(|A|+\mathbb{1}\right)^{-1}\right\| \leq C$$

for some constant C, which readily implies pure absolutely continuous spectrum of H within Δ.

We shall not prove this theorem but rather refer the reader to [CyconKirschFroeseSimon].

1.6.5 The non-zero index method

Here we describe the fact that if

$$\operatorname{index}\left(\Lambda U \Lambda + \Lambda^{\perp}\right) \neq 0$$

for some projection Λ and some unitary U, then $\sigma(U) = \sigma_{\mathrm{ac}}(U) = \mathbb{S}^1$. In particular if $U = \mathrm{e}^{\mathrm{i}2\pi A}$ for some self-adjoint A then by the spectral mapping theorem, $\sigma_{\mathrm{ac}}(A) = [0, 1]$ or a translate of this interval.

1.7 Linear response theory: the Kubo formula

We now want to derive various formulas for the DC conductivity of a system using perturbation theory. In order to do so we first derive a general form of perturbation theory known as the Kubo linear response formula [Kub57]. The main reason for this hurdle is the following

Example 1.57. We are interested in calculating the electric conductivity of, say,

$$H = (P - A)^2 - E_0 X_j$$

which is a magnetic system with electric field of strength E_0 on the jth axis. If we have e.g. constant magnetic field with

$$A = X_2 e_1$$

then there is no dependence on X_1 (if $j = 2$) in the Hamiltonian, so the spectrum would not be discrete in this case. Hence we cannot use Rayleigh-Schroedinger perturbation theory.

Hence even though the perturbations we will consider (the electric field) eventually do *not* depend on time, for regularizing purposes we consider them being ramped up with time very gradually. The order of limits prescribed by this procedure is :

$$\boxed{\text{First the perturbation becomes constant in time and only then small in norm.}} \tag{1.17}$$

Before proceeding we explain why traces with the Fermi projection $P \equiv \chi_{(-\infty, E_F)}(H)$ are of use to us.

1.7.1 Density matrices

Usually one talks about states of quantum mechanical systems as vectors ψ in a separable Hilbert space $\mathcal{H}$ with $\|\psi\| = 1$. Equivalently we could speak about rank-1 projections:

$$P := \psi \otimes \psi^* .$$

Then the quantum expectation value of the observable A on the state ψ is given by

$$\langle \psi, A\psi \rangle = \operatorname{tr}(PA) \equiv \operatorname{tr}(\psi \otimes \psi^* A) .$$

Sometimes it is useful however to speak of a classical statistical *mixture* of states: let $N \in \mathbb{N}$ and $\{ p_i \}_{i=1}^N \subseteq [0,1]$ such that $\sum_{i=1}^N p_i = 1$. Let also $\{ \psi_i \}_{i=1}^N \subseteq \mathcal{H}$ be some ONB of some subspace. Then

$$\rho := \sum_{i=1}^N p_i \psi_i \otimes \psi_i^*$$

is such a statistical mixture of states. Note that

$$\operatorname{tr}(\rho) = \sum_{i=1}^N p_i \operatorname{tr}(\psi_i \otimes \psi_i^*) = 1$$

and actually

$$\langle \varphi, \rho\varphi \rangle = \sum_{i=1}^N p_i \left| \langle \psi_i, \varphi \rangle \right|^2 \geq 0 \qquad (\varphi \in \mathcal{H})$$

so $\rho \geq 0$. This leads us to the

Definition 1.58 (Density matrix)**.** A *density matrix* ρ on a separable Hilbert space is a positive trace-class operator of trace 1.

1.7.2 The many-body Fermionic ground state in single-particle universe

In quantum mechanics, the state of a particle is described by a vector in a Hilbert space $\mathcal{H}$ (or a density matrix, as we have just seen). Conversely, to talk about the state of M *distinguishable* particles simultaneously, we need to consider a vector in the M-fold tensor product Hilbert space $\bigotimes_{j=1}^M \mathcal{H}$. However, if we have M *indistinguishable* particles which are *Fermions,* which is the situation for electrons in a solid, then the state of these particles is actually a vector in the M-fold exterior product Hilbert space $\bigwedge_{j=1}^M \mathcal{H}$, since the state must be anti-symmetric with respect to exchange of any two particles (*as a basic axiom of quantum mechanics*).

If the single-particle Hamiltonian $H = H^* \in \mathcal{B}(\mathcal{H})$ is acting on each particle separately, then the many-body Hamiltonian is given by

$$\mathrm{d}\Gamma(H) := \sum_{j=1}^M \mathbb{1}^{\wedge(j-1)} \wedge H \wedge \mathbb{1}^{\wedge(M-j)}\,,$$

i.e., the single particle Hamiltonian acts on the jth particle and doesn't do anything on all other particles.

Then, if we are interested in the *many-body expectation value of a non-interacting single-particle observable*, say, B, we would first raise it to the many-body Hilbert space just as above:

$$B \mapsto \sum_{j=1}^M \mathbb{1}^{\wedge(j-1)} \wedge B \wedge \mathbb{1}^{\wedge(M-j)} =: \mathrm{d}\Gamma(B)$$

and then if our system was in the state $\Psi \in \bigwedge_{j=1}^M \mathcal{H}$, we would calculate

$$\langle \Psi, \mathrm{d}\Gamma(B)\Psi \rangle\,.$$

Now, if Ψ itself is a product state, i.e., $\Psi = \psi_1 \wedge \cdots \wedge \psi_M$, where $\{ \psi_j \}_{j=1}^M$ is an orthonormal collection, then this simplifies to

$$\begin{aligned}\langle \Psi, \mathrm{d}\Gamma(B)\Psi \rangle &= \sum_{j=1}^M \langle \psi_j, B\psi_j \rangle \\ &= \operatorname{tr}\left(\sum_{i=1}^M \psi_i \otimes \psi_i^* B \right)\end{aligned}$$

where we recognize $\sum_{i=1}^M \psi_i \otimes \psi_i^*$ as the projection operator onto the space spanned by the orthonormal set $\{ \psi_j \}_{j=1}^M$.

Now, say our Hamiltonian of the solid we wish to describe is $H \in \mathcal{B}(\mathcal{H})$, and say its eigenstates are $\{ \varphi_j \}_{j\in\mathbb{N}}$ (ordered so that φ_1 has the lowest energy, etc). Since no two Fermions can occupy the same quantum mechanical state (this is the

Pauli exclusion principle), if we fill the solid with M electrons, the ground state (i.e., the state of least energy, *at zero temperature*) is the one where the M electrons occupy the M first levels of H, i.e., $\varphi_1, \dots, \varphi_M$. The corresponding state on the many-body Hilbert space $\bigwedge \mathcal{H}$ (the exterior algebra generated by $\mathcal{H}$) is thus

$$\varphi_1 \wedge \cdots \wedge \varphi_M$$

(which is called *the Slater determinant*).

In conclusion we recognize that the many-body zero-temperature expectation value of a non-interacting observable B in a the ground state corresponding to M filled electrons is $\operatorname{tr}(P_M B)$ where $P_M := \sum_{j=1}^{M} \varphi_j \otimes \varphi_j^*$. More generally, if we work in infinite volume we have an infinite number of electrons and it is more judicious to speak of the Fermi energy E_F: that energy of the most energetic electron in the system. Then the appropriate expression is the preempted $\operatorname{tr}(P_F B)$ where $P_F \equiv \chi_{(-\infty, E_F)}(H)$.

Indeed, in infinite volume the range of this operator is infinite dimensional. In order to define this operator rigorously one has to apply the measurable functional calculus of bounded self-adjoint operators, see [RS80]. It will turn out that the Fermi projection P_F contains most of the properties we care about in regards to topological insulators. At non-zero temperatures the Fermi-Dirac distribution should be used--we won't make use of this here.

1.7.3 Electric conductivity

We wish to study insulators, for which we would like to calculate their electric conductance, which is phenomenologically defined via Ohms law:

$$\sigma = \frac{I}{V}$$

with I being the current and V the voltage. More generally, the conductivity σ is defined as the matrix relating the current density j with the electric field as follows:

$$j = \sigma E\,.$$

In principle each of these calculations of σ depends on the Fermi energy μ to which we fill the system.

> **Definition 1.59.** An electric insulator at Fermi energy μ is a material filled to μ whose conductivity matrix at that energy is zero on the diagonal:
>
> $$\sigma_{ii}(\mu) = 0\,.$$

Why do we only talk about the diagonal conductivity will become clear later when we consider the Hall conductivity.

In the physics literature, for historical and possibly physical reasons, one usually separates the objects of study in an experimental setup where there is a material (a solid) which is described by a Hamiltonian H and the external driving electric field. Hence, if we calculate the conductivity associated with H alone, it should be zero (since it would typically have no spontaneous currents) and only once we *perturb* with an external electric field it does it actually make to calculate σ. Thus, we are at the task of perturbation theory, by, say a constant electric field. As we know from undergraduate quantum mechanics, this means adding a term of the form

$$E_0 X_i$$

if the field is of strength E_0 in direction $i = 1, \dots, d$.

Typically, however, the type of perturbation theory taught in undergraduate quantum mechanics (Rayleigh-Schrödinger perturbation theory) is inappropriate for most systems we want to deal with, since it only deals with systems with discrete spectrum (finitely degenerate isolated eigenvalues). Also, generally one likes to do perturbation theory of the more general density matrices. The general theory under which this is done is called *linear response theory* [Kub91].

1.7.4 Linear response theory

As we have said the perturbation we are mostly concerned with is something proportional to the position operator and the observable should be the current density, i.e.,

$$j_i = n\mathrm{i}\,[H, X_i] \qquad (i = 1, \dots, d)$$

where n is the density of particles. Indeed, H being the generator of time-translations, $\mathrm{i}\,[H, X_i]$ is associated with $\frac{\mathrm{d}}{\mathrm{d}t} X_i$, i.e., the velocity.

Furthermore, the perturbations we shall consider are *not* constant in time. Instead, they will be turned on very slowly from being zero at the beginning of time.

Theorem 1.60. *(The Kubo formula) Assume a system governed by H is in state described by density matrix ρ_0. Assume further that it is perturbed by the time-dependent operator $\varepsilon f(t)A$, i.e.,*

$$\tilde{H}(t) := H + \varepsilon f(t) A$$

where $f : \mathbb{R} \to [0,1]$ is some smooth time-modulation function which obeys $f(-\infty) = 0$ and $f(0) = 1$, $\varepsilon > 0$ is some small order parameter, and A is a time-independent self-adjoint operator. Then the first order (in ε) coefficient of the expectation value of an observable $B = B^*$ for which $\mathrm{tr}(\rho_0 B) = 0$ to the perturbation at time zero is given by

$$\chi_{BA} := -\mathrm{i}\int_{-\infty}^{0} \mathrm{tr}\left(\mathrm{e}^{-\mathrm{i}tH} B \mathrm{e}^{\mathrm{i}tH}[A, \rho_0]\right) f(t)\, \mathrm{d}t\,. \tag{1.18}$$

Remark 1.61. Note that this corresponds to the *wrong order of limits in (1.17)*. We *would have liked* instead to calculate the limit first of $f \to 1$ and only then $\varepsilon \to 0$. Unfortunately addressing this problem is yet a completely different story. We refer the reader to [Teufel et al NESS etc].

Proof. The state of the system at time t is governed by the Schrödinger equation for the density matrix, which is

$$\mathrm{i}\dot{\rho}(t) = [H + \varepsilon f(t)A, \rho(t)]$$

with initial condition $\rho(-\infty) =: \rho_0$. We assume that ρ_0 is an equilibrium state for H in the sense that

$$[H, \rho_0] = 0\,.$$

We write explicitly the first order term as

$$\rho(t) = \rho_0 + \varepsilon\rho_1(t)$$

where ρ_0 is independent of time since the zero order in ε has no time dependence in the Hamiltonian. Hence

$$\begin{aligned}\varepsilon\mathrm{i}\dot{\rho}_1(t) &= [H + \varepsilon f(t)A, \rho_0 + \varepsilon\rho_1(t)] = \varepsilon[H, \rho_1(t)] + \varepsilon f(t)[A, \rho_0] + \mathcal{O}\left(\varepsilon^2\right)\\ &= \varepsilon H^\times \rho_1(t) + \varepsilon f(t) A^\times \rho_0 + \mathcal{O}\left(\varepsilon^2\right)\end{aligned}$$

where we used the notation $O^\times(\cdot) \equiv [O, \cdot]$ (sometimes also denoted by the adjoint notation ad_O for O)

Claim. $e^{a^\times} b = e^a b e^{-a}$

Proof. One can proceed either in a pedestrian way by computing the explicit expression for $(a^\times)^n$ (make guess and proof by induction) or by defining

$$F(t) := e^{ta} b e^{-ta} \qquad \forall t \in \mathbb{R}$$

and

$$G(t) := e^{ta^\times} b \qquad \forall t \in \mathbb{R}$$

Next note that F and G both solve the differential equation

$$\tilde{F}'(t) = a^\times \tilde{F}(t)$$

with initial condition $\tilde{F}(0) = b$. Since the solution to a first order ordinary differential equation is unique, $F = G$ and in particular $F(1) = G(1)$. □

Claim. The solution for ρ_1 is given by:

$$\begin{aligned}\rho_1(t) &= i\int_{-\infty}^{t}\exp\left(-i\left(t-t'\right)H^{\times}\right)\varepsilon A^{\times}\rho_0 f\left(t'\right)dt' \\ &\equiv i\int_{-\infty}^{t}\exp\left(-i\left(t-t'\right)H\right)\varepsilon\left[A,\,\rho_0\right]\exp\left(i\left(t-t'\right)H\right)f\left(t'\right)dt'\end{aligned}$$

Proof. Using the fact that

$$\frac{d}{dx}\int_{a(x)}^{b(x)}f(x,\,y)\,dy = f(x,\,b(x))\,b'(x)-f(x,\,a(x))\,a'(x)+\int_{a(x)}^{b(x)}\left[\partial_x f(x,\,y)\right]dy$$

we have

$$\begin{aligned}&\partial_t i\int_{-\infty}^{t}\exp\left(-i\left(t-t'\right)H^{\times}\right)\varepsilon A^{\times}\rho_0 f\left(t'\right)dt' \\ =\ & \mathrm{i}\varepsilon A^{\times}\rho_0 f(t)+\mathrm{i}\int_{-\infty}^{t}\exp\left(-i\left(t-t'\right)H^{\times}\right)(-\mathrm{i})\,H^{\times}\varepsilon A^{\times}\rho_0 f\left(t'\right)dt' \\ =\ & \mathrm{i}\varepsilon A^{\times}\rho_0 f(t)+H^{\times}\int_{-\infty}^{t}\exp\left(-i\left(t-t'\right)H^{\times}\right)\varepsilon A^{\times}\rho_0 f\left(t'\right)dt'\end{aligned}$$

using the fact that

$$\left[\exp\left(-i\left(t-t'\right)H^{\times}\right),\,H^{\times}\right] = 0$$

and also note that the initial value is obeyed: $\Delta\rho(-\infty)=0$. ☐

Then we have

$$\begin{aligned}\langle B\rangle_{\rho(0)} &\equiv Tr\left[\rho(0)\,B\right] \\ &= Tr\left[\left(\rho_0+\varepsilon\rho_1(0)+\mathcal{O}\left(\varepsilon^2\right)\right)B\right] \\ &= \underbrace{Tr\left[\rho_0 B\right]}_{\langle B\rangle_{\rho_0}}+\varepsilon\underbrace{Tr\left[\rho_1(0)\,B\right]}_{\lambda\chi_{BA}}+\mathcal{O}\left(\varepsilon^2\right)\end{aligned}$$

so that

$$\begin{aligned}\chi_{BA} &= \frac{1}{\varepsilon}Tr\left[\rho_1(0)\,B\right] \\ &= \frac{1}{\varepsilon}Tr\left[i\int_{-\infty}^{0}\exp\left(-i\left(0-t'\right)H^{\times}\right)\varepsilon A^{\times}\rho_0 f\left(t'\right)dt' B\right] \\ &= i\int_{-\infty}^{0}Tr\left\{\left[\exp\left(itH^{\times}\right)\left(A^{\times}\rho_0\right)\right]B\right\}f(t)\,dt \\ &\equiv i\int_{-\infty}^{0}Tr\left[\exp(itH)\left[A,\,\rho_0\right]\exp(-itH)\,B\right]f(t)\,dt \\ &= i\int_{-\infty}^{0}Tr\left[\exp(-itH)\,B\exp(itH)\left[A,\,\rho_0\right]\right]f(t)\,dt \\ &= i\int_{0}^{\infty}Tr\left[\exp(itH)\,B\exp(-itH)\left[A,\,\rho_0\right]\right]f(-t)\,dt\end{aligned}$$

We now take care of the limit:

$$\lim_{\varepsilon\to 0}\chi_{BA} = \lim_{\varepsilon\to 0}i\int_{0}^{\infty}Tr\left[\exp(itH)\,B\exp(-itH)\left[A,\,\rho_0\right]\right]\exp(-\varepsilon t)\,dt$$

We now use Lebesgue's dominated convergence theorem ([Rud86] pp. 26) with the dominating function being $t \mapsto |Tr[\exp(itH) B \exp(-itH)[A, \rho_0]]|$ (need to show it is L^1) to take the limit $\varepsilon \to 0$ into the integrand and obtain our result. □

1.8 Zero temperature DC conductivity

1.8.1 time-reversal invariant case

We now want to apply the formula (1.18) in order to calculate the conductivity of a system. As explained above, the appropriate initial density matrix ρ_0 to use is the Fermi projection, i.e.,

$$\rho_0 \quad = \quad P \equiv \chi_{(-\infty,\mu]}(H) \, .$$

At non-zero temperature one replaces $\chi_{(-\infty,\mu]}$ with the Fermi-Dirac distribution:

$$f_{\mathrm{FD}}(E) \quad \equiv \quad \frac{1}{1+\mathrm{e}^{\beta(E-\mu)}}$$

where $\beta \equiv \frac{1}{k_{\mathrm{B}}T}$ with k_B the Boltzmann constant and T the temperature. Of course

$$\lim_{\beta\to\infty} f_{\mathrm{FD}} \quad = \quad \chi_{(-\infty,\mu]} \, .$$

The observable B should be the current density, which is related to the velocity operator in direction i, so we shall take

$$B \quad = \quad \mathrm{i}[H, X_i] \, .$$

The perturbation shall be the electric field in direction j, i.e., X_j, so that all together we find that to first order in the electric field,

$$\sigma_{ij}(\mu) = \lim_{f\to 1} \mathrm{tr} \int_{-\infty}^{0} \mathrm{e}^{-\mathrm{i}tH} \mathrm{i}[H, X_i] \mathrm{e}^{+\mathrm{i}tH} \mathrm{i}[X_j, P] f(t) \, \mathrm{d}t \, . \tag{1.19}$$

The reason why we take the limit is that eventually we are interested in the static case, where the perturbation is *not* time dependent (or alternatively in the adiabatic limit where the perturbation is turned on infinitely slowly). We shall make the choice $f(t) = \mathrm{e}^{\varepsilon t}$ and take the limit $\varepsilon \to 0^+$.

To proceed further, we shall also make use of the notion of time-reversal in quantum mechanics. Since this hasn't been introduced yet, let us formally

Definition 1.62 (Time-reversal)**.** Time-reversal Θ is an anti-unitary operator $\Theta : \mathcal{H} \to \mathcal{H}$. That means it is anti-$\mathbb{C}$-linear:

$$\Theta(\alpha\psi + \varphi) \quad = \quad \overline{\alpha}\Theta(\psi) + \Theta(\varphi) \qquad (\alpha \in \mathbb{C}, \psi, \varphi \in \mathcal{H})$$

and obeys

$$\langle \Theta\psi, \Theta\varphi \rangle_{\mathcal{H}} \quad \equiv \quad \langle \varphi, \psi \rangle_{\mathcal{H}} \qquad (\psi, \varphi \in \mathcal{H}) \, .$$

Generally in condensed matter physics, $\Theta^2 = -\mathbb{1}$ for Fermions and $\Theta^2 = +\mathbb{1}$ for Bosons by the spin-statistics theorem coming from QFT. A Hamiltonian H is said to be time-reversal invariant (with respect to the fixed time-reversal operator Θ) iff

$$[H, \Theta] \quad = \quad 0 \, .$$

Theorem 1.63. *If H is time-reversal invariant as in Definition 1.62 then*

$$\sigma_{ij}(\mu) \quad = \quad \lim_{\varepsilon\to 0^+} \frac{\varepsilon^2}{\pi} \sum_{x\in\mathbb{Z}^d} x_i x_j \lim_{L\to\infty} \frac{1}{(2L+1)^d} \sum_{y\in\mathbb{Z}^d : \|y\|_1 \le L} |G(y+x, y; \mu + \mathrm{i}\varepsilon)|^2 \, . \tag{1.20}$$

We first note that generically, the operator within the trace appearing in (1.19) is *not* expected to be trace-class. For that reason, we should rather work with the

Definition 1.64 (Trace-per-unit-volume)**.** Given an operator $A\in\mathcal{B}\left(\ell^2\left(\mathbb{Z}^d\right)\otimes\mathbb{C}^N\right)$, we define its trace-per-unit-volume (if the limit exists) as

$$\widetilde{\operatorname{tr}}\left(A\right) \equiv \lim_{L\to\infty}\frac{1}{\left(2L+1\right)^d}\sum_{x\in\mathbb{Z}^d:\|x\|_1\leq L}\operatorname{tr}_{\mathbb{C}^N}\left(\langle\delta_x,A\delta_x\rangle\right)$$

where $\|x\|_1\equiv\sum_{j=1}^d|x_j|$ and $\operatorname{tr}_{\mathbb{C}^N}\left(\langle\delta_x,A\delta_x\rangle\right)$ means the trace within $\mathbb{C}^N$ of the $N\times N$ matrix $\langle\delta_x,A\delta_x\rangle$.

Proof. In the proof below we assume $N=1$ for simplicity. Given the comment above regarding the trace-class property, our starting point is the following modification of (1.19):

$$\sigma_{ij}\left(\mu\right)=\lim_{\varepsilon\to0^+}\int_{-\infty}^0\widetilde{\operatorname{tr}}\left(\mathrm{e}^{-\mathrm{i}tH}\mathrm{i}\left[H,X_i\right]\mathrm{e}^{+\mathrm{i}tH}\mathrm{i}\left[X_j,P\right]\right)\mathrm{e}^{\varepsilon t}\mathrm{d}t$$

with $P\equiv\chi_{(-\infty,\mu]}\left(H\right)$. We assume that the limit involved in $\widetilde{\operatorname{tr}}$ exists and start off by re-writing the regulator as

$$\mathrm{e}^{\varepsilon t} = \partial_t\left(\frac{\mathrm{e}^{\varepsilon t}-1}{\varepsilon}\right)$$

to perform integration by parts and find

$$\sigma_{ij}\left(\mu\right) = -\lim_{\varepsilon\to0^+}\int_{-\infty}^0\left(\partial_t\widetilde{\operatorname{tr}}\left(\mathrm{e}^{-\mathrm{i}tH}\mathrm{i}\left[H,X_i\right]\mathrm{e}^{+\mathrm{i}tH}\mathrm{i}\left[X_j,P\right]\right)\right)\frac{\mathrm{e}^{\varepsilon t}-1}{\varepsilon}\,.$$

Now

$$\begin{aligned}\partial_t\widetilde{\operatorname{tr}}\left(\mathrm{e}^{-\mathrm{i}tH}\mathrm{i}\left[H,X_i\right]\mathrm{e}^{+\mathrm{i}tH}\mathrm{i}\left[X_j,P\right]\right) &= \partial_t\widetilde{\operatorname{tr}}\left(\mathrm{i}\left[H,X_i\right]\mathrm{e}^{+\mathrm{i}tH}\mathrm{i}\left[X_j,P\right]\mathrm{e}^{-\mathrm{i}tH}\right)\\ &= \partial_t\widetilde{\operatorname{tr}}\left(\mathrm{i}\left[H,X_i\right]\mathrm{i}\left[\mathrm{e}^{+\mathrm{i}tH}X_j\mathrm{e}^{-\mathrm{i}tH},P\right]\right)\\ &= \widetilde{\operatorname{tr}}\left(\mathrm{i}\left[H,X_i\right]\mathrm{i}\left[\mathrm{e}^{+\mathrm{i}tH}\mathrm{i}\left[H,X_j\right]\mathrm{e}^{-\mathrm{i}tH},P\right]\right)\\ &= \widetilde{\operatorname{tr}}\left(\mathrm{i}\left[H,X_i\right]\mathrm{e}^{+\mathrm{i}tH}\mathrm{i}\left[\mathrm{i}\left[H,X_j\right],P\right]\mathrm{e}^{-\mathrm{i}tH}\right)\\ &\overset{V_j:=\mathrm{i}[H,X_j]}{=} \widetilde{\operatorname{tr}}\left(V_i\mathrm{e}^{+\mathrm{i}tH}\mathrm{i}\left[V_j,P\right]\mathrm{e}^{-\mathrm{i}tH}\right)\,.\end{aligned}$$

Let us write

$$f\left(H\right) = \int_{E\in\mathbb{R}}f\left(E\right)\mathrm{d}Q\left(E\right)$$

where Q is the projection-valued measure associated to H, and f is any bounded measurable function. Then

$$\begin{aligned}\widetilde{\operatorname{tr}}\left(V_i\mathrm{e}^{+\mathrm{i}tH}\mathrm{i}\left[V_j,P\right]\mathrm{e}^{-\mathrm{i}tH}\right) &= \widetilde{\operatorname{tr}}\left(V_i\int_{\lambda_1\in\mathbb{R}}\mathrm{d}Q\left(\lambda_1\right)\mathrm{e}^{+\mathrm{i}tH}\mathrm{i}\left[V_j,P\right]\mathrm{e}^{-\mathrm{i}tH}\int_{\lambda_2\in\mathbb{R}}\mathrm{d}Q\left(\lambda_2\right)\right)\\ &= \int_{\lambda_1\in\mathbb{R}}\int_{\lambda_2\in\mathbb{R}}\mathrm{e}^{\mathrm{i}t(\lambda_1-\lambda_2)}\widetilde{\operatorname{tr}}\left(V_i\mathrm{d}Q\left(\lambda_1\right)\mathrm{i}\left[V_j,P\right]\mathrm{d}Q\left(\lambda_2\right)\right)\\ &= \int_{\lambda_1\in\mathbb{R}}\int_{\lambda_2\in\mathbb{R}}\mathrm{e}^{\mathrm{i}t(\lambda_1-\lambda_2)}\mathrm{i}\left(g\left(\lambda_2\right)-g\left(\lambda_1\right)\right)\widetilde{\operatorname{tr}}\left(V_i\mathrm{d}Q\left(\lambda_1\right)V_j\mathrm{d}Q\left(\lambda_2\right)\right)\,.\end{aligned}$$

Here we are using

$$g\left(\lambda\right) := \chi_{(-\infty,\mu]}\left(\lambda\right)\,.$$

Moreover, we also write

$$\mathrm{e}^{\varepsilon t}-1 = t\int_0^\varepsilon\mathrm{e}^{\eta t}\mathrm{d}\eta$$

to get

$$\sigma_{ij}(\mu) \quad = \quad -\mathrm{i}\lim_{\varepsilon\to 0^+}\int_{t=-\infty}^{0}\int_{\lambda_1\in\mathbb{R}}\int_{\lambda_2\in\mathbb{R}}\mathrm{e}^{\mathrm{i}t(\lambda_1-\lambda_2)}\left(g(\lambda_2)-g(\lambda_1)\right)\widetilde{\mathrm{tr}}\left(V_i\mathrm{d}Q(\lambda_1)V_j\mathrm{d}Q(\lambda_2)\right)\frac{t}{\varepsilon}\int_0^{\varepsilon}\mathrm{e}^{\eta t}\mathrm{d}\eta\,.$$

The time integral we can do explicitly to get

$$\int_{t=-\infty}^{0}\mathrm{d}tt\mathrm{e}^{\mathrm{i}t(\lambda_1-\lambda_2-\mathrm{i}\eta)} = \frac{1}{(\lambda_1-\lambda_2-\mathrm{i}\eta)^2}$$

so we get

$$\sigma_{ij}(\mu) \quad = \quad \mathrm{i}\lim_{\varepsilon\to 0^+}\int_{\lambda_1,\lambda_2\in\mathbb{R}}\int_{\eta=0}^{\varepsilon}\frac{\mathrm{d}\eta}{\varepsilon}\frac{1}{(\lambda_1-\lambda_2-\mathrm{i}\eta)^2}\left(g(\lambda_1)-g(\lambda_2)\right)\mathrm{d}m_{ij}(\lambda_1,\lambda_2)$$

where we define the velocity measure

$$\mathrm{d}m_{ij}(\lambda_1,\lambda_2) \quad := \quad \widetilde{\mathrm{tr}}\left(V_i\mathrm{d}Q(\lambda_1)V_j\mathrm{d}Q(\lambda_2)\right)$$

and by $P(\lambda)$ we mean now the *function*

$$\mathbb{R}\ni\lambda\mapsto\chi_{(-\infty,\mu]}(\lambda)\,.$$

Now for well-behaved functions f we may replace

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\int_{\eta=0}^{\varepsilon}f(\eta)\,\mathrm{d}\eta \quad &= \quad \partial_\varepsilon|_{\varepsilon=0}\int_{\eta=0}^{\varepsilon}f(\eta)\,\mathrm{d}\eta \\ &= \quad \lim_{\eta\to 0^+}f(\eta)\,.\end{aligned}$$

Moreover, we have the so-called Kramers-Kronig relation [Sha23] (Corollary 7.61)

$$\lim_{\varepsilon\to 0^+}\frac{1}{x\pm\mathrm{i}\varepsilon} \quad \overset{\mathcal{D}}{=} \quad \mp\mathrm{i}\pi\delta(x)+\mathscr{P}\left(\frac{1}{x}\right)$$

where $\mathscr{P}$ is the Cauchy principal value of an integral. If we take the derivative of this relation w.r.t. x we get

$$\lim_{\varepsilon\to 0^+}-\frac{1}{(x\pm\mathrm{i}\varepsilon)^2} \quad \overset{\mathcal{D}}{=} \quad \mp\mathrm{i}\pi\delta'(x)+\mathscr{P}'\left(\frac{1}{x}\right)\,.$$

Finally, if our Hamiltonian is *time-reversal invariant*, then using Lemma 1.65 right below we get

$$\mathrm{d}m_{ij}(\lambda_1,\lambda_2) \quad = \quad \mathrm{d}m_{ij}(\lambda_2,\lambda_1)\,.$$

Then the function $\mathscr{P}'\left(\frac{1}{x}\right)$ is even (seen from the derivative of the Kramers-Kronig relation) so that we integrate the odd function of λ_1,λ_2

$$\left(g(\lambda_1)-g(\lambda_2)\right)\mathrm{d}m_{ij}(\lambda_1,\lambda_2)$$

against $\mathscr{P}'$ we get zero. we are thus left only with the δ' term to get

$$\sigma_{ij}(\mu) \quad = \quad \pi\int_{\lambda_1,\lambda_2\in\mathbb{R}}\delta'(\lambda_1-\lambda_2)\left(g(\lambda_1)-g(\lambda_2)\right)\mathrm{d}m_{ij}(\lambda_1,\lambda_2)\,.$$

Next we write

$$\delta'(\lambda_1-\lambda_2) \quad \overset{\mathcal{D}}{=} \quad \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\delta(\lambda_1-\lambda_2+\varepsilon)-\delta(\lambda_1-\lambda_2)\right)$$

and we get

$$\begin{aligned}
\sigma_{ij}(\mu) &= \lim_{\varepsilon\to0^+}\frac{\pi}{\varepsilon}\int_{\lambda_1}\left[\left(g(\lambda_1)-g(\lambda_1+\varepsilon)\right)\mathrm{d}m_{ij}(\lambda_1,\lambda_1+\varepsilon)\right]-\left[\left(g(\lambda_1)-g(\lambda_1)\right)\mathrm{d}m_{ij}(\lambda_1,\lambda_1)\right]\\
&= \pi\int_{\lambda_1}\partial_{\lambda_1}\chi_{(-\infty,\mu)}(\lambda_1)\,\mathrm{d}m_{ij}(\lambda_1,\lambda_1)\\
&= \pi\int_{\lambda_1}\delta(\lambda_1-\mu)\,\mathrm{d}m_{ij}(\lambda_1,\lambda_1)\\
&= \pi\int_{\lambda_1}\int_{\lambda_2}\delta(\lambda_1-\mu)\,\delta(\lambda_2-\mu)\,\delta\mathrm{d}m_{ij}(\lambda_1,\lambda_2)\\
&= \pi\lim_{\varepsilon\to0^+}\int_{\lambda_1}\int_{\lambda_2}\delta_\varepsilon(\lambda_1-\mu)\,\delta_\varepsilon(\lambda_2-\mu)\,\delta\mathrm{d}m_{ij}(\lambda_1,\lambda_2)\\
&= \pi\lim_{\varepsilon\to0^+}\int_{\lambda_1}\int_{\lambda_2}\delta_\varepsilon(\lambda_1-\mu)\,\delta_\varepsilon(\lambda_2-\mu)\,\delta\mathrm{d}m_{ij}(\lambda_1,\lambda_2)\\
&= \pi\widetilde{\operatorname{tr}}\left(\delta_\varepsilon(H-\mu)\,[H,X_i]\,\delta_\varepsilon(H-\mu)\,[X_j,H]\right)\,.
\end{aligned}$$

Here we have used the identity of distributions for the approximate delta function

$$\delta_\varepsilon(H-\mu) = \frac{1}{\pi}\mathbb{Im}\left\{R(\mu+\mathrm{i}\varepsilon)\right\}\,.$$

We now have

$$\begin{aligned}
&= \pi\widetilde{\operatorname{tr}}\left(\frac{1}{\pi}\mathbb{Im}\left\{R(\mu+\mathrm{i}\varepsilon)\right\}[H,X_i]\,\frac{1}{\pi}\mathbb{Im}\left\{R(\mu+\mathrm{i}\varepsilon)\right\}[X_j,H]\right)\\
&= \frac{\varepsilon^2}{\pi}\widetilde{\operatorname{tr}}\left(R(\mu+\mathrm{i}\varepsilon)\,R(\mu-\mathrm{i}\varepsilon)\,[H,X_i]\,R(\mu-\mathrm{i}\varepsilon)\,R(\mu+\mathrm{i}\varepsilon)\,[X_j,H]\right)\\
&= \frac{\varepsilon^2}{\pi}\widetilde{\operatorname{tr}}\left(R(\mu+\mathrm{i}\varepsilon)\,R(\mu-\mathrm{i}\varepsilon)\,[H,X_i]\,R(\mu-\mathrm{i}\varepsilon)\,R(\mu+\mathrm{i}\varepsilon)\,[X_j,H]\right)\\
&= \frac{\varepsilon^2}{\pi}\widetilde{\operatorname{tr}}\left(R(\mu-\mathrm{i}\varepsilon)\,[H,X_i]\,R(\mu-\mathrm{i}\varepsilon)\,R(\mu+\mathrm{i}\varepsilon)\,[X_j,H]\,R(\mu+\mathrm{i}\varepsilon)\right)\\
&= \frac{\varepsilon^2}{\pi}\widetilde{\operatorname{tr}}\left(\left[R(\mu-\mathrm{i}\varepsilon),X_i\right]\left[X_j,R(\mu+\mathrm{i}\varepsilon)\right]\right)\,.
\end{aligned}$$

We proceed by plugging in

$$\mathbb{1} = \sum_{x\in\mathbb{Z}^d}\delta_x\otimes\delta_x^*$$

and the definition of the trace-per-unit-volume to get

$$\begin{aligned}
\sigma_{ij}(\mu) &= \lim_{\varepsilon\to0^+}\frac{\varepsilon^2}{\pi}\widetilde{\operatorname{tr}}\left(\left[R(\mu-\mathrm{i}\varepsilon),X_i\right]\left[X_j,R(\mu+\mathrm{i}\varepsilon)\right]\right)\\
&= \lim_{\varepsilon\to0^+}\frac{\varepsilon^2}{\pi}\lim_{L\to\infty}\frac{1}{(2L+1)^d}\sum_{y\in\mathbb{Z}^d:\|y\|_1\leq L}\sum_{x\in\mathbb{Z}^d}\langle\delta_y,[R(\mu-\mathrm{i}\varepsilon),X_i]\,\delta_x\rangle\langle\delta_x,[X_j,R(\mu+\mathrm{i}\varepsilon)]\,\delta_y\rangle\\
&= \lim_{\varepsilon\to0^+}\frac{\varepsilon^2}{\pi}\lim_{L\to\infty}\frac{1}{(2L+1)^d}\sum_{y\in\mathbb{Z}^d:\|y\|_1\leq L}\sum_{x\in\mathbb{Z}^d}G(y,x;\mu-\mathrm{i}\varepsilon)\,(x_i-y_i)\,(x_j-y_j)\,G(x,y;\mu+\mathrm{i}\varepsilon)\\
&= \lim_{\varepsilon\to0^+}\frac{\varepsilon^2}{\pi}\lim_{L\to\infty}\frac{1}{(2L+1)^d}\sum_{y\in\mathbb{Z}^d:\|y\|_1\leq L}\sum_{x\in\mathbb{Z}^d}(x_i-y_i)\,(x_j-y_j)\,|G(x,y;\mu+\mathrm{i}\varepsilon)|^2\\
&= \lim_{\varepsilon\to0^+}\frac{\varepsilon^2}{\pi}\sum_{x\in\mathbb{Z}^d}x_ix_j\lim_{L\to\infty}\frac{1}{(2L+1)^d}\sum_{y\in\mathbb{Z}^d:\|y\|_1\leq L}|G(y+x,y;\mu+\mathrm{i}\varepsilon)|^2
\end{aligned}$$

□

Lemma 1.65. *Assume H is time-reversal invariant as in Definition 1.62 and let $\mathrm{d}m_{ij}(\lambda_1,\lambda_2)$ be the associated velocity measure*

$$\mathrm{d}m_{ij}(\lambda_1,\lambda_2)\equiv\widetilde{\operatorname{tr}}\left(V_i\mathrm{d}Q(\lambda_1)V_j\mathrm{d}Q(\lambda_2)\right)$$

where

$$V_i \quad := \quad \mathrm{i}\left[H,X_i\right]$$

is the velocity operator in the ith direction and Q is the projection-valued measure associated to H.

Then

$$\mathrm{d}m_{ij}(\lambda_1,\lambda_2) \quad = \quad \mathrm{d}m_{ij}(\lambda_2,\lambda_1)\,.$$

Proof. TODO □

Corollary 1.66 (Random ergodic operators)**.** *We will have a thorough discussion later about random ergodic operators, but let us just remark here that if H is actually a random ergodic operator, so that Birkhoff's theorem applies on it (in the sense that space averages may be exchanged for disorder averages) then we get*

$$\sigma_{ij}(\mu)=\lim_{\varepsilon\to0^+}\frac{\varepsilon^2}{\pi}\sum_{x\in\mathbb{Z}^d}x_ix_j\mathbb{E}\left[\left|G(x,0;\mu+\mathrm{i}\varepsilon)\right|^2\right]\,. \tag{1.21}$$

Proof. We employ Birkhoff's theorem

$$\lim_{L\to\infty}\frac{1}{(2L+1)^d}\sum_{x\in\mathbb{Z}^d:\|x\|_1\leq L}\langle\delta_x,f(H)\,\delta_x\rangle=\mathbb{E}\left[\langle\delta_0,f(H)\,\delta_0\rangle\right]$$

where f is any measurable function of the Hamiltonian. We thus start from the last displayed equation in the above proof to get

$$\widetilde{\operatorname{tr}}\left(\left[R(\mu-\mathrm{i}\varepsilon),X_i\right]\left[X_j,R(\mu+\mathrm{i}\varepsilon)\right]\right)=\mathbb{E}\left[\langle\delta_0,\left[R(\mu-\mathrm{i}\varepsilon),X_i\right]\left[X_j,R(\mu+\mathrm{i}\varepsilon)\right]\delta_0\rangle\right]\,.$$

With that we have

$$\begin{aligned}
\sigma_{ij}(\mu) \quad &= \quad \lim_{\varepsilon\to0^+}\frac{\varepsilon^2}{\pi}\mathbb{E}\left[\langle\delta_0,\left[R(\mu-\mathrm{i}\varepsilon),X_i\right]\left[X_j,R(\mu+\mathrm{i}\varepsilon)\right]\delta_0\rangle\right]\\
&= \quad \lim_{\varepsilon\to0^+}\frac{\varepsilon^2}{\pi}\mathbb{E}\left[\left\langle\delta_0,\left[R(\mu-\mathrm{i}\varepsilon),X_i\right]\sum_{x\in\mathbb{Z}^d}\delta_x\otimes\delta_x^*\left[X_j,R(\mu+\mathrm{i}\varepsilon)\right]\delta_0\right\rangle\right]\\
&= \quad \lim_{\varepsilon\to0^+}\frac{\varepsilon^2}{\pi}\sum_{x\in\mathbb{Z}^d}\mathbb{E}\left[\langle\delta_0,\left[R(\mu-\mathrm{i}\varepsilon),X_i\right]\delta_x\rangle\langle\delta_x,\left[X_j,R(\mu+\mathrm{i}\varepsilon)\right]\delta_0\rangle\right]\\
&\overset{X_i\delta_0=0}{=} \quad \lim_{\varepsilon\to0^+}\frac{\varepsilon^2}{\pi}\sum_{x\in\mathbb{Z}^d}\mathbb{E}\left[\langle\delta_0,R(\mu-\mathrm{i}\varepsilon)X_i\delta_x\rangle\langle\delta_x,X_jR(\mu+\mathrm{i}\varepsilon)\delta_0\rangle\right]\\
&= \quad \lim_{\varepsilon\to0^+}\frac{\varepsilon^2}{\pi}\sum_{x\in\mathbb{Z}^d}x_ix_j\mathbb{E}\left[G(0,x;\mu-\mathrm{i}\varepsilon)\,G(x,0;\mu+\mathrm{i}\varepsilon)\right]\,.
\end{aligned}$$

But $H=H^*$ so $G(x,y;z)=\overline{G(y,x;\overline{z})}$ and hence

$$\sigma_{ij}(\mu) \quad = \quad \lim_{\varepsilon\to0^+}\frac{\varepsilon^2}{\pi}\sum_{x\in\mathbb{Z}^d}x_ix_j\mathbb{E}\left[\left|G(x,0;\mu+\mathrm{i}\varepsilon)\right|^2\right]$$

which is what we were trying to show. □

Remark 1.67 (Gapped systems are insulators). It is clear that if $\mu \notin \sigma(H)$, then $\sigma_{ij}(\mu) = 0$. Indeed, in that case we may invoke the Combes-Thomas estimate Theorem 1.18 to obtain

$$\sup_{\varepsilon>0} |G(x, y; \mu + \mathrm{i}\varepsilon)|^2 \quad \leq \quad \frac{4}{\delta^2} \exp(-2\delta\tilde{\mu}\|x - y\|) \qquad (x, y \in \mathbb{Z}^d)$$

where $\delta := \operatorname{dist}(\mu, \sigma(H)) > 0$ is the gap size. Since this is an estimate uniform in $\varepsilon > 0$, we get summability in the x variable before even taking the limit $\varepsilon \to 0^+$. Then end result is then

$$\sigma_{ij}(\mu) \quad = \quad \lim_{\varepsilon \to 0^+} \varepsilon^2 \times \left(\text{something uniformly bounded as } \varepsilon \to 0^+\right) = 0\,.$$

Remark 1.68 (DC conductivity for ballistic motion is infinite). What might happen if we have ballistic motion? For instance, can we show that $\sigma_{ij}(\mu) = 0$ if $\mu \in \sigma_{\mathrm{ac}}(H)$? As a case study, take the discrete Laplacian in $1D$, whence we have

$$\sigma_{11}(\mu) \quad = \quad \frac{\varepsilon^2}{\pi} \sum_{x \in \mathbb{Z}} x^2 \left| \left((-\Delta - \mu \mathbb{1} - \mathrm{i}\varepsilon \mathbb{1})^{-1} \right)_{x,0} \right|^2 .$$

To proceed we have two options, we could either first calculate

$$(-\Delta - z\mathbb{1})^{-1}_{x,0} \quad \equiv \quad \frac{1}{2\pi} \int_{k=0}^{2\pi} \mathrm{e}^{\mathrm{i}kx} \frac{1}{2 - 2\cos(k) - z} \mathrm{d}k$$

where $z = \mu + \mathrm{i}\varepsilon$ and say, μ lies in the middle of the spectrum, say, at 2, and doing a residue calculation. Instead, we may bring this integral to momentum space as

$$\begin{aligned} \sum_{x \in \mathbb{Z}} x^2 \left| \left((-\Delta - \mu \mathbb{1} - \mathrm{i}\varepsilon \mathbb{1})^{-1} \right)_{x,0} \right|^2 \quad &= \quad \left\langle \delta_0, R(\overline{z}) X^2 R(z) \delta_0 \right\rangle_{\ell^2} \\ &= \quad \left\langle \mathcal{F}\delta_0, \mathcal{F} R(\overline{z}) \mathcal{F}^* \mathcal{F} X^2 \mathcal{F}^* \mathcal{F} R(z) \mathcal{F}^* \mathcal{F} \delta_0 \right\rangle_{L^2} . \end{aligned}$$

We now re-call from the proof of Proposition 1.27 that

$$\begin{aligned} \mathcal{F} X^2 \mathcal{F}^* \quad &= \quad -\partial_k^2 \\ \mathcal{F}\delta_0 \quad &= \quad k \mapsto 1 \end{aligned}$$

so that (with $z = 2 + \mathrm{i}\varepsilon$, say)

$$\begin{aligned} \sum_{x \in \mathbb{Z}} x^2 \left| \left((-\Delta - \mu \mathbb{1} - \mathrm{i}\varepsilon \mathbb{1})^{-1} \right)_{x,0} \right|^2 \quad &= \quad -\frac{1}{2\pi} \int_{k=0}^{2\pi} \frac{1}{2 - 2\cos(k) - \overline{z}} \partial^2 \frac{1}{2 - 2\cos(k) - z} \mathrm{d}k \\ &\overset{\text{Mathematica}}{=} \quad \frac{2}{\varepsilon^3 \sqrt{4 + \varepsilon^2}} . \end{aligned}$$

We see clearly that as $\varepsilon \to 0^+$, the expression

$$\lim_{\varepsilon \to 0^+} \quad \frac{\varepsilon^2}{\pi} \times \frac{2}{\varepsilon^3 \sqrt{4+\varepsilon^2}} \quad = \infty\,.$$

This is not an accident: periodic operators in general will exhibit *infinite* DC conductivity, i.e., zero resistivity.

1.8.2 The general case: IQHE application

Before proceeding we make an important modification:

Definition 1.69 (Switch function)**.** A switch function on the jth axis ($j = 1, \ldots, d$) is a projection

$$\Lambda_j \equiv \chi_{\mathbb{N}}(X_j)$$

to the jth positive half-space.

We want to replace X_j with Λ_j so that will turn out to yield trace class operators. For the perturbation (i.e., the application of the electric field) the justification is easy. It replaces the constant field with a delta field. For the observable, it means calculating the amount of charge accumulated on the half-space rather than velocity.

Theorem 1.70. *For two-dimensional systems that do* not *have time-reversal-invariance, such as integer quantum Hall systems, if μ is within a spectral gap of H, one can bring (1.19) to the form*

$$\sigma_{ij}(\mu) = \mathrm{itr}\left(P\left[\left[\Lambda_i, P\right], \left[\Lambda_j, P\right]\right]\right) . \tag{1.22}$$

Here, Λ_j is a projection operator onto the positive half-space defined by the jth axis:

$$(\Lambda_j \psi)_x \equiv \begin{cases} \psi_x & x_j \geq 1 \\ 0 & x_j \leq 0 . \end{cases}$$

Part of the statement of the theorem is that the above expression is indeed trace-class in two-dimensions (in higher dimensions it is not *and one should rather use the trace per unit volume).*

We delay the proof of this statement until we go on to talk about the Chern number of integer quantum Hall systems.

2 Random operators and Anderson localization

In this chapter we set up the necessary machinery for discussing the phenomenon of Anderson localization: this is the set up of random ergodic operators. We shall then present different proofs of this fact in various regimes and dimensions and conclude by presenting the big open problem of delocalization.

The theory of localization started with the ground breaking work of Anderson [And58]. Roughly speaking it says that if electrons are placed in a sufficiently disordered medium–neglecting electron-electron interactions–they will get "stuck" in confined regions rather than flow throughout space (compare this with translation-invariant media where Bloch theorem says that electrons are blind to the crystal structure and flow through it freely). One important consequence is that the DC electrical conductivity at the corresponding Fermi energy is zero (1.21), which means we should associate such materials with insulators. Mathematically the first proof of localization appeared in [FS83]; a simpler, different proof appeared in [AM93] which was further developed in [AG98], allowing for the understanding of the role of localization in the plateaus of the IQHE.

2.1 Why random operators?

Anderson's strategy to understand a disordered material was to toss coins in order to generate a random potential, and make statements which hold *almost surely* with respect to the probability distribution of the coins or alternatively statements about expectations (w.r.t. disorder) of physical quantities. While an actual experiment is performed on *one single* material (and hence corresponding to a deterministic Hamiltonian), the theory should describe the outcome of an average over *many* experiments so that such theoretical statements about inherently random objects could actually describe (an ensemble of) experiments. The individual macroscopic sample contains in itself many microscopic subsamples, and hence the averaging. Indeed, the actual process with which disorder is formed in materials is likely described by some probability distribution (ultimately relating to a quantum stochastic process) and our probabilistic model is merely a (gross) simplification of the real one. Another philosophical justification for this approach is via Wigner's random matrix theory. It says that in the absence of better knowledge about the actual physical laws, we pretend the unknown part of the model is given by a collection of random variables. General physical principles (e.g. locality) will then give constraints on these random variables (e.g., their independence). For an introduction to random operators, see [AW15].

2.2 Basic setup for random operators

2.2.1 Abstract definitions

A probability space is a triplet $(\Omega, \mathcal{F}, \mathbb{P})$ where Ω is a set (of possible *basic* events), $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ is a sigma-algebra[1] and

$$\mathbb{P} : \mathcal{F} \to [0,1]$$

is a probability measure. On such a probability measure we put additional structure as follows

[1] Recall a sigma-algebra is a collection of subsets of Ω containing Ω which is closed under complements and countable unions. The smallest sigma-algebra is $\{\varnothing, \Omega\}$ and the largest one is 2^{Ω}.

Definition 2.1 (Measure-preserving morphism)**.** A map $T:\Omega\to\Omega$ is called *measure-preserving* iff

$$\mathbb{P}(S) \quad = \quad \mathbb{P}\left(T^{-1}(S)\right) \qquad (S\in\mathscr{F})$$

where $T^{-1}(S)$ is the pre-image of S under T. The tuple $(\Omega,\mathscr{F},\mathbb{P},T)$ is called a *measure-preserving dynamical system.*

Definition 2.2. Let G be a group and that $T:G\to\operatorname{Aut}(\Omega)$ is a a group morphism, where $\operatorname{Aut}(\Omega)$ is the group of automorphisms of Ω (i.e., each element $T(g)$ is measure-preserving). Then the tuple $(\Omega,\mathscr{F},\mathbb{P},T)$ is called a *measure-preserving G-dynamical system.*

Definition 2.3 (invariant RV, ergodic dynamical systems)**.** A measurable map $X:\Omega\to\mathbb{R}$ is called *a random variable.* A random variable on a measure-preserving G-dynamical system $(\Omega,\mathscr{F},\mathbb{P},T)$ is called *invariant* iff

$$X\circ T(g)=X \qquad (g\in G)\ .$$

A measure-preserving G-dynamical system $(\Omega,\mathscr{F},\mathbb{P},T)$ is called *ergodic* iff every invariant random variable is constant $\mathbb{P}$-almost-surely.

Definition 2.4 (Random operator)**.** Let $(\Omega,\mathscr{F},\mathbb{P})$ be a probability space. That means that Ω is a measure space, $\mathscr{F}$ is a given sigma-algebra on it and $\mathbb{P}:\mathscr{F}\to[0,1]$ is a probability measure. Let $\mathscr{H}$ be a fixed separable Hilbert space. A random (self-adjoint) operator A is a weakly-measurable function

$$A:\Omega\to\{\,B\in\mathcal{B}(\mathscr{H})\mid B=B^*\,\}$$

i.e., for any $\varphi,\psi\in\mathscr{H}$, for any measurable $f:\mathbb{R}\to\mathbb{C}$

$$\Omega\ni\omega\mapsto\langle\varphi,f(A(\omega))\psi\rangle\in\mathbb{C}$$

is a measurable function.

Definition 2.5 (Ergodic operator)**.** A random operator $A:\Omega\to\mathcal{B}(\mathscr{H})$ (where Ω has the structure of a measure-preserving G-dynamical system $(\Omega,\mathscr{F},\mathbb{P},T)$) is called *ergodic* iff for all $g\in G$, and for all $\omega\in\Omega$, $A(\omega)$ and $A(T(g)(\omega))$ are unitary conjugates (the unitary may well depend on both ω and g).

Theorem 2.6 (Birkhoff)**.** *Let an ergodic measure-preserving $\mathbb{Z}^d$-dynamical system $(\Omega,\mathscr{F},\mathbb{P},T)$ be given. Let $X\in L^1(\Omega,\mathbb{P})$ be a random variable. Then the following limit exists $\mathbb{P}$-almost-surely and equals*

$$\lim_{L\to\infty}\frac{1}{(2L+1)^d}\sum_{x\in\mathbb{Z}^d:\|x\|_1\leq L}X(T_x\omega) \quad = \quad \mathbb{E}[X]\ .$$

Theorem 2.7 (Pastur)**.** *Let an ergodic measure-preserving $\mathbb{Z}^d$-dynamical system $(\Omega,\mathscr{F},\mathbb{P},T)$ be given and $H=H^*:\Omega\to\mathcal{B}(\mathscr{H})$ be an ergodic random self-adjoint operator. Then there are (deterministic) subset $s,s_\sharp\subseteq\mathbb{R}$ such that $\mathbb{P}$-almost-surely,*

$$\sigma_\sharp(H(\omega)) \quad = \quad s_\sharp$$

where $\sharp$ is either nothing (in which case we mean the entire spectrum) or pp,sc,ac.

Proof. [TODO: fix this] Consider, for any $a<b\in\mathbb{R}$ the map

$$\Omega\ni\omega \quad \mapsto \quad \dim\left(\operatorname{im}\left(\chi_{(a,b)}(H(\omega))\right)\right)\in[0,\infty]$$

which is measurable. Since H is presumed ergodic, these functions are invariant under translations. Indeed, we have

$$\begin{aligned}\dim\left(\mathrm{im}\left(\chi_{(a,b)}\left(H\left(T_x\omega\right)\right)\right)\right) &= \mathrm{tr}\left(\chi_{(a,b)}\left(H\left(T_x\omega\right)\right)\right)\\ &= \mathrm{tr}\left(\chi_{(a,b)}\left(U_x^* H\left(\omega\right)U_x\right)\right)\\ &= \mathrm{tr}\left(U_x^*\chi_{(a,b)}\left(H\left(\omega\right)\right)U_x\right)\\ &= \mathrm{tr}\left(\chi_{(a,b)}\left(H\left(\omega\right)\right)\right)\,.\end{aligned}$$

So this is an invariant random variable and since our system is ergodic, it implies there are constants $\alpha_{(a,b)} \in [0,\infty]$ such that

$$\mathbb{P}\left[\left\{\,\dim\left(\mathrm{im}\left(\chi_{(a,b)}\left(H\left(\omega\right)\right)\right)\right) = \alpha_{(a,b)}\,\right\}\right] \quad = \quad 1\,.$$

Since $\sigma\left(H\left(\omega\right)\right)$ is identified as the essential support of the spectral projections of $H\left(\omega\right)$, we identify

$$s \quad := \quad \left\{\,E\in\mathbb{R}\,\middle|\,\forall a,b\in\mathbb{Q}: a<E<b,\,\alpha_{(a,b)}>0\,\right\}\,.$$

We choose rational end points to make sure the countable intersection still has probability one:

$$\bigcap_{a,b\in\mathbb{Q}:\alpha_{(a,b)}>0}\left\{\,\dim\left(\mathrm{im}\left(\chi_{(a,b)}\left(H\left(\omega\right)\right)\right)\right) = \alpha_{(a,b)}\,\right\} \quad\subseteq\quad \left\{\,\sigma\left(H\left(\omega\right)\right) = s\,\right\}\,.$$

□

2.2.2 Concrete application: the Anderson model

We now consider the main setup which will concern us. We are interested in random operators H_ω on $\ell^2\left(\mathbb{Z}^d\right)$ which are of the form

$$H_\omega := -\Delta + \lambda V_\omega\left(X\right) \tag{2.1}$$

where $-\Delta$ is the discrete Laplacian, $\lambda>0$ is a coupling constant, and

$$V_\omega\left(x\right) := \omega_x \qquad \left(x\in\mathbb{Z}^d\right)$$

where $\left\{\,\omega_x\,\right\}_{x\in\mathbb{Z}^d}$ is a point in the random configuration space

$$\Omega \quad := \quad \left\{\,\omega:\mathbb{Z}^d\to\mathbb{R}\text{ measurable}\,\right\}\cong\mathbb{R}^{\mathbb{Z}^d}\,.$$

Moreover, we are interested in the following product measure

$$\int_{\omega\in\Omega} f\left(\omega\right)\mathrm{d}\mathbb{P}\left(\omega\right) \quad := \quad \prod_{x\in\mathbb{Z}^d}\int_{\omega_x\in\mathbb{R}}\mathrm{d}\mu\left(\omega_x\right)f\left(\omega\right)$$

where μ is a fixed probability measure on $\mathbb{R}$. Formally we write

$$\mathbb{P} \quad = \quad \mu^{\otimes\mathbb{Z}^d}\,.$$

We say that in this case, the stochastic process $\left\{\,\omega_x\,\right\}_{x\in\mathbb{Z}^d}$ is *iid*: it is independent and identically distributed (according to the "single site" probability measure μ). We usually ask that μ obeys some regularity condition, for example,

Definition 2.8 (uniform τ-Hoelder continuity)**.** Let $\tau\in(0,1]$. The probability measure $\mu:\mathcal{B}\left(\mathbb{R}\right)\to[0,1]$ ($\mathcal{B}\left(\mathbb{R}\right)$ being Borel measurable subsets of $\mathbb{R}$) is said to be uniformly τ-Hoelder continuous iff there exists some constant $C_\mu>0$ such that

$$\mu\left(J\right)\leq C_\mu\left|J\right|^\tau \qquad \left(J\subseteq\mathbb{R}\text{ interval with }\left|J\right|\leq 1\right)$$

where $|J|$ is the Lebesgue measure of J.

In this case, the group we are interested in is

$$G := \mathbb{Z}^d$$

i.e., the group of lattice translations:

$$T:\mathbb{Z}^d \quad\to\quad \mathrm{Aut}\left(\Omega\right)$$

is defined as

$$(T_x)(\omega) \quad := \quad \omega_{\cdot - x} \qquad \left(x \in \mathbb{Z}^d\right) .$$

These shifts are measure preserving since Ω is merely a product space with the product measure. The unitary transformation which relates H_ω with $H_{T_x\omega}$ is of course lattice translations (recall they commute with $-\Delta$).

Remark 2.9. It is appropriate to look at independent identically distributed random potential values due to the homogeneous (*in distribution*) nature of materials: we presume that on the whole they obey the same laws of physics throughout space. We can of course generalize this to decaying correlations etc. We avoid doing so here unless otherwise specified.

Theorem 2.10 (Kunz-Souillard)**.** *If we normalize $-\Delta$ such that*

$$\sigma(-\Delta) = [-2d, 2d]$$

then $\mathbb{P}$-almost-surely, the spectrum of the Anderson model (2.1)

$$\sigma(-\Delta + \lambda V_\omega(X)) \quad = \quad [-2d, 2d] + \lambda \operatorname{supp}(\mu) .$$

Here we mean the set addition as

$$A + B \quad := \quad \left\{ E + \tilde{E} \in \mathbb{R} \,\middle|\, E \in A, \tilde{E} \in B \right\}$$

and

$$\operatorname{supp}(\mu) \quad \equiv \quad \{ u \in \mathbb{R} \mid \forall \varepsilon > 0, \mu(B_\varepsilon(u)) > 0 \} .$$

Note we have

$$\overline{\{ V_\omega(x) \mid x \in \mathbb{Z}^d \}} = \operatorname{supp}(\mu)$$

$\mathbb{P}$-almost-surely.

Proof. [TODO: fix this] This statement shall be proven in two steps: $\subseteq$ and $\supseteq$. Let us begin with the former. Let

$$E \notin [-2d, 2d] + \lambda \operatorname{supp}(\mu) .$$

That means that

$$\operatorname{dist}(E, \lambda \operatorname{supp}(\mu)) > 2d .$$

But then,

$$-\Delta + \lambda V_\omega(X) - E\mathbb{1} \quad = \quad (\lambda V_\omega(X) - E\mathbb{1}) \left(\mathbb{1} - (\lambda V_\omega(X) - E\mathbb{1})^{-1} \Delta\right)$$

and we have

$$\begin{aligned} \left\| - (\lambda V_\omega(X) - E\mathbb{1})^{-1} \Delta \right\| \quad &\leq \quad \left\| (\lambda V_\omega(X) - E\mathbb{1})^{-1} \right\| \|-\Delta\| \\ &< \quad 1 \end{aligned}$$

so the operator

$$\left(\mathbb{1} - (\lambda V_\omega(X) - E\mathbb{1})^{-1} \Delta\right)$$

is invertible and hence

$$E \quad \notin \quad \sigma(-\Delta + \lambda V_\omega(X)) .$$

For the other inclusion, let $E \in [-2d, 2d] = \sigma(-\Delta)$. We thus build a Weyl sequence [Sha24] for $-\Delta$: for any $\varepsilon > 0$ there exists some $\psi \in \ell^2\left(\mathbb{Z}^d\right)$ with $\|\psi\| = 1$ such that

$$\|(-\Delta - E\mathbb{1})\psi\| \leq \varepsilon .$$

Let us assume for a moment that ψ is supported within a large finite box Λ (otherwise approximate and use locality of $-\Delta$). Then if $\tilde{E} \in \operatorname{supp}(\mu)$, we must have

$$\mathbb{P}\left[\left\{ \omega\in\Omega \,\middle|\, \sup_{x\in\Lambda}\left|\lambda\omega_x-\tilde{E}\right|<\varepsilon \right\}\right] \quad = \quad \prod_{x\in\Lambda}\mu\left(B_\varepsilon\left(\tilde{E}\right)\right)>0\,.$$

For such ω's, we have

$$\left\|\left(-\Delta+\lambda V_\omega(X)-\left(E+\tilde{E}\right)\mathbb{1}\right)\psi\right\| \leq \|(-\Delta-E\mathbb{1})\psi\|+\left\|\left(V_\omega(X)-\tilde{E}\mathbb{1}\right)\psi\right\| \leq 2\varepsilon$$

so that actually ψ is a Weyl sequence for $-\Delta+V_\omega(X)$ and hence

$$\mathbb{P}\left[\left\{ \omega\in\Omega \,\middle|\, \operatorname{dist}\left(E+\tilde{E},\sigma\left(-\Delta+\lambda V_\omega(X)\right)\right)<2\varepsilon \right\}\right] \quad > \quad 0$$

and so by ergodicity, must equal 1. □

2.3 The main results known so far and conjectures

2.3.1 Criteria for localization

Here we survey various criteria for localization. Some imply the others automatically (as we discuss momentarily).

Let $\omega \mapsto H_\omega$ be an ergodic random operator on $\ell^2\left(\mathbb{Z}^d\right)$ and $E\in\mathbb{R}$ be an energy value. We have

1. *Spectral localization*: There exists some $\varepsilon>0$ such that

$$B_\varepsilon(E)\cap\sigma(H) \quad = \quad B_\varepsilon(E)\cap\sigma_{\text{pp}}(H)$$

almost-surely (or an analogous statement about the almost sure spectrum).

2. *Decay of eigenfunctions*: If $H\psi=E\psi$ then there exists some $C,\mu\in(0,\infty)$ such that

$$|\psi(x)| \quad \leq \quad C\mathrm{e}^{-\mu\|x\|} \qquad \left(x\in\mathbb{Z}^d\right)$$

almost-surely. Presumably it is impossible that this happens merely at a *single* energy and one should rather ask that this holds for every eigenfunction with energy $\tilde{E}\in B_\varepsilon(E)$ for some $\varepsilon>0$.

3. *High inverse participation ratio*: In finite boxes $\Lambda\subseteq\mathbb{Z}^d$, if $H\psi=E\psi$ with $\|\psi\|=1$, then for any $x\in\Lambda$, $|\psi(x)|^2$ could take the values between $0,\frac{1}{|\Lambda|^{\frac{1}{2}}},1$. If the state is fully localized in one position, there would be a single $x_0\in\Lambda$ where $|\psi(x_0)|=1$ and otherwise if it is fully delocalized, it would be completely spread out throughout space so that

$$|\psi(x)| \quad \approx \quad \frac{1}{|\Lambda|^{\frac{1}{2}}} \qquad (x\in\Lambda)$$

so that $\sum_{x\in\Lambda}|\psi(x)|^2=1$. Hence, to measure how "flat" the wave-function is, we introduce

$$\operatorname{IPR}(\psi):=\sum_{x\in\Lambda}|\psi(x)|^4\,. \tag{2.2}$$

If $\operatorname{IPR}(\psi)\approx 1$ we say the state is localized. However, if it is fully delocalized we expect

$$\operatorname{IPR}(\psi)\approx\sum_{x\in\Lambda}\left(|\Lambda|^{-\frac{1}{2}}\right)^4=|\Lambda|\,|\Lambda|^{-2}=|\Lambda|^{-1}$$

which should be tiny if $|\Lambda|$ is large.

4. *Localization of transport*[AG98]: The diagonal elements of the zero-temperature DC conductivity matrix vanish

$$\sigma_{ii}(E)\equiv\lim_{\varepsilon\to0^+}\frac{\varepsilon^2}{\pi}\sum_{x\in\mathbb{Z}^d}x_i^2\,\mathbb{E}\left[|G(x,0;E+\mathrm{i}\varepsilon)|^2\right]=0 \qquad (i=1,\dots,d)\,.$$

5. *Localization of position*: The second moments of the position operator evolved with time around bounded. That is, there exists some $\varepsilon > 0$ such that

$$\sup_{t>0} \mathbb{E}\left[\left|\left\langle \chi_{B_\varepsilon(E)}(H)\delta_0, \mathrm{e}^{\mathrm{i}tH} X_i X_j \mathrm{e}^{-\mathrm{i}tH} \chi_{B_\varepsilon(E)}(H)\delta_0\right\rangle\right|\right] < \infty\,.$$

6. *Dynamical localization*[AM93]: The probability to reach far away places via time evolution decays with distance, uniformly in time. I.e., there exists some $\varepsilon > 0$ such that there exist $C, \mu \in (0,\infty)$ with which

$$\mathbb{E}\left[\sup_{t>0}\left|\left\langle \delta_x, \mathrm{e}^{-\mathrm{i}tH}\chi_{B_\varepsilon(E)}(H)\delta_y\right\rangle\right|\right] \quad\leq\quad C\mathrm{e}^{-\mu\|x-y\|} \qquad \left(x, y \in \mathbb{Z}^d\right)\,.$$

7. *The fractional moment condition*[AM93]: There exists some $\varepsilon > 0$, $s \in (0,1)$, $C, \mu \in (0,\infty)$ such that

$$\sup_{\eta>0, \tilde{E}\in B_\varepsilon(E)} \mathbb{E}\left[\left|G\left(x, y; \tilde{E} + \mathrm{i}\eta\right)\right|^s\right] \quad\leq\quad C\mathrm{e}^{-\mu\|x-y\|} \qquad \left(x, y \in \mathbb{Z}^d\right)\,.$$

8. *The second moment condition*[Gra94]: There exists some $\varepsilon > 0$ and $C, \mu \in (0,\infty)$ such that

$$\sup_{\eta>0, \tilde{E}\in B_\varepsilon(E)} \eta\mathbb{E}\left[\left|G\left(x, y; \tilde{E} + \mathrm{i}\eta\right)\right|^2\right] \leq C\mathrm{e}^{-\mu\|x-y\|} \qquad \left(x, y \in \mathbb{Z}^d\right)\,. \tag{2.3}$$

9. *The many-body ground state exhibits decay of correlations* [AG98]: As we have seen above, we should associate

$$P \quad\equiv\quad \chi_{(-\infty,E)}(H)$$

with the many-body ground state reduced one-particle density matrix of the system filled to Fermi energy E. Then we expect decay of correlations in the many-body ground state to manifest itself as follows: there exists some $C, \mu \in (0,\infty)$ such that

$$\mathbb{E}\left[\left|\left\langle \delta_x, P\delta_y\right\rangle\right|\right] \quad\leq\quad C\mathrm{e}^{-\mu\|x-y\|} \qquad \left(x, y \in \mathbb{Z}^d\right)\,.$$

10. *The bounded measurable functional calculus exhibits exponential decay* [AG98]: More generally and abstractly, there exists some $\varepsilon > 0$ such that if $B_1(B_\varepsilon(E))$ is the space of measurable functions $f : \mathbb{R} \to \mathbb{C}$ which obey $\|f\|_\infty \leq 1$ as well as being constant above and below $B_\varepsilon(E)$ (with possibly different constants) then there exist constants $C, \mu \in (0,\infty)$ such that

$$\mathbb{E}\left[\sup_{f\in B_1(B_\varepsilon(E))}\left\|\left\langle \delta_x, f(H)\delta_y\right\rangle\right\|\right] \quad\leq\quad C\mathrm{e}^{-\mu\|x-y\|} \qquad \left(x, y \in \mathbb{Z}^d\right)\,.$$

11. *Poisson statistics* [Min96]: Coming from the direction of random matrices, there appears to be a dichotomy in the stochastic process of gaps between gaps of eigenvalues as follows. Let H_N be a matrix resulting from H restricted to a finite box with N sites (eventually $N \to \infty$). Then we consider the random measure

$$\mathcal{n}_N(B) := \operatorname{tr}\left(\chi_B\left(N\left(H_N - E\mathbb{1}\right)\right)\right) \qquad (B \subseteq \mathbb{R})\,. \tag{2.4}$$

Then $\mathcal{n}_N$ converges, as $N \to \infty$, to a Poisson point process on $\mathbb{R}$ with intensity equal to the local density of states at E times the Lebesgue measure.

2.3.2 Criteria for delocalization

Unfortunately for delocalization we have way less conditions. We merely state

Let $\omega \mapsto H_\omega$ be an ergodic random operator on $\ell^2\left(\mathbb{Z}^d\right)$ and $E \in \mathbb{R}$ be an energy value. We have

1. *Spectral delocalization*: There exists some $\varepsilon > 0$ such that

$$B_\varepsilon(E) \cap \sigma(H) \quad=\quad B_\varepsilon(E) \cap \sigma_{\mathrm{ac}}(H)$$

almost-surely (or an analogous statement about the almost sure spectrum).

2. *Low inverse participation ratio*: In finite boxes $\Lambda \subseteq \mathbb{Z}^d$, if $H\psi = E\psi$ with $\|\psi\| = 1$, then

$$\mathrm{IPR}(\psi) \quad \approx \quad \frac{1}{|\Lambda|}$$

where the inverse participation ratio was defined in (2.2).

3. *Delocalization of transport*: The diagonal elements of the zero-temperature DC conductivity matrix are finite and non-zero:

$$\sigma_{ii}(E) \equiv \lim_{\varepsilon \to 0^+} \frac{\varepsilon^2}{\pi} \sum_{x \in \mathbb{Z}^d} x_i^2 \mathbb{E}\left[|G(x, 0; E + \mathrm{i}\varepsilon)|^2\right] > 0 \qquad (i = 1, \dots, d) .$$

4. *Delocalization of position*: The second moments of the position operator evolved with time around unbounded. That is, there exists some $\varepsilon > 0$ such that

$$\sup_{t>0} \mathbb{E}\left[\left|\left\langle \chi_{B_\varepsilon(E)}(H)\,\delta_0, \mathrm{e}^{\mathrm{i}tH} X_i X_j \mathrm{e}^{-\mathrm{i}tH} \chi_{B_\varepsilon(E)}(H)\,\delta_0 \right\rangle\right|\right] = \infty .$$

We could also boost this to diffusion if we ask that the quantity behaves linearly in t.

5. *The many-body ground state exhibits no decay of correlations*:

$$\sum_{x \in \mathbb{Z}^d} \mathbb{E}\left[|\langle \delta_0, P\delta_x \rangle|\right] \quad = \quad \infty .$$

6. *GUE statistics* : The measure defined above as (2.4) converges, as $N \to \infty$, to the GUE statistics point process: The joint probability density of the eigenvalues $\{ E_j \}_j$ is given by

$$\frac{1}{Z} \exp\left(-\frac{1}{2}\sum_{j=1}^{N} E_j^2\right) \prod_{i<j} |E_i - E_j|^2$$

where Z is a normalization constant. In particular eigenvalues repel as the density is zero for $E_i = E_j$.

2.3.3 Established mathematical facts

Consider the Anderson model

$$H_\omega \quad = \quad -\Delta + \lambda V_\omega(X)$$

on $\ell^2(\mathbb{Z}^d)$ and $\lambda > 0$ the coupling strength. Here $\{ \omega_x \}_{x \in \mathbb{Z}^d}$ is an IID sequence of real random variables. Then

1. *Complete localization in 1D*: For $d = 1$, for *any* $\lambda > 0$, the system is localized at *all* energies (see [KLS90] and references therein).

2. *Complete localization at high* λ: For $d \in \mathbb{N}_{\geq 1}$, there exists some $\lambda_c > 0$ such that if $\lambda \geq \lambda_c$ then the system is localized at *all* energies (see [FS83, AM93]).

3. *Localization at arbitrary* λ *for extreme energies*: For $d \in \mathbb{N}_{\geq 1}$, given $\lambda > 0$, there exists some non-empty subset $S_\lambda \subseteq \sigma(H)$ such that the system is localized for all energies within S_λ. The set S_λ will typically lie near the boundaries of the spectrum (see [FS83, AM93]).

2.3.4 Conjectures

For the same Anderson model, one conjectures that

1. *Complete localization in 2D*: For $d = 2$, for *any* $\lambda > 0$, the system is localized at *all* energies.

2. *Delocalization for 3D and higher*: For $d \geq 3$, there exists some $\lambda_c > 0$ such that if $\lambda \leq \lambda_c$, there exists some energies where the system is delocalized. These energies will typically be in the middle of the spectrum.

Establishing either one of these statements would mean a huge breakthrough in mathematical physics.

2.4 The a-priori bound

A basic tool in the approach to Anderson localization we will consider is the a-priori bound, developed by Aizenman and Molchanov [AM93]. It is built on the following basic observation: the average of the Greens function may not exist, because it is a singularity that behaves like $\frac{1}{x}$ at $x=0$, which is *not* integrable. However, as it turns out, *a fractional moment* of the Greens function is just as good at controlling many dynamical properties, and that object *is* integrable at the origin. Indeed,

$$\int_{x=-1}^{1} \frac{1}{|x|^s} \mathrm{d}x \quad = \quad \frac{2}{1-s} \qquad (s<1)\ .$$

To begin the analysis, we state the basic tool from linear algebra, the Schur complement.

Lemma 2.11. *Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ be a $\mathbb{Z}_2$-grading of a Hilbert space and let*

$$L \quad = \quad \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

be a block operator on $\mathcal{H}$. Then if D is invertible and the Schur operator

$$S \quad := \quad A - BD^{-1}C : \mathcal{H}_1 \to \mathcal{H}_1$$

is invertible, we have

$$L^{-1} \quad = \quad \begin{bmatrix} S^{-1} & -S^{-1}BD^{-1} \\ -D^{-1}CS^{-1} & D^{-1}+D^{-1}CS^{-1}BD^{-1} \end{bmatrix}\ .$$

Proof. Note the identity

$$\begin{aligned}
\begin{bmatrix} \mathbb{1}_1 & BD^{-1} \\ 0 & \mathbb{1}_2 \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} \mathbb{1}_1 & 0 \\ D^{-1}C & \mathbb{1}_2 \end{bmatrix} \quad &= \quad \begin{bmatrix} \mathbb{1}_1 & BD^{-1} \\ 0 & \mathbb{1}_2 \end{bmatrix} \begin{bmatrix} S & 0 \\ C & D \end{bmatrix} \\
&= \quad \begin{bmatrix} S - BD^{-1}C & B \\ C & D \end{bmatrix} \\
&\overset{S\equiv A-BD^{-1}C}{=} \quad \begin{bmatrix} A & B \\ C & D \end{bmatrix}\ .
\end{aligned}$$

But the matrices $\begin{bmatrix} \mathbb{1}_1 & BD^{-1} \\ 0 & \mathbb{1}_2 \end{bmatrix}, \begin{bmatrix} \mathbb{1}_1 & 0 \\ D^{-1}C & \mathbb{1}_2 \end{bmatrix}$ are both invertible regardless of B, D, C:

$$\begin{bmatrix} \mathbb{1}_1 & BD^{-1} \\ 0 & \mathbb{1}_2 \end{bmatrix}^{-1} \quad = \quad \begin{bmatrix} \mathbb{1}_1 & -BD^{-1} \\ 0 & \mathbb{1}_2 \end{bmatrix}, \qquad \begin{bmatrix} \mathbb{1}_1 & 0 \\ D^{-1}C & \mathbb{1}_2 \end{bmatrix}^{-1} = \begin{bmatrix} \mathbb{1}_1 & 0 \\ -D^{-1}C & \mathbb{1}_2 \end{bmatrix}\ .$$

Hence we may take the inverse of the previous identity to get

$$\begin{aligned}
\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \quad &= \quad \left(\begin{bmatrix} \mathbb{1}_1 & BD^{-1} \\ 0 & \mathbb{1}_2 \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} \mathbb{1}_1 & 0 \\ D^{-1}C & \mathbb{1}_2 \end{bmatrix} \right)^{-1} \\
&= \quad \begin{bmatrix} \mathbb{1}_1 & 0 \\ -D^{-1}C & \mathbb{1}_2 \end{bmatrix} \begin{bmatrix} S^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix} \begin{bmatrix} \mathbb{1}_1 & -BD^{-1} \\ 0 & \mathbb{1}_2 \end{bmatrix} \\
&= \quad \begin{bmatrix} S^{-1} & 0 \\ -D^{-1}CS^{-1} & D^{-1} \end{bmatrix} \begin{bmatrix} \mathbb{1}_1 & -BD^{-1} \\ 0 & \mathbb{1}_2 \end{bmatrix} \\
&= \quad \begin{bmatrix} S^{-1} & -S^{-1}BD^{-1} \\ -D^{-1}CS^{-1} & D^{-1}CS^{-1}BD^{-1}+D^{-1} \end{bmatrix}\ .
\end{aligned}$$

□

In analyzing the properties of

$$G(x,y;z) \quad \equiv \quad (H - z\mathbb{1})^{-1}_{xy}$$

it turns out it is useful to isolate the dependence of $G(x,y;z)$ on just ω_x and ω_y, and integrate on them explicitly, before performing all other integrations. This leads to the following theorem, taken from [Gra94] but also appears in [AM93]:

Theorem 2.12 (Aizenman-Molchanov, Graf)**.** *There exists some $s \in (0,1)$ such that*

$$\sup_{x,y\in\mathbb{Z}^d, z\in\mathbb{C}:\mathbb{Im}\{z\}>0} \mathbb{E}\left[|G(x,y;z)|^s\right] \quad < \quad \infty\,.$$

As a warm up, let us first study the case $x = y$. In this case, we define

$$\mathcal{H}_1 \quad := \quad \operatorname{span}(\{\,\delta_x\,\})$$

and $\mathcal{H}_2 := \mathcal{H}_1^\perp$. Clearly the diagonal part of H restricted to $\mathcal{H}_1$ is precisely

$$\lambda\omega_x$$

whereas the off-diagonal part includes all hopping terms from the Laplacian that may lead into or out of x: We write this generally as

$$H \quad = \quad \begin{bmatrix} \lambda\omega_x & P_x(-\Delta)P_x^\perp \\ P_x^\perp(-\Delta)P_x & \tilde{H} \end{bmatrix}$$

where $P_x := \delta_x \otimes \delta_x^*$ and $\tilde{H} \equiv P_x^\perp H P_x^\perp \in \mathcal{B}\left(\ell^2\left(\mathbb{Z}^d \setminus \{\,x\,\}\right)\right)$ is a random operator that, by definition, does not depend on the variable ω_x. Then by Lemma 2.11

$$(H - z\mathbb{1})^{-1}_{xx} = \frac{1}{\lambda\omega_x - z - P_x(-\Delta)P_x^\perp\left(\tilde{H} - z\mathbb{1}_{\mathcal{H}_2}\right)^{-1}P_x^\perp(-\Delta)P_x} \qquad (z \in \mathbb{C} : \mathbb{Im}\{z\} > 0)\,. \tag{2.5}$$

This hinges on verifying the invertibility of the Schur operator as well as $\tilde{H} - z\mathbb{1}$. Let us check those: $\tilde{H} \equiv P_x^\perp H P_x^\perp$ is a self-adjoint operator so if $\mathbb{Im}\{z\} > 0$ it is automatically invertible. As for the Schur operator, since $\mathbb{Im}\{z\} > 0$ and

$$z \quad \mapsto \quad \left\langle \varphi, (H - z\mathbb{1})^{-1}\psi \right\rangle$$

is a Herglotz function (see [Sha24]) then necessarily it has a positive imaginary part. This implies that

$$\mathbb{Im}\left\{\lambda\omega_x - z - P_x(-\Delta)P_x^\perp\left(\tilde{H} - z\mathbb{1}_{\mathcal{H}_2}\right)^{-1}P_x^\perp(-\Delta)P_x\right\} > 0$$

and is hence invertible. Hence we are justified in employing the Schur complement.

The particular form of the operator

$$P_x(-\Delta)P_x^\perp\left(\tilde{H} - z\mathbb{1}_{\mathcal{H}_2}\right)^{-1}P_x^\perp(-\Delta)P_x$$

is unimportant except that it is *independent* of ω_x, by construction. Just for fun let us study it anyway. We begin with the discrete Laplacian:

$$-\Delta\delta_z = 2d\delta_z - \sum_{y\sim z}\delta_y$$

then

$$\langle\delta_w, -\Delta\delta_z\rangle = 2d\delta_{z,w} - \sum_{y\sim z}\delta_{wy}$$

and so

$$\begin{aligned} P_x^\perp(-\Delta)P_x &= P_x^\perp(-\Delta)\delta_x \otimes \delta_x^* \\ &= P_x^\perp\left(2d\delta_x - \sum_{y\sim x}\delta_y\right)\otimes\delta_x^* \\ &= -\sum_{y\sim x}P_x^\perp\delta_y \otimes \delta_x^* \\ &= -\sum_{y\sim x}\delta_y \otimes \delta_x^* \end{aligned}$$

and similarly

$$\begin{aligned} P_x (-\Delta) P_x^\perp &= \left(-\sum_{y\sim x} \delta_y \otimes \delta_x^*\right)^* \\ &= -\sum_{y\sim x} \delta_x \otimes \delta_y^* . \end{aligned}$$

Hence

$$\begin{aligned} P_x (-\Delta) P_x^\perp \left(\tilde{H} - z\mathbb{1}_{\mathcal{H}_2}\right)^{-1} P_x^\perp (-\Delta) P_x &= \sum_{y\sim x}\sum_{z\sim x} \delta_x \otimes \delta_y^* \left(\tilde{H} - z\mathbb{1}_{\mathcal{H}_2}\right)^{-1} \delta_z \otimes \delta_x^* \\ &= P_{xx} \sum_{y\sim x}\sum_{z\sim x} \left(\tilde{H} - z\mathbb{1}_{\mathcal{H}_2}\right)^{-1}_{yz} \end{aligned}$$

We thus find

$$(H - z\mathbb{1})^{-1}_{xx} = \frac{1}{\lambda\omega_x - z - \sum_{y\sim x}\sum_{z\sim x} \left(P_x^\perp H P_x^\perp - z\mathbb{1}_{\mathcal{H}_2}\right)^{-1}_{yz}} .$$

We emphasize again, we will *not* make use of any information about

$$\sum_{y\sim x}\sum_{z\sim x} \left(P_x^\perp H P_x^\perp - z\mathbb{1}_{\mathcal{H}_2}\right)^{-1}_{yz}$$

except that it is independent of ω_x.

Theorem 2.13 (a-priori bound, diagonal version (Aizenman-Molchanov))**.** *For any $s < \tau$, we have*

$$\sup_{x\in\mathbb{Z}^d} \sup_{z\in\mathbb{C}:\mathbb{Im}\{z\}>0} \mathbb{E}\left[|G(x,x;z)|^s\right] < \frac{\tau}{\tau - s} C_\mu^{\frac{s}{\tau}} \left(\frac{2}{\lambda}\right)^s$$

where $C_\mu < \infty$ is the constant of regularity of the single-site probability measure μ which is assumed to be tau-Holder regular as in Definition 2.8.

Proof. Thanks to (2.5) we find that

$$G(x,x;z) = \frac{1}{\lambda\omega_x - w}$$

for some $w \in \mathbb{C}$ which is independent of ω_x. Hence in taking the expectation $\mathbb{E}$ which is essentially an integral over *all* variables $\{\, \omega_z \,\}_{z\in\mathbb{Z}^d}$, we may first integrate over ω_x before all other variables. Hence we must bound

$$\sup_{w\in\mathbb{C}} \int_{\omega_x\in\mathbb{R}} \frac{1}{|\lambda\omega_x - w|^s} \mathrm{d}\mu(\omega_x)$$

where μ obeys some τ-Hoelder regularity as in Definition 2.8. To that end, let us estimate

$$\int_{\omega_x\in\mathbb{R}} \frac{1}{|\lambda\omega_x - w|^s} \mathrm{d}\mu(\omega_x) \le D + \int_{\omega_x\in\mathbb{R}:\frac{1}{|\lambda\omega_x - w|^s}\ge D} \frac{1}{|\lambda\omega_x - w|^s} \mathrm{d}\mu(\omega_x)$$

which holds for *any* $D > 0$. The reason for separating into above and below D is in order to regularize the second term, as will become apparent momentarily. The second term then may be rewritten using the so-called *layer-cake representation*

$$\int_{x:f(x)\ge t} f(x)\,\mathrm{d}\mu(x) = \int_{t'=t}^{\infty} \mu(\{\, x\in\mathbb{R} \mid f(x) \ge t' \,\})\,\mathrm{d}t' .$$

Indeed, the two expressions are equal by re-writing

$$f(x) = \int_{t=0}^{f(x)} \mathrm{d}t = \int_{t=0}^{\infty} \chi_{[0,f(x)]}(t)\,\mathrm{d}t = \int_{t=0}^{\infty} \chi_{\{\, y\in\mathbb{R} \mid f(y)>t \,\}}(x)\,\mathrm{d}t$$

and so

$$\begin{aligned}\int_{x:f(x)\geq t} f(x)\,\mathrm{d}\mu(x) &= \int_{x:f(x)\geq t}\int_{t'=0}^{\infty} \chi_{\{\, y\in\mathbb{R} \mid f(y)>t' \,\}}(x)\,\mathrm{d}t'\mathrm{d}\mu(x) \\ &= \int_{x}\int_{t'=t}^{\infty} \chi_{\{\, y\in\mathbb{R} \mid f(y)>t' \,\}}(x)\,\mathrm{d}t'\mathrm{d}\mu(x) \\ &= \int_{t'=t}^{\infty}\int_{x} \chi_{\{\, y\in\mathbb{R} \mid f(y)>t' \,\}}(x)\,\mathrm{d}\mu(x)\mathrm{d}t' \\ &= \int_{t'=t}^{\infty} \mu(\{\, y\in\mathbb{R} \mid f(y)>t' \,\})\,\mathrm{d}t'\,.\end{aligned}$$

But now,

$$\begin{aligned}|\lambda\omega_x - w|^{-s} > t &\iff |\lambda\omega_x - w|^{-1} > t^{\frac{1}{s}} \\ &\iff |\lambda\omega_x - w| < t^{-\frac{1}{s}} \\ &\iff \left|\omega_x - \frac{1}{\lambda}w\right| < \frac{1}{\lambda}t^{-\frac{1}{s}} \\ &\Longleftarrow \omega_x \in B_{\frac{1}{\lambda}t^{-\frac{1}{s}}}\left(\frac{1}{\lambda}\mathbb{Re}\{w\}\right)\,.\end{aligned}$$

The τ-Hoelder regularity then implies

$$\mu\left(\left\{\, |\lambda\omega_x - w|^{-s} > t \,\right\}\right) \leq C\left(2\frac{1}{\lambda}t^{-\frac{1}{s}}\right)^{\tau}$$

so that

$$\begin{aligned}\int_{\omega_x\in\mathbb{R}:\frac{1}{|\lambda\omega_x - w|^s}\geq D} \frac{1}{|\lambda\omega_x - w|^s}\mathrm{d}\mu(\omega_x) &= \int_{t'=D}^{\infty} \mu\left(\left\{\, \frac{1}{|\lambda\omega_x - w|^s} > t' \,\right\}\right)\mathrm{d}t' \\ &\leq \int_{t'=D}^{\infty} C\left(2\frac{1}{\lambda}t'^{-\frac{1}{s}}\right)^{\tau}\mathrm{d}t' \\ &= C\left(\frac{2}{\lambda}\right)^{\tau}\int_{t'=D}^{\infty} t'^{-\frac{\tau}{s}}\mathrm{d}t' \\ &= C\left(\frac{2}{\lambda}\right)^{\tau}\frac{D^{1-\frac{\tau}{s}}s}{\tau - s} \qquad (\text{if } s<\tau)\,.\end{aligned}$$

Together we find

$$\int_{\omega_x\in\mathbb{R}} \frac{1}{|\lambda\omega_x - w|^s}\mathrm{d}\mu(\omega_x) \leq D + \underbrace{\frac{C}{\frac{\tau}{s}-1}\left(\frac{2}{\lambda}\right)^{\tau}}_{=:\frac{\tilde{C}}{\alpha}} D^{1-\frac{\tau}{s}}\,.$$

Note that $s<\tau$ so $-\alpha := 1-\frac{\tau}{s} < 0$ and hence, even though $D>0$ was arbitrary, we actually have

$$\inf_{D>0}\left(D + \frac{\tilde{C}}{\alpha}D^{-\alpha}\right) = \left(1+\frac{1}{\alpha}\right)\tilde{C}^{\frac{1}{1+\alpha}}\,.$$

Hence

$$\int_{\omega_x\in\mathbb{R}} \frac{1}{|\lambda\omega_x - w|^s}\mathrm{d}\mu(\omega_x) \leq \frac{\tau}{\tau - s}C_{\mu}^{\frac{s}{\tau}}\left(\frac{2}{\lambda}\right)^{s}\,. \tag{2.6}$$

Since this upper bound is independent of w, integrating over all other variables and taking $\sup_{z\in\mathbb{C}}$ does not change the bound. We also see that it is the regularity of μ which dictates the allowed values of s: any $s\in(0,\tau)$ would do. ☐

Remark 2.14. Of course once we now that $\mathbb{E}[X^s]<\infty$ for some $s\in(0,1)$ then the same holds for all $s'\in(0,s)$. Indeed this is merely a consequence of the Hoelder inequality: Let $p:=\frac{s}{s'}>1$ and q so that $\frac{1}{p}+\frac{1}{q}=1$. Then

$$\begin{aligned}\mathbb{E}\left[X^{s'}\right] &= \mathbb{E}\left[X^{s'}\cdot 1\right]\\ &\leq \mathbb{E}[X^s]^{\frac{1}{p}}\,\mathbb{E}[1^q]^{\frac{1}{q}}\,.\end{aligned}$$

Proof of Theorem 2.12. We now turn to the proof that

$$\sup_{\varepsilon>0}\sup_{x,y\in\mathbb{Z}^d}\mathbb{E}\left[|G(x,y;E+\mathrm{i}\varepsilon)|^s\right] < \infty \qquad (E\in\mathbb{R})\,.$$

We shall follow [Gra94]. We begin by extracting the dependence of $G(x,y;z)$ on both ω_x and ω_y. Assuming that $x\neq y$, we need to study the rank-2 perturbation theory. Let us rewrite

$$\mathcal{H} = \mathcal{H}_{xy}\oplus\mathcal{H}_{xy}^{\perp}$$

where

$$\mathcal{H}_{xy} := \operatorname{im}(P_{xy})$$

with $P_x\equiv\delta_x\otimes\delta_x^*$ and $P_{xy}\equiv P_x+P_y$. With this notation, we may write

$$H = \begin{bmatrix}\lambda\omega_x P_x+\lambda\omega_y P_y & P_{xy}(-\Delta)P_{xy}^{\perp}\\ P_{xy}^{\perp}(-\Delta)P_{xy} & P_{xy}^{\perp}HP_{xy}^{\perp}\end{bmatrix}.$$

If $\mathbb{Im}\,z>0$ then again thanks to the Herglotz property, $P_{xy}^{\perp}(H-z\mathbb{1})P_{xy}^{\perp}$ will be invertible and so will the Schur operator

$$\lambda\omega_x P_x+\lambda\omega_y P_y - zP_{xy} - P_{xy}(-\Delta)\left(P_{xy}^{\perp}(H-z\mathbb{1})P_{xy}^{\perp}\right)^{-1}(-\Delta)P_{xy}\,.$$

Hence we find thanks to the Schur complement Lemma 2.11 that

$$P_{xy}(H-z\mathbb{1})^{-1}P_{xy} = \frac{1}{\lambda}\left(\begin{bmatrix}\omega_x & 0\\ 0 & \omega_y\end{bmatrix}+M\right)^{-1}$$

for some 2×2 matrix $\lambda M = -zP_{xy}-P_{xy}(-\Delta)\left(P_{xy}^{\perp}(H-z\mathbb{1})P_{xy}^{\perp}\right)^{-1}(-\Delta)P_{xy}$ with the following properties:

1. It has a positive imaginary part $\mathbb{Im}\{M\}>0$ thanks to the Herglotz property.
2. It does depend on $z\in\mathbb{C}$ and on $\omega_{\tilde{x}}$ for all $\tilde{x}\neq x,y$.

Hence if we manage to come up with an upper bound by integrating over only ω_x,ω_y and uniformly in M,z we'd be finished. We have

$$M := \begin{bmatrix}m_{xx} & m_{xy}\\ m_{yx} & m_{yy}\end{bmatrix}$$

and

$$\begin{aligned}\mathbb{Im}\{M\} &\equiv \frac{1}{2\mathrm{i}}(M-M^*)\\ &= \frac{1}{2\mathrm{i}}\left(\begin{bmatrix}m_{xx} & m_{xy}\\ m_{yx} & m_{yy}\end{bmatrix}-\begin{bmatrix}\overline{m_{xx}} & \overline{m_{yx}}\\ \overline{m_{xy}} & \overline{m_{yy}}\end{bmatrix}\right)\\ &= \begin{bmatrix}\mathbb{Im}\{m_{xx}\} & \frac{1}{2\mathrm{i}}(m_{xy}-\overline{m_{yx}})\\ \frac{1}{2\mathrm{i}}(m_{yx}-\overline{m_{xy}}) & \mathbb{Im}\{m_{yy}\}\end{bmatrix}.\end{aligned}$$

Hence

$$\lambda G(x,y;z) = \left[\left(\begin{bmatrix}\omega_x & 0\\ 0 & \omega_y\end{bmatrix}+\begin{bmatrix}m_{xx} & m_{xy}\\ m_{yx} & m_{yy}\end{bmatrix}\right)^{-1}\right]_{\text{top right corner}} = \frac{-m_{xy}}{(m_{xx}+\omega_x)(m_{yy}+\omega_y)-m_{xy}m_{yx}}\,.$$

Using the trivial $|w|\geq|\mathbb{Re}\{w\}|$ or $|w|\geq|\mathbb{Im}\{w\}|$ and the notation $\tilde{\omega}_x:=\omega_x+\mathbb{Re}\{m_{xx}\}$ and $\tilde{\omega}_y:=\omega_y+\mathbb{Re}\{m_y\}$ we get the two possible estimates

$$\lambda|G(x,y;z)| \leq \frac{|m_{xy}|}{|\mathbb{Re}\{(m_{xx}+\omega_x)(m_{yy}+\omega_y)-m_{xy}m_{yx}\}|} = \frac{|m_{xy}|}{|\tilde{\omega}_x\tilde{\omega}_y-\mathbb{Im}\{m_{xx}\}\mathbb{Im}\{m_y\}-\mathbb{Re}\{m_{xy}m_{yx}\}|}$$

as well as

$$\lambda|G(x,y;z)| \leq \frac{|m_{xy}|}{|\tilde{\omega}_x\,\mathbb{Im}\{m_{yy}\}+\tilde{\omega}_y\,\mathbb{Im}\{m_{xx}\}-\mathbb{Im}\{m_{xy}m_{yx}\}|}\,. \tag{2.7}$$

Moreover, since $\mathbb{Im}\{M\}>0$, we have

$$\begin{aligned}0 &< \det(\mathbb{Im}\{M\})\\ &= \mathbb{Im}\{m_{xx}\}\mathbb{Im}\{m_{yy}\}+\frac{1}{4}(m_{xy}-\overline{m_{yx}})(m_{yx}-\overline{m_{xy}})\\ &= \mathbb{Im}\{m_{xx}\}\mathbb{Im}\{m_{yy}\}+\frac{1}{4}\left(m_{xy}m_{yx}+\overline{m_{yx}m_{xy}}-|m_{xy}|^2-|m_{yx}|^2\right)\\ &= \mathbb{Im}\{m_{xx}\}\mathbb{Im}\{m_{yy}\}+\frac{1}{2}\mathbb{Re}\{m_{xy}m_{yx}\}-\frac{1}{4}\left(|m_{xy}|^2+|m_{yx}|^2\right)\end{aligned}$$

Case 1: Assume that

$$\max(\{\,|\mathbb{Im}\{m_{xx}\}|,|\mathbb{Im}\{m_{yy}\}|\,\}) \overset{\star}{<} \frac{1}{2}|m_{xy}|\,. \tag{2.8}$$

Then

$$\begin{aligned}c^2 &:= \mathbb{Im}\{m_{xx}\}\mathbb{Im}\{m_{yy}\}+\mathbb{Re}\{m_{xy}m_{yx}\}\\ &\overset{\det(\mathbb{Im}\{M\})>0}{>} \frac{1}{2}\left(|m_{xy}|^2+|m_{yx}|^2\right)-\mathbb{Im}\{m_{xx}\}\mathbb{Im}\{m_{yy}\}\\ &\overset{\star}{>} \frac{1}{4}|m_{xy}|^2\,.\end{aligned}$$

Hence

$$\lambda|G(x,y;z)| \leq \frac{2c}{|\tilde{\omega}_x\tilde{\omega}_y-c^2|}=\frac{2c^{-1}}{|c^{-2}\tilde{\omega}_x\tilde{\omega}_y-1|}\,.$$

Next, define $f(w):=\frac{1}{w}\min\left(\left\{\,1,w^2\,\right\}\right)$. Then we claim

$$|ab-1| \geq \min(\{\,|a-f(b)|,|b-f(a)|\,\}) \qquad (a,b\in\mathbb{R})\,. \tag{2.9}$$

Indeed, if $a^2\geq 1$ then

$$|b-f(a)|=\left|b-\frac{1}{a}\right|\leq|ab-1|\,.$$

Similarly if $b^2\geq 1$. If, however, both $a^2,b^2<1$ then

$$(a-f(b))^2=(a-b)^2$$

and

$$\left(b-f\left(a\right)\right)=\left(a-b\right)^{2}$$

which equals

$$\left(a-b\right)^{2}=\left(ab-1\right)^{2}-\left(1-a^{2}\right)\left(1-b^{2}\right)<\left(ab-1\right)^{2}\,.$$

We conclude that Section 2.4 holds. We use it as

$$\begin{aligned}\left|ab-1\right|^{-1} &\leq \frac{1}{\min\left(\left\{\,\left|a-f\left(b\right)\right|,\left|b-f\left(a\right)\right|\,\right\}\right)}\\ &= \max\left(\left\{\,\left|a-f\left(b\right)\right|^{-1},\left|b-f\left(a\right)\right|^{-1}\,\right\}\right)\\ &\leq \left|a-f\left(b\right)\right|^{-1}+\left|b-f\left(a\right)\right|^{-1}\,.\end{aligned}$$

We conclude that

$$\begin{aligned}\lambda\left|G\left(x,y;z\right)\right| &\leq \frac{2c^{-1}}{\left|c^{-2}\tilde{\omega}_{x}\tilde{\omega}_{y}-1\right|}\\ &\leq 2c^{-1}\left(\left|c^{-1}\tilde{\omega}_{x}-f\left(c^{-1}\tilde{\omega}_{y}\right)\right|^{-1}+\left|c^{-1}\tilde{\omega}_{y}-f\left(c^{-1}\tilde{\omega}_{x}\right)\right|^{-1}\right)\\ &= 2\left(\left|\tilde{\omega}_{x}-cf\left(c^{-1}\tilde{\omega}_{y}\right)\right|^{-1}+\left|\tilde{\omega}_{y}-cf\left(c^{-1}\tilde{\omega}_{x}\right)\right|^{-1}\right)\,.\end{aligned}$$

But we have just seen above in (2.6) that

$$\int\left|\omega_{x}-z\right|^{-s}\mathrm{d}\mu\left(\omega_{x}\right)\quad<\quad\frac{\tau}{\tau-s}C_{\mu}^{\frac{s}{\tau}}2^{s}\,.$$

Case 2: Conversely, if (2.8) we must have

$$\left|\mathbb{Im}\left\{m_{\alpha\alpha}\right\}\right|\geq\frac{1}{2}\left|m_{xy}\right|\qquad\left(\alpha=x\vee\alpha=y\right)\,.$$

Assume that $\alpha=y$. Then using Section 2.4 we get

$$\begin{aligned}\lambda\left|G\left(x,y;z\right)\right| &\leq \frac{\left|m_{xy}\right|}{\left|\tilde{\omega}_{x}\,\mathbb{Im}\left\{m_{yy}\right\}+\tilde{\omega}_{y}\,\mathbb{Im}\left\{m_{xx}\right\}-\mathbb{Im}\left\{m_{xy}m_{yx}\right\}\right|}\\ &\leq \frac{2\left|\mathbb{Im}\left\{m_{yy}\right\}\right|}{\left|\tilde{\omega}_{x}\,\mathbb{Im}\left\{m_{yy}\right\}+\tilde{\omega}_{y}\,\mathbb{Im}\left\{m_{xx}\right\}-\mathbb{Im}\left\{m_{xy}m_{yx}\right\}\right|}\\ &= \frac{2}{\left|\tilde{\omega}_{x}+\frac{1}{\mathbb{Im}\left\{m_{yy}\right\}}\left(\tilde{\omega}_{y}\,\mathbb{Im}\left\{m_{xx}\right\}-\mathbb{Im}\left\{m_{xy}m_{yx}\right\}\right)\right|}\end{aligned}$$

and again we know how to estimate the s moment of this. □

2.5 Sub-harmonicity in space

The next ingredient we will need is a basic statement about integral kernels of operators, called *sub-harmonicity*. The basic statement is essentially that if a kernel decays faster than the massive Laplacian would then it exhibits exponential decay (because the massive Laplacian does).

Lemma 2.15 (Subharmonicity implies exponential decay)**.** *Assume that an integral kernel* $B:\mathbb{Z}^d\times\mathbb{Z}^d\to[0,\infty)$ *obeys the sub-harmonicity bound*

$$B_{xy}\leq\gamma\sum_{u\sim x}B_{u,y}\qquad\left(x,y\in\mathbb{Z}^d\right)\tag{2.10}$$

for some $\gamma<\frac{1}{2d}$*;* $u\sim x$ *means* u,x *share an edge on* $\mathbb{Z}^d$*. Then*

$$B_{xy}\leq\frac{2}{m}\mathrm{e}^{-\frac{1}{2}m\|x-y\|}\qquad\left(x,y\in\mathbb{Z}^d\right)$$

with $m:=\frac{1}{\gamma}-2d$*.*

Proof. With the Laplacian $-\Delta$ defined so that

$$\sigma\left(-\Delta\right)=[0,4d]$$

we have

$$-\Delta=2d\mathbb{1}-A$$

where A is the adjacency matrix with $\sigma\left(A\right)=[-2d,2d]$ and

$$\left(A\psi\right)_x=\sum_{y\sim x}\psi_y\qquad\left(x\in\mathbb{Z}^d\right)\,.$$

With this notation,

$$\sum_{u\sim x}B_{u,y}\quad\equiv\quad\left(AB\right)_{xy}$$

and so we find that (2.10) is equivalent to

$$\begin{aligned}B_{xy}&\leq\gamma\left(AB\right)_{xy}\\&\Updownarrow\\\left[\left(\mathbb{1}-\gamma A\right)B\right]_{xy}&\leq 0\\&\Updownarrow\\\left[\left(\frac{1}{\gamma}\mathbb{1}-A\right)B\right]_{xy}&\leq 0\\&\downarrow\\\left[\left(\frac{1}{\gamma}\mathbb{1}-A\right)B\right]_{xy}&\leq\delta_{xy}\,.\end{aligned}$$

But we may re-write the operator as

$$\begin{aligned}\frac{1}{\gamma}\mathbb{1}-A&=\left(2d+\frac{1}{\gamma}-2d\right)\mathbb{1}-A\\&=-\Delta+\left(\frac{1}{\gamma}-2d\right)\mathbb{1}\,.\end{aligned}$$

The condition $\gamma<\frac{1}{2d}$ implies that the mass term $m:=\frac{1}{\gamma}-2d$ is positive, so we find by Theorem 1.18 that

$$\left(-\Delta+m\mathbb{1}\right)^{-1}_{xy}\leq\frac{2}{m}\exp\left(-\tilde{\mu}m\|x-y\|\right)\qquad\left(x,y\in\mathbb{Z}^d\right)\,.$$

Since the integral kernel obeys $(-\Delta + m\mathbb{1})^{-1}_{xy} \geq 0$, we learn that

$$B_{xy} \leq \frac{2}{m} \exp\left(-\tilde{\mu} m \|x-y\|\right) \qquad \left(x, y \in \mathbb{Z}^d\right) .$$

To see that $(-\Delta + m\mathbb{1})^{-1}_{xy} \geq 0$, note that $-\Delta \geq 0$ so $-\Delta + m\mathbb{1} > m\mathbb{1}$ and hence $(-\Delta + m\mathbb{1})^{-1} \geq \frac{1}{4d+m}\mathbb{1}$. Indeed, the integral kernel of the Laplacian is positive for all positions. To see this, we write

$$\begin{aligned} (-\Delta + m\mathbb{1})^{-1}_{xy} &= \int_{t=0}^{\infty} \left(\mathrm{e}^{-t(-\Delta+m\mathbb{1})}\right)_{xy} \mathrm{d}t \\ &= \int_{t=0}^{\infty} \mathrm{e}^{-tm} \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{k\in\mathbb{T}^d} \mathrm{e}^{\mathrm{i}k\cdot(x-y)-t\mathcal{E}(k)} \mathrm{d}k\mathrm{d}t \end{aligned}$$

where $\mathcal{E}(k) \equiv 2d - \sum_{j=1}^{d} 2\cos(k_j)$. Hence

$$\begin{aligned} (-\Delta + m\mathbb{1})^{-1}_{x0} &= \int_{t=0}^{\infty} \mathrm{e}^{tm}\mathrm{e}^{-2dt} \prod_{j=1}^{d} \left(\frac{1}{\sqrt{2\pi}} \int_{k_j=0}^{2\pi} \mathrm{e}^{\mathrm{i}k_j x_j - 2t\cos(k_j)} \mathrm{d}k_j\right) \mathrm{d}t \\ &= \int_{t=0}^{\infty} \mathrm{e}^{tm}\mathrm{e}^{-2dt} \prod_{j=1}^{d} \left(\sqrt{2\pi} I_{x_j}(2t)\right) \mathrm{d}t \end{aligned}$$

where $I_{x_j}(2t)$ is the modified Bessel function of order x_j, which is known to be positive, for instance using the representation

$$I_\alpha(2t) = \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(m+\alpha+1)} t^{2m+\alpha} .$$

□

Another related result is

Lemma 2.16. *Assume that* $f : \mathbb{Z}^d \to [0, \infty)$ *with* $\|f\|_\infty < \infty$ *obeys*

$$f(x) \leq g(x) + \sum_{y\in\mathbb{Z}^d} K(x,y) f(y) \qquad \left(x \in \mathbb{Z}^d\right) \tag{2.11}$$

for some kernel $K : \mathbb{Z}^d \times \mathbb{Z}^d \to [0, \infty)$ *which obeys*

$$\sup_{x\in\mathbb{Z}^d} \sum_{y\in\mathbb{Z}^d} K(x,y) \quad < \quad 1$$

as well as

$$r \quad := \quad \sup_{x\in\mathbb{Z}^d} \sum_{y\in\mathbb{Z}^d} \frac{W(x)}{W(y)} K(x,y) < 1$$

and

$$b \quad := \quad \sum_{x\in\mathbb{Z}^d} W(x) g(x) < \infty$$

for some $W : \mathbb{Z}^d \to [0, \infty)$.

Then

$$\sum_{x\in\mathbb{Z}^d} W(x) f(x) \quad \leq \quad \frac{b}{1-r} .$$

Proof. We apply the estimate (2.11) repeatedly many times to get

$$\begin{aligned} f &\leq g+Kf \\ &\leq g+K\left(g+Kf\right) \\ &\leq \cdots \\ &\leq \sum_{j=0}^{n-1}K^{j}g+K^{n}f \qquad \left(n\in\mathbb{N}_{\geq1}\right)\,. \end{aligned}$$

Now,

$$\begin{aligned} \left|K^{n}f\left(x\right)\right| &= \left|\sum_{x_{1},\dots,x_{n}}K\left(x,x_{1}\right)K\left(x_{1},x_{2}\right)\cdots K\left(x_{n-1},x_{n}\right)f\left(x_{n}\right)\right| \\ &\leq \sum_{x_{1},\dots,x_{n}}\left|K\left(x,x_{1}\right)\right|\left|K\left(x_{1},x_{2}\right)\right|\cdots\left|K\left(x_{n-1},x_{n}\right)\right|\left|f\left(x_{n}\right)\right| \\ &\leq \left\Vert f\right\Vert_{\infty}\left(\sup_{x}\sum_{y}\left|K\left(x,y\right)\right|\right)^{n} \\ &\to 0 \qquad \left(n\to\infty\right)\,. \end{aligned}$$

Hence

$$f\left(x\right) \leq \sum_{j=0}^{\infty}\left(K^{j}g\right)\left(x\right)\,.$$

Thus

$$\begin{aligned} \sum_{x}W\left(x\right)f\left(x\right) &\leq \sum_{x}W\left(x\right)\sum_{j=0}^{\infty}\left(K^{j}g\right)\left(x\right) \\ &= \sum_{j=0}^{\infty}\sum_{x}W\left(x\right)\sum_{x_{1},\dots,x_{j}}K\left(x,x_{1}\right)\cdots K\left(x_{j-1},x_{j}\right)g\left(x_{j}\right) \\ &= \sum_{j=0}^{\infty}\sum_{x}\sum_{x_{1},\dots,x_{j}}\frac{W\left(x\right)}{W\left(x_{1}\right)}K\left(x,x_{1}\right)\cdots\frac{W\left(x_{j-1}\right)}{W\left(x_{j}\right)}K\left(x_{j-1},x_{j}\right)W\left(x_{j}\right)g\left(x_{j}\right) \\ &\leq \sum_{j=0}^{\infty}r^{j}b \\ &= \frac{b}{1-r}\,. \end{aligned}$$

□

We learn in particular that for any $x\in\mathbb{Z}^{d}$

$$W\left(x\right)f\left(x\right) \leq \sum_{\tilde{x}}W\left(\tilde{x}\right)f\left(\tilde{x}\right)\leq\frac{b}{1-r}$$

i.e.,

$$f\left(x\right) \leq \frac{b}{1-r}\frac{1}{W\left(x\right)} \qquad \left(x\in\mathbb{Z}^{d}\right)\,.$$

2.6 The decoupling lemma

In the sequel we will use the so-called decoupling lemma which goes as follows:

Lemma 2.17 (Decoupling lemma)**.** *Let* $s\in(0,\tau)$ *and* $\alpha,\beta\in\mathbb{C}$. *Then*

$$\int_{\omega\in\mathbb{R}}\frac{|\omega-\alpha|^{s}}{|\omega-\beta|^{s}}\mathrm{d}\mu(\omega)\quad\geq\quad\frac{\tau}{\tau-s}C^{\frac{s}{\tau}}2^{s+1}\int_{\omega\in\mathbb{R}}\frac{1}{|\omega-\beta|^{s}}\mathrm{d}\mu(\omega)$$

where C *is a constant independent of* α,β *coming from the a-prior bound.*

Proof. We first claim that

$$|v-\beta|^{-s}+|u-\beta|^{-s}\leq\frac{|v|^{s}}{|v-\beta|^{s}}\left(|u|^{-s}+|u-\beta|^{-s}\right)+\frac{|u|^{s}}{|u-\beta|^{s}}\left(|v|^{-s}+|v-\beta|^{-s}\right)\tag{2.12}$$

for all $u,v,\beta\in\mathbb{C}$ unless the denominators vanish. To see this, multiply by $|v-\beta|^{s}|u-\beta|^{s}$ and re-arrange to get the equivalent claim

$$0\quad\leq\quad\left(|v|^{s}|u|^{-s}-1\right)|u-\beta|^{s}+\left(|u|^{s}|v|^{-s}-1\right)|v-\beta|^{s}+|u|^{s}+|v|^{s}\,.$$

Since this expression is symmetric in $u\leftrightarrow v$, suffice to show it for $|u-\beta|\geq|v-\beta|$. By the triangle inequality, we have

$$\begin{aligned}|u-\beta|^{s}&=\left(|v-\beta+u-v|\right)^{s}\\&\leq\left(|v-\beta|+|u|+|v|\right)^{s}\\&\leq|v-\beta|^{s}+|u|^{s}+|v|^{s}\end{aligned}$$

or $|u|^{s}+|v|^{s}\geq|u-\beta|^{s}-|v-\beta|^{s}$. Applying this we find that our equivalent claim reduces to

$$\begin{aligned}&\left(|v|^{s}|u|^{-s}-1\right)|u-\beta|^{s}+\left(|u|^{s}|v|^{-s}-1\right)|v-\beta|^{s}+|u|^{s}+|v|^{s}\\\geq\;&\left(|v|^{s}|u|^{-s}-1\right)|u-\beta|^{s}+\left(|u|^{s}|v|^{-s}-1\right)|v-\beta|^{s}+|u-\beta|^{s}-|v-\beta|^{s}\\=\;&|v|^{s}|u|^{-s}|u-\beta|^{s}+\left(|u|^{s}|v|^{-s}-2\right)|v-\beta|^{s}\\\geq\;&|v|^{s}|u|^{-s}|v-\beta|^{s}+\left(|u|^{s}|v|^{-s}-2\right)|v-\beta|^{s}\\=\;&\left(|v|^{s}|u|^{-s}+|u|^{s}|v|^{-s}-2\right)|v-\beta|^{s}\\\geq\;&0\end{aligned}$$

where in the last line we have used $t+\frac{1}{t}\geq2$ for all $t>0$. Hence (2.12) is proven. We use it by replacing v with $v-\alpha$, u with $u-\alpha$ and finally replace β with $\beta-\alpha$ to get

$$|v-\beta|^{-s}+|u-\beta|^{-s}\quad\leq\quad\frac{|v-\alpha|^{s}}{|v-\beta|^{s}}\left(|u-\alpha|^{-s}+|u-\beta|^{-s}\right)+\frac{|u-\alpha|^{s}}{|u-\beta|^{s}}\left(|v-\alpha|^{-s}+|v-\beta|^{-s}\right).$$

We then integrate on both u,v with respect to μ (using $\mu(\mathbb{R})=1$) (after renaming some variables and dividing by 2)

$$\int_{v}|v-\beta|^{-s}\,\mathrm{d}\mu(v)\quad\leq\quad\left(\int_{v}\frac{|v-\alpha|^{s}}{|v-\beta|^{s}}\mathrm{d}\mu(v)\right)\int_{u}\left(|u-\alpha|^{-s}+|u-\beta|^{-s}\right)\mathrm{d}\mu(u)\,.$$

On the latter integral on the RHS we use the same proof as in Section 2.4, in particular (2.6), to get

$$\int_{v}|v-\beta|^{-s}\,\mathrm{d}\mu(v)\leq\frac{\tau}{\tau-s}C^{\frac{s}{\tau}}2^{s+1}\int_{v}\frac{|v-\alpha|^{s}}{|v-\beta|^{s}}\mathrm{d}\mu(v)\,.$$

□

There is a converse type of decoupling lemma that we shall also need

Lemma 2.18. *For the regular probability measure μ, assume further that it has some fractional moment in the sense that for some $s \in (0,1)$,*

$$B_s := \int_{v\in\mathbb{R}} |v|^s \, \mathrm{d}\mu(v) < \infty .$$

Then there exits some $s \in (0,1)$ and constant $D_s < \infty$ (independent of α below) such that

$$\int_{v\in\mathbb{R}} \frac{|v|^s}{|\lambda v-\alpha|^s} \mathrm{d}\mu(v) \leq D_s \int_{v\in\mathbb{R}} \frac{1}{|\lambda v-\alpha|^s} \mathrm{d}\mu(v) \qquad (\alpha\in\mathbb{C}, \lambda>0) . \tag{2.13}$$

Proof. Using the Cauchy-Schwarz inequality, we have

$$\int_{v\in\mathbb{R}} \frac{|v|^s}{|\lambda v-\alpha|^s} \mathrm{d}\mu(v) \leq \sqrt{B_{2s}} \sqrt{\int_{v\in\mathbb{R}} |\lambda v-\alpha|^{-2s} \, \mathrm{d}\mu(v)} .$$

Now by assumption the first factor is bounded whereas the second one was shown to be bounded above in Theorem 2.13. We get an upper bound of the form

$$\int_{v\in\mathbb{R}} \frac{|v|^s}{|\lambda v-\alpha|^s} \mathrm{d}\mu(v) \quad \leq \quad \sqrt{B_{2s}} \sqrt{\frac{\tau}{\tau-2s} C_\mu^{\frac{2s}{\tau}} \left(\frac{2}{\lambda}\right)^{2s}} .$$

Conversely, we seek a lower bound on

$$\begin{aligned}
\int_{v\in\mathbb{R}} |\lambda v-\alpha|^{-s} \, \mathrm{d}\mu(v) \quad &\geq \quad \int_{v:|\lambda v|\leq Q} |\lambda v-\alpha|^{-s} \, \mathrm{d}\mu(v) \\
&\geq \quad \int_{v:|\lambda v|\leq Q} \frac{1}{(|\lambda v|+|\alpha|)^s} \mathrm{d}\mu(v) \\
&\geq \quad \frac{1}{(Q+|\alpha|)^s} \int_{v:|\lambda v|\leq Q} \mathrm{d}\mu(v) \\
&= \quad \frac{1}{(Q+|\alpha|)^s} \mu\left(\left\{ v\in\mathbb{R} \middle| |v| \leq \frac{Q}{\lambda} \right| \right\}\right) \\
&= \quad \frac{1}{(Q+|\alpha|)^s} \left(1-\mu\left(\left\{ v\in\mathbb{R} \middle| |v| \geq \frac{Q}{\lambda} \right| \right\}\right)\right) .
\end{aligned}$$

Now, by Markov's inequality,

$$\mu\left(\left\{ v\in\mathbb{R} \middle| |v| \geq \frac{Q}{\lambda} \right| \right\}\right) \quad \leq \quad \frac{B_{2s}}{\left(\frac{Q}{\lambda}\right)^{2s}} .$$

Hence if we pick Q such that

$$\begin{aligned}
\frac{B_{2s}}{\left(\frac{Q}{\lambda}\right)^{2s}} \quad &:= \quad \frac{1}{2} \\
Q \quad &= \quad \left(2\lambda^{-2s} B_{2s}\right)^{\frac{1}{2s}} .
\end{aligned}$$

We find

$$\int_{v\in\mathbb{R}} |\lambda v-\alpha|^{-s} \, \mathrm{d}\mu(v) \quad \geq \quad \frac{1}{2} \frac{1}{\left(\left(2\lambda^{-2s}B_{2s}\right)^{\frac{1}{2s}}+|\alpha|\right)^s} .$$

If $|\alpha| \leq \left(2\lambda^{-2s}B_{2s}\right)^{\frac{1}{2s}}$ we get

$$\left(2\lambda^{-2s}B_{2s}\right)^{\frac{1}{2s}} + |\alpha| \quad \leq \quad 2\left(2\lambda^{-2s}B_{2s}\right)^{\frac{1}{2s}}$$

and hence

$$\int_{v\in\mathbb{R}}|\lambda v-\alpha|^{-s}\,\mathrm{d}\mu(v)\quad\geq\quad\frac{1}{2}\left(2\left(2\lambda^{-2s}B_{2s}\right)^{\frac{1}{2s}}\right)^{-s}$$

and we need to define D_s so that

$$D_s\frac{1}{2}\left(2\left(2\lambda^{-2s}B_{2s}\right)^{\frac{1}{2s}}\right)^{-s}\quad\overset{!}{\geq}\quad\sqrt{B_{2s}}\sqrt{\frac{\tau}{\tau-2s}C_\mu^{\frac{2s}{\tau}}\left(\frac{2}{\lambda}\right)^{2s}}\,.$$

If, on the other hand, $|\alpha|\geq\left(2\lambda^{-2s}B_{2s}\right)^{\frac{1}{2s}}$, we can estimate

$$\begin{aligned}\int_{v\in\mathbb{R}}\frac{|v|^s}{|\lambda v-\alpha|^s}\mathrm{d}\mu(v)\quad&\leq\quad\left(\frac{2}{|\alpha|}\right)^s\int_{|v|\leq\frac{|\alpha|}{2}}|v|^s\,\mathrm{d}\mu(v)+\left(\frac{2}{|\alpha|}\right)^s\int_{|v|\geq\frac{|\alpha|}{2}}\frac{|v|^{2s}}{|\lambda v-\alpha|^s}\mathrm{d}\mu(v)\\&\leq\quad\left(\frac{2}{|\alpha|}\right)^sB_s+\left(\frac{2}{|\alpha|}\right)^s(B_{2s}+\cdots)\\&=:\quad\left(\frac{2}{|\alpha|}\right)^sM_s\,.\end{aligned}$$

Then we ask that

$$D_s\frac{1}{2}\frac{1}{\left(\left(2\lambda^{-2s}B_{2s}\right)^{\frac{1}{2s}}+|\alpha|\right)^s}\quad\overset{!}{\geq}\quad\left(\frac{2}{|\alpha|}\right)^sM_s$$

which can clearly be fulfilled for large $|\alpha|$. □

Example 2.19 (Gaussian distribution)**.** Consider the case where

$$\frac{\mathrm{d}\mu(v)}{\mathrm{d}v}\quad=\quad\frac{1}{\sqrt{\pi}}\exp\left(-v^2\right)\,.$$

2.7 Complete localization at sufficiently strong disorder

We are now ready to prove complete localization (i.e., at *all* energies) for sufficiently strong disorder using all the ingredients at our disposal.

Theorem 2.20 (Aizenman-Molchanov 1993)**.** *Let $H=-\Delta+\lambda V_\omega(X)$ be the Anderson model on $\ell^2\left(\mathbb{Z}^d\right)$ with $\lambda>0$ and $\{\,\omega_x\,\}_{x\in\mathbb{Z}^d}$ an IID sequence with common single site measure μ which obeys Definition 2.8. If*

$$\lambda>\left(\frac{2d}{\frac{\tau}{\tau-s}C^{\frac{s}{\tau}}2^{s+1}}\right)^{\frac{1}{s}}$$

then for any $E\in\mathbb{R}$ there exist $C,\mu\in(0,\infty)$ and $s\in(0,1)$ such that

$$\sup_{\eta>0}\mathbb{E}\left[|G(x,y;E+\mathrm{i}\eta)|^s\right]\quad\leq\quad C\mathrm{e}^{-\mu\|x-y\|}\qquad\left(x,y\in\mathbb{Z}^d\right)\,.$$

In particular, we have exponential decay of the fractional moments of the Greens function for all *energies.*

Proof. We begin by writing

$$(-\Delta+\lambda V-z\mathbb{1})\,R(z)\quad\equiv\quad\mathbb{1}$$

for $R(z) \equiv (-\Delta + \lambda V - z\mathbb{1})^{-1}$. Separating this equation to its diagonal and non-diagonal parts we find

$$(2d - z + \lambda\omega_x)\, G(x, y; z) = \delta_{xy} + \sum_{x' \sim x} G(x', y; z)$$

Raise this to some power $s \in (0,1)$ after taking absolute values to find

$$\begin{aligned} |2d - z + \lambda\omega_x|^s\, |G(x, y; z)|^s &= \left|\delta_{xy} + \sum_{x' \sim x} G(x', y; z)\right|^s \\ &\le \left(\delta_{xy} + \sum_{x' \sim x} |G(x', y; z)|\right)^s . \qquad (\text{x}\mapsto x^s \text{ monotone increasing}) \end{aligned}$$

Next, note that if $a, b \ge 0$ then $(a+b)^s \le a^s + b^s$. Indeed, we have

$$\begin{array}{ccc} a+b & \ge & a \\ & \updownarrow & \\ (a+b)^{s-1} & \le & a^{s-1} \end{array}$$

and so

$$a^s = a a^{s-1} \ge a\,(a+b)^{s-1} .$$

Similarly $b^s \ge b\,(a+b)^{s-1}$ and so adding those two inequalities we find

$$a^s + b^s \ge (a+b)^s .$$

Hence

$$|2d - z + \lambda\omega_x|^s\, |G(x, y; z)|^s \le \delta_{xy} + \sum_{x' \sim x} |G(x', y; z)|^s .$$

Now take $\mathbb{E}[\cdot]$ of both sides of the equation to get

$$\mathbb{E}\left[|2d - z + \lambda\omega_x|^s\, |G(x, y; z)|^s\right] \le \delta_{xy} + \sum_{x' \sim x} \mathbb{E}\left[|G(x', y; z)|^s\right] .$$

Now we want to integrate the LHS only over ω_x to get a lower bound. For that we need the explicit dependence of $G(x, y; z)$ on ω_x. Similarly to how we handled $G(x, x; z)$ in Section 2.4, we find

$$(H - z\mathbb{1})^{-1} \quad = \quad \begin{bmatrix} \lambda\omega_x & P_x(-\Delta) \\ (-\Delta)P_x & \tilde{H} \end{bmatrix}^{-1}$$

and hence

$$\begin{aligned} G(x, y; z) &= \left[-\left(\lambda\omega_x - z - \sum_{y' \sim x}\sum_{z' \sim x}\left(P_x^\perp H P_x^\perp - z\mathbb{1}_{P_x^\perp}\right)^{-1}_{y'z'}\right)^{-1} P_x(-\Delta)P_x^\perp\left(P_x^\perp H P_x^\perp - z\mathbb{1}_{P_x^\perp}\right)^{-1} P_x^\perp\right]_{xy} \\ &= -\left(\lambda\omega_x - z - \sum_{y' \sim x}\sum_{z' \sim x}\left(P_x^\perp H P_x^\perp - z\mathbb{1}_{P_x^\perp}\right)^{-1}_{y'z'}\right)^{-1} \left[(-\Delta)P_x^\perp\left(P_x^\perp H P_x^\perp - z\mathbb{1}_{P_x^\perp}\right)^{-1}\right]_{xy} \\ &= \left(\lambda\omega_x - z - \sum_{y' \sim x}\sum_{z' \sim x}\left(P_x^\perp H P_x^\perp - z\mathbb{1}_{P_x^\perp}\right)^{-1}_{y'z'}\right)^{-1} \sum_{x' \sim x}\left(P_x^\perp H P_x^\perp - z\mathbb{1}_{P_x^\perp}\right)^{-1}_{x'y} . \end{aligned}$$

Again the particular form of this expression is unimportant, since we only care where ω_x dependence appears. In that sense, we shall write

$$G(x, y; z) = \frac{1}{\lambda\omega_x - \alpha}\beta\,. \tag{2.14}$$

Using now Lemma 2.17 we find then that, with $M:=\frac{\tau}{\tau-s}C^{\frac{s}{\tau}}2^{s+1}$

$$\begin{aligned}\mathbb{E}\left[\left|2d-z+\lambda\omega_x\right|^s\left|G\left(x,y;z\right)\right|^s\right] &= \mathbb{E}\left[\int_{\omega_x\in\mathbb{R}}\mathrm{d}\mu\left(\omega_x\right)\left|2d-z+\lambda\omega_x\right|^s\left|\lambda\omega_x-\alpha\right|^{-s}\beta^s|\omega_x\right]\\ &\geq \lambda^s M\mathbb{E}\left[\int_{\omega_x\in\mathbb{R}}\mathrm{d}\mu\left(\omega_x\right)\left|\lambda\omega_x-\alpha\right|^{-s}\beta^s|\omega_x\right]\\ &= \lambda^s M\mathbb{E}\left[\left|G\left(x,y;z\right)\right|^s\right]\,.\end{aligned}$$

Collecting everything together we have

$$\lambda^s M\mathbb{E}\left[\left|G\left(x,y;z\right)\right|^s\right] \leq \delta_{xy}+\sum_{x'\sim x}\mathbb{E}\left[\left|G\left(x',y;z\right)\right|^s\right]\,.$$

Let us define $g\left(x,y\right):=\mathbb{E}\left[\left|G\left(x,y;z\right)\right|^s\right]\geq 0$. Then we have

$$\begin{aligned}\left(\lambda^s Mg-Ag\right)_{xy} &\leq \delta_{xy}\\ \left(-\Delta g+\left(\lambda^s M-2d\right)g\right)_{xy} &\leq \delta_{xy}\,.\end{aligned}$$

Now from Lemma 2.15 we learn that if $\lambda^s M>2d$ we have exponential decay, as desired. □

2.8 Localization at weak disorder and extreme energies

Another regime in which localization may be established is at arbitrarily small λ but at extreme energies. We first start with a technical lemma

Using Lemma 2.18 and the Combes-Thomas estimate Theorem 1.18, we find:

Theorem 2.21. *If $\lambda>0$ there exists some $E_c\left(\lambda\right)\in\mathbb{R}$ such that if $E\geq E_c\left(\lambda\right)$ then there exists some $s\in\left(0,1\right)$ and $C,\mu\in\left(0,\infty\right)$ such that*

$$\sup_{\varepsilon>0}\mathbb{E}\left[\left|G\left(x,y;E+\mathrm{i}\varepsilon\right)\right|^s\right] \leq C\mathrm{e}^{-\mu\|x-y\|} \qquad \left(x,y\in\mathbb{Z}^d\right)\,.$$

Proof. We start by writing the resolvent identity between the operators

$$H_\omega = -\Delta+\lambda V_\omega\left(X\right)$$

and

$$H_0 := -\Delta\,.$$

It yields, at some $z\in\mathbb{C}\setminus\mathbb{R}$,

$$\begin{aligned}R_\omega\left(z\right) &= R_0\left(z\right)+R_0\left(z\right)\left(H_0-H_\omega\right)R_\omega\left(z\right)\\ &= R_0\left(z\right)-R_0\left(z\right)\lambda V_\omega\left(X\right)R_\omega\left(z\right)\,.\end{aligned}$$

Taking the x,y matrix elements, expectation w.r.t some $s\in\left(0,1\right)$ moment and using the triangle inequality, we find

$$\mathbb{E}\left[\left|G_\omega\left(x,y;z\right)\right|^s\right] \leq \left|G_0\left(x,y;z\right)\right|^s+\lambda^s\sum_{\tilde{x}\in\mathbb{Z}^d}\left|G_0\left(x,\tilde{x};z\right)\right|^s\mathbb{E}\left[\left|\omega_{\tilde{x}}\right|^s\left|G_\omega\left(\tilde{x},y;z\right)\right|^s\right]\,.$$

We already know the dependence of $G_\omega\left(\tilde{x},y;z\right)$ on $\omega_{\tilde{x}}$ from our study of finite rank perturbation theory above in (2.14):

$$G_\omega\left(\tilde{x},y;z\right) = \frac{1}{\lambda\omega_x-\alpha}\beta$$

for some $\alpha, \beta \in \mathbb{C}$ which are *independent* of ω_x. Using the regularity condition (2.13) we then find

$$\mathbb{E}\left[\left|G_\omega(x,y;z)\right|^s\right] \leq \left|G_0(x,y;z)\right|^s + \lambda^s D_s \sum_{\tilde{x}\in\mathbb{Z}^d} \left|G_0(x,\tilde{x};z)\right|^s \mathbb{E}\left[\left|G_\omega(\tilde{x},y;z)\right|^s\right] .$$

Using now Theorem 1.18, assuming $\mathbb{Re}\{z\} \notin \sigma(-\Delta)$, we find that there exists some $\delta(z) > 0$ such that

$$\left|G_0(x,y;z)\right| \leq \frac{2}{\delta(z)} \exp\left(-\tilde{\mu}\delta(z)\|x-y\|\right) \qquad \left(x,y\in\mathbb{Z}^d\right) .$$

Hence

$$\mathbb{E}\left[\left|G_\omega(x,y;z)\right|^s\right] \leq \frac{2^s}{\delta(z)^s} \exp\left(-s\tilde{\mu}\delta(z)\|x-y\|\right) + \frac{2^s}{\delta(z)^s} D_s \lambda^s \sum_{\tilde{x}\in\mathbb{Z}^d} \exp\left(-s\tilde{\mu}\delta(z)\|x-\tilde{x}\|\right) \mathbb{E}\left[\left|G_\omega(\tilde{x},y;z)\right|^s\right] .$$

Such a condition implies in itself exponential decay of $\mathbb{E}\left[\left|G_\omega(x,y;z)\right|^s\right]$ if λ is sufficiently small. Indeed, let us re-write it as

$$f(x) := \mathbb{E}\left[\left|G_\omega(x,y;z)\right|^s\right]$$

to get

$$f(x) \leq Q\mathrm{e}^{-\nu\|x-y\|} + Q\lambda^s \sum_{\tilde{x}} \mathrm{e}^{-\nu\|\tilde{x}-x\|} f(\tilde{x})$$

for some $Q < \infty$ and $\nu > 0$. Let us define

$$\begin{aligned} W(x) &:= \exp\left(+\xi\|x-y\|\right) \\ K(x,y) &:= Q\lambda^s \mathrm{e}^{-\nu\|x-y\|} \\ g(x,y) &:= Q\mathrm{e}^{-\nu\|x-y\|} \end{aligned}$$

with which we verify:

$$\begin{aligned} \sup_{x\in\mathbb{Z}^d} \sum_{y\in\mathbb{Z}^d} K(x,y) &= \sup_{x\in\mathbb{Z}^d} \sum_{y\in\mathbb{Z}^d} Q\lambda^s \mathrm{e}^{-\nu\|x-y\|} \\ &= Q\lambda^s \left(\sum_{x\in\mathbb{Z}^d} \mathrm{e}^{-\nu\|x\|}\right) \\ &\overset{!}{<} 1 \end{aligned}$$

which implies

$$\lambda < \left[Q\left(\sum_{x\in\mathbb{Z}^d} \mathrm{e}^{-\nu\|x\|}\right)\right]^{\frac{1}{s}} .$$

Moreover,

$$r \equiv \sup_{x\in\mathbb{Z}^d} \sum_{y\in\mathbb{Z}^d} \exp\left(+\xi\|x-y\|\right) Q\lambda^s \mathrm{e}^{-\nu\|x-y\|}$$

is indeed smaller than 1 iff $\xi < \nu$ and λ is even smaller. Finally,

$$b := \sum_{x\in\mathbb{Z}^d} \exp\left(+\xi\|x-y\|\right) Q\mathrm{e}^{-\nu\|x-y\|} < \infty .$$

Hence Lemma 2.16 applies and we find

$$f(x) \quad \leq \quad \frac{b}{1-r}\exp\left(-\xi\|x-y\|\right) .$$

□

2.9 What about localization at the edges of the Laplacian's spectrum?

The argument we presented just above manages to establish localization when

$$E \quad \notin \quad \sigma(-\Delta)$$

essentially using the Combes-Thomas estimate for $-\Delta$. It turns out that localization also holds for $E \in \sigma(-\Delta)$ but at its fringes, but through a somewhat more intricate mechanism: very low density of states due to the so-called *Lifschitz tails*. To better study this phenomenon, we need to study *finite volume restrictions* of H onto boxes

$$\Lambda_L := [-L, L]^d \cap \mathbb{Z}^d .$$

The boundary conditions don't matter a lot, it turns out, so pick Dirichlet for simplicity. Denote by H_L the operator H restricted to $\ell^2(\Lambda_L)$ with Dirichlet boundary conditions. Clearly H_L is just a finite matrix which has $(2L+1)^d$ eigenvalues.

We do not show the full argument but just explain the mechanism which enables this other form of localization. The full argument will be found in [AW15], Corollary 11.6:

1. First of all, through a finite rank perturbation theory argument,

$$|G(x,y;z)| \quad \leq \quad C\,|G_L(x,y;z)|$$

and

$$|G_L(x,y;z)| \leq C\,|G_L(0,Le_j;z)|$$

for some constant C independent of disorder and independent of L.

2. Sufficiently fast polynomial decay of $|G_L(0,Le_j;z)|$ implies that actually it has exponential decay.

3. H_L has *very low density of eigenvalues* at the fringes of $\sigma(-\Delta)$. This is called Lifschitz tails.

2.9.1 Low density of states implies polynomial decay of Greens function

Fix some energy $E \in \mathbb{R}$, size of box $L \in \mathbb{N}$ and $\delta := CL^{-\beta}$ for some $C < \infty$ and $\beta \in (0,1)$. Using the deterministic Combes-Thomas estimate, we know that on the set of realizations ω such that

$$\Omega(E,\delta) := \{\, \omega \in \Omega \mid \operatorname{dist}(\sigma(H_L), E) > \delta \,\}$$

we have exponential decay of the Greens function:

$$|G_{L,\omega}(0,x;E)| \leq \frac{2}{\delta}\exp(-c\delta\|x\|) \qquad (x \in \Lambda_L, \omega \in \Omega(E,\delta)) .$$

Hence we get

$$\begin{aligned}\mathbb{E}\left[|G_{L,\omega}(0,x;E)|^s\right] \quad &\leq \quad \mathbb{E}\left[|G_{L,\omega}(0,x;E)|^s \chi_{\Omega(E,\delta)}\right] + \mathbb{E}\left[|G_{L,\omega}(0,x;E)|^s \chi_{\Omega(E,\delta)^c}\right] \\ &\leq \quad \frac{2^s}{\delta^s}\exp(-cs\delta\|x\|) + \mathbb{E}\left[|G_{L,\omega}(0,x;E)|^{sp}\right]^{\frac{1}{p}} \mathbb{P}\left[\Omega(E,\delta)^c\right]^{1-\frac{1}{p}}\end{aligned}$$

for some $p > 1$ to make Hoelder work. To deal with $\mathbb{E}\left[|G_{L,\omega}(0,x;E)|^{sp}\right]^{\frac{1}{p}}$ we use the a-priori bound Theorem 2.12 (which necessitates $p < \frac{\tau}{s}$). Hence if we managed to show that

$$\mathbb{P}\left[\left\{\, \omega \in \Omega \;\middle|\; \operatorname{dist}(\sigma(H_L), E) \leq CL^{-\beta} \,\right\}\right] \leq \tilde{C}L^{-\alpha} \tag{2.15}$$

then we would have the estimate

$$\mathbb{E}\left[|G_{L,\omega}(0,Le_j;E)|^s\right] \leq \frac{2^s}{C^s}L^{\beta s}\exp\left(-csCL^{1-\beta}\right) + C\left(\tilde{C}L^{-\alpha}\right)^{1-\frac{1}{p}}$$

which yields *polynomial* decay in L.

2.9.2 Fast enough polynomial decay implies exponential decay

[TODO]: This proceeds by a finite rank perturbation argument.

2.9.3 The Lifschitz tails argument

[TODO]: Establish (2.15) for energies in the fringes of $\sigma(-\Delta)$.

2.10 Complete localization in one dimension for arbitrary strength of disorder

In this chapter we show yet another mechanism of localization: complete localization in one-dimension. We largely follow [CPSS22] in its mechanism but restrict our attention to the one-dimensional Anderson model instead of the random band matrices treated in that reference. Another standard technique (which we do not present here) is the transfer matrix approach. It is harder and gives *less* quantitative information, but it came beforehand, see [KLS90]. It is also more robust.

Consider then

$$H \quad := \quad -\Delta + \lambda V_\omega (X)$$

on $\ell^2(\mathbb{Z})$. To facilitate the discussion, it will useful to restrict this to $[1,L]\cap\mathbb{Z}$ and thus discuss the $L\times L$ matrix

$$H_L := \begin{bmatrix} 2+\lambda\omega_1 & -1 & & & \\ -1 & 2+\lambda\omega_2 & -1 & & \\ & -1 & \cdots & & \\ & & & 2+\lambda\omega_{L-1} & -1 \\ & & & -1 & 2+\lambda\omega_L \end{bmatrix}.$$

Furthermore assume also that ω_j has a standard Gaussian distribution (for simplicity):

$$\exp\left(-\pi\omega_x^2\right)\mathrm{d}\omega_x\,.$$

We have seen above that it is only necessary to study the corner element

$$G_L(1,L;E) = \left[(H_L - E\mathbb{1})^{-1}\right]_{1,L}$$

and establish that its fractional moment has exponential decay with L. Note that since we have a finite matrix we do not need to go off the real axis to get an invertible operator: almost surely any given E does not hit an eigenvalue of H_L so we may contend ourselves to real E. Hence our goal is to establish

$$\mathbb{E}\left[|G_L(1,L;E)|^s\right] \quad \leq \quad \mathrm{e}^{-cL}$$

for some $c>0$.

2.10.1 Lower bound on fluctuations implies exponential decay

Let us begin with a remark about Hoelder's inequality: For $0<r<s<1$ and $Y\geq 0$ we have

$$\mathbb{E}\left[Y^r\right] \quad \leq \quad \left(\mathbb{E}\left[Y^s\right]\right)^{\frac{r}{s}}\,.$$

We shall require a strengthening of this into

Lemma 2.22. *Let $0<r<s<1$ and $Y\geq 0$. Then*

$$\mathbb{E}\left[Y^r\right] \quad = \quad \left(\mathbb{E}\left[Y^s\right]\right)^{\frac{r}{s}} \exp\left(-\int_0^s f_{r,s}(q)\,\mathbb{Var}_q\left[\log(Y)\right]\mathrm{d}q\right)$$

where

$$f_{r,s}(q) := \frac{1}{s}\min(\{\,r,q\,\})\,(s-\max(\{\,r,q\,\})) \qquad (q\in(0,s))$$

and

$$\mathbb{Var}_q[X] \quad := \quad \mathbb{E}_q\left[(X-\mathbb{E}_q[X])^2\right], \qquad \mathbb{E}_q[Z] \equiv \frac{\mathbb{E}\left[Z\mathrm{e}^{qX}\right]}{\mathbb{E}\left[\mathrm{e}^{qX}\right]}\,.$$

Proof. [TODO] □

If we apply this lemma on $Y := |G_L(1, L; E)|$ we find, using Theorem 2.12, that thanks to $f_{rs}(q) > 0$, it suffices to prove

$$\mathbb{Var}_q [\log(|G_L(1, L; E)|)] \gtrsim C_q L$$

for some constant $C_q > 0$ where $q \in (0, s)$.

To get a lower bound on fluctuations, we shall use the basic

Lemma 2.23. *Let X be a real-valued RV distributed according to a probability measure $\mathbb{P}$ and such that there are some*

$$0 < \alpha < a$$

and $\varepsilon \in (0, 1)$, $\beta \in (0, \infty)$ with which

$$\mathbb{P}[\{ |X| \leq \alpha \}] \leq \beta \sqrt{\mathbb{P}[\{ X \geq a \}] \mathbb{P}[\{ X \leq -a \}]} + \varepsilon \, .$$

Then

$$\mathbb{E}\left[X^2\right] \geq \frac{1 - \varepsilon}{1 + \frac{1}{2}\beta} \alpha^2 \, .$$

Proof. TODO □

2.10.2 Factorizing the Greens function

Symplectic transfer matrices For the sake of completeness, let us first study the usual factorization, which is using the transfer matrix. We have the eigenequation

$$\begin{aligned} H\psi &= E\psi \\ -\psi_{x+1} - \psi_{x-1} + (2d + \lambda\omega_x)\psi_x &= E\psi_x \\ \psi_{x+1} &= -(E - 2d - \lambda\omega_x)\psi_x - \psi_{x-1} \, . \end{aligned}$$

If we define

$$\Psi_x := \begin{bmatrix} \psi_{x+1} \\ \psi_x \end{bmatrix}$$

then we find the Schroedinger equation is equivalent to

$$\Psi_x = \begin{bmatrix} -(E - 2d - \lambda\omega_x) & -1 \\ 1 & 0 \end{bmatrix} \Psi_{x-1} \qquad (x \in \mathbb{Z}) \, .$$

We call the matrix

$$A_x(E) := \begin{bmatrix} -(E - 2d - \lambda\omega_x) & -1 \\ 1 & 0 \end{bmatrix}$$

a *transfer matrix*. We note a few properties of it:

1. It has real entries for the usual Anderson model at real energies.
2. It obeys the symplectic condition $A_x(E)^T \Omega A_x(E) = \Omega$ for $\Omega = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ which ultimately is a consequence of probability preservation in quantum mechanics. The symplectic condition implies that its eigenvalues are symmetric about the unit circle.

From the Schroedinger equation it is apparent that the product of many transfer matrices controls the eigenfunctions as

$$\Psi_x = A_x(E) \cdots A_2(E) \Psi_1 \, .$$

To understand better the decay and growth properties of these matrices, we study the *Lyapunov exponents*

$$\gamma(E) := \lim_{x \to \infty} \frac{1}{x} \log(\|A_x(E) \cdots A_2(E)\|)$$

where we note that due the symplectic condition, it is only the top singular value that is necessary to extract since the bottom one is symmetric about the unit circle. By (abstract arguments we do not want to get into, see manuscript by Lacroix) we also have almost-surely

$$\gamma(E) = \lim_{x\to\infty}\frac{1}{x}\mathbb{E}\left[\log\left(\|A_x(E)\cdots A_2(E)\|\right)\right] .$$

It turns out that the Greens function can similarly be enlarged as

$$\mathcal{G}(x,y;E) = \begin{bmatrix} G(x+1,y;E) \\ G(x,y;E)\end{bmatrix}$$

in order to yield

$$\mathcal{G}(x,y;E) = A_x(E)\cdots A_{y-2}(E)\,\mathcal{G}(y-1,y;E) \qquad (x\le y-1) .$$

Thanks to this identity and the a-priori bound Theorem 2.12, after some manipulations it is sufficient to prove that

$$\gamma(E) > 0 . \tag{2.16}$$

Indeed, [TODO: explain how].

Establishing (2.16) is covered by Furstenberg's theory [GMP77] which shows that the Lyapunov spectrum of sufficiently rich sequences of random matrices is *simple* and hence avoids zero by the symplectic condition. We shall *not* take that route here.

The self-adjoint factorization Instead of the symplectic factorization of the Greens function, we instead factorize the Greens function using Gaussian elimination.

Claim 2.24. We have

$$G_L(1,L;E) = \Gamma_1^{-1}\cdots\Gamma_L^{-1}$$

where

$$\begin{aligned}\Gamma_1 &:= \lambda\omega_1 - E\\ \Gamma_j &:= \lambda\omega_j - E - \Gamma_{j-1}^{-1} \qquad (j=2,\dots,L) .\end{aligned}$$

Proof. Consider H_{L-1} as an $L\times L$ matrix with 0 added:

$$H_{L-1}\oplus 0 .$$

Then the resolvent identity yields

$$(H_L - z\mathbb{1})^{-1} = (H_{L-1}\oplus 0 - z\mathbb{1})^{-1} + (H_{L-1}\oplus 0 - z\mathbb{1})^{-1}(H_{L-1}\oplus 0 - H_L)(H_L - z\mathbb{1})^{-1} .$$

But now, take the $1,L$ matrix elements. Since $H_{L-1}\oplus 0$ does not couple the site L with the ret of the matrix, that matrix element will be zero. Hence

$$G_L(1,L;z) = \sum_{j,k=1}^{L}\left[(H_{L-1}\oplus 0 - z\mathbb{1})^{-1}\right]_{1,j}(H_{L-1}\oplus 0 - H_L)_{j,k}\,G_L(k,L;z) .$$

Moreover,

$$H_{L-1}\oplus 0 - H_L = \begin{bmatrix} 0 & 0 & & & \\ 0 & 0 & 0 & & \\ & 0 & \cdots & & \\ & & & 0 & 1 \\ & & & 1 & -(2+\lambda\omega_L)\end{bmatrix}$$

and $\left[(H_{L-1}\oplus 0 - z\mathbb{1})^{-1}\right]_{1,j} = 0$ if $j=L$, whereas by the above, $(H_{L-1}\oplus 0 - H_L)_{j,k} = 0$ if $j\le L-1$ but then

$k=L$, so we get

$$\begin{aligned} G_L(1,L;z) &= \left[(H_{L-1}\oplus 0 - z\mathbb{1})^{-1}\right]_{1,L-1} G_L(L,L;z) \\ &= G_{L-1}(1,L-1;z)\, G_L(L,L;z)\ . \end{aligned}$$

Iterating this identity $L-1$ more times we find

$$G_L(1,L;z) \quad = \quad G_1(1,1;z)\cdots G_{L-1}(L-1,L-1;z)\, G_L(L,L;z)\ .$$

Hence, let us define

$$\Gamma_j \quad := \quad G_j(j,j;z)^{-1}\ .$$

Let us now use the Schur complement formula Lemma 2.11 on H_j (decomposing $\mathbb{C}^j = \mathbb{C}^{j-1}\oplus\mathbb{C}$) to find

$$G_j(j,j;z) \quad = \quad \left(\lambda\omega_j - z - G_{j-1}(j-1,j-1;z)^{-1}\right)^{-1}\ .$$

□

2.10.3 The change of variable argument

Since the numbers Γ_j are also real, we make a change of variable

$$\omega_j \mapsto \Gamma_j\ .$$

Since the dependence of the Γ_j is only on the past, the determinant of the Jacobian is identity and we find now the distribution of the random variables

$$\exp\left(-E(\Gamma)\right)\mathrm{d}\Gamma_1\cdots\mathrm{d}\Gamma_L$$

where

$$E(\Gamma) \quad := \quad \pi\lambda^{-2}\left(\Gamma_1+E\right)^2 + \pi\lambda^{-2}\sum_{j=2}^{L}\left(\Gamma_j + E + \Gamma_{j-1}^{-1}\right)^2$$

and

$$X \quad := \quad \log\left(\left|\Gamma_1^{-1}\cdots\Gamma_L^{-1}\right|\right)\ .$$

Let us define a collective change of variables on $\{\,\Gamma_j\,\}_j$ as follows

$$\Gamma_j^{\pm} := \exp\left(\pm\delta F_j\right)\Gamma_j \qquad (j=1,\cdots,L)$$

where δ, F_j are to be determined. Then

$$\begin{aligned} X^{\pm} &= \log\left(\left|\Gamma_1^{\pm -1}\cdots\Gamma_L^{\pm -1}\right|\right) \\ &= X \pm \delta F \end{aligned}$$

with $F := \sum_j F_j$. Moreover, we also have

$$X = \frac{1}{2}X^{+} + \frac{1}{2}X^{-}\ .$$

We are interested, thanks to Lemma 2.23, in

$$\begin{array}{rcl} \left|\overline{X}\right| & \leq & \alpha \\ \left|X-\mathbb{E}_q[X]\right| & \leq & \alpha \\ \mathbb{E}_q[X]-\alpha & \leq X \leq & \mathbb{E}_q[X]+\alpha \\ \mathbb{E}_q[X]-\alpha & \leq X^{\pm}\mp\delta F \leq & \mathbb{E}_q[X]+\alpha \\ \mathbb{E}_q[X]\pm\delta F-\alpha & \leq X^{\pm} \leq & \mathbb{E}_q[X]\pm\delta F+\alpha \\ \pm\delta F-\alpha & \leq \overline{X^{\pm}} \leq & \pm\delta F+\alpha\,. \end{array}$$

Finally, we estimate

$$\mathbb{P}_q\left[\left\{\, \left|\overline{X}\right| \leq \alpha \,\right\} \cap M\right] \quad = \quad \frac{\mathbb{E}\left[\mathrm{e}^{qX}\right]}{Z}\int_{\Gamma\in\mathbb{R}^L\cap M}\mathrm{e}^{qX-E(\Gamma)}\chi_{\left\{\, \left|\overline{X}\right|\leq\alpha \,\right\}}\left(\Gamma\right)\mathrm{d}\Gamma\,.$$

We now make the following replacements in this integral

1. $X = \frac{1}{2}X^+ + \frac{1}{2}X^-$.
2. $\mathcal{R}\left(\Gamma\right) := \frac{1}{2}E\left(\Gamma^+\right) + \frac{1}{2}E\left(\Gamma^-\right) - E\left(\Gamma\right)$.
3. $\chi_{\left\{\, \left|\overline{X}\right|\leq\alpha \,\right\}}\left(\Gamma\right) = \sqrt{\chi_{\left\{\, \delta F-\alpha\leq\overline{X^+}\leq\delta F+\alpha \,\right\}}\left(\Gamma\right)\chi_{\left\{\, -\delta F-\alpha\leq\overline{X^-}\leq-\delta F+\alpha \,\right\}}\left(\Gamma\right)}$.
4. If $\eta_\pm : \Gamma\mapsto\Gamma^\pm$ then let $J^\pm := \left|\det\left(\mathcal{D}\eta^\pm\right)\right|$ be the Jacobian.

With all of these, we get, with $R_M := \left\|\mathrm{e}^{\mathcal{R}(\Gamma)}\frac{1}{\sqrt{J^+J^-}}\right\|_{L^\infty(M)}$,

$$\begin{aligned}
\mathbb{P}_q\left[\left\{\, \left|\overline{X}\right| \leq \alpha \,\right\} \cap M\right]\frac{Z}{\mathbb{E}\left[\mathrm{e}^{qX}\right]} &= \int_{\Gamma\in M}\mathrm{e}^{\mathcal{R}(\Gamma)}\prod_{\sigma\in\{\,\pm\,\}}\sqrt{\mathrm{e}^{qX^\sigma-E(\Gamma^\sigma)}\chi_{\left\{\, \sigma\delta F-\alpha\leq\overline{X^\sigma}\leq\sigma\delta F+\alpha \,\right\}}\left(\Gamma\right)J^\sigma\frac{1}{J^\sigma}}\mathrm{d}\Gamma \\
&\leq R_M\int_{\Gamma\in M}\prod_{\sigma\in\{\,\pm\,\}}\sqrt{\mathrm{e}^{qX^\sigma-E(\Gamma^\sigma)}\chi_{\left\{\, \sigma\delta F-\alpha\leq\overline{X^\sigma}\leq\sigma\delta F+\alpha \,\right\}}\left(\Gamma\right)J^\sigma}\mathrm{d}\Gamma \\
&\overset{\text{CS}}{\leq} R_M\sqrt{\prod_{\sigma\in\{\,\pm\,\}}\int_{\Gamma\in M}\mathrm{e}^{qX^\sigma-E(\Gamma^\sigma)}\chi_{\left\{\, \sigma\delta F-\alpha\leq\overline{X^\sigma}\leq\sigma\delta F+\alpha \,\right\}}\left(\Gamma\right)J^\sigma\mathrm{d}\Gamma}
\end{aligned}$$

Now we would like to apply the change of variables formula, but the set

$$\left\{\, \delta F - \alpha \leq \overline{X^+} \leq \delta F + \alpha \,\right\}$$

depends in a complicated way on F, and through that Γ. Instead of using that, we note that this set is a subset of

$$\left\{\, \delta\inf_M F - \alpha \leq \overline{X^+} \,\right\}$$

and similarly for the other sign. Hence this set is defined non-randomly, and we may apply the change of variables formula to get

$$\mathbb{P}_q\left[\left\{\, \left|\overline{X}\right| \leq \alpha \,\right\} \cap M\right]\frac{Z}{\mathbb{E}\left[\mathrm{e}^{qX}\right]} \leq R_M\sqrt{\prod_{\sigma\in\{\,\pm\,\}}\int_{\Gamma\in M}\mathrm{e}^{qX-E(\Gamma)}\chi_{\left\{\, \sigma\delta\inf_M F-\alpha\leq\overline{X} \,\right\}}\left(\Gamma\right)\mathrm{d}\Gamma}$$

where we have applied

$$\int_{\mathbb{R}^L} f\circ\eta_\sigma J^\sigma\mathrm{d}\Gamma = \int f\mathrm{d}\Gamma\,.$$

Finally, we estimate

$$\mathbb{P}_q\left[\left\{\, \left|\overline{X}\right| \leq \alpha \,\right\}\right] \quad = \quad \mathbb{P}_q\left[\left\{\, \left|\overline{X}\right| \leq \alpha \,\right\} \cap M\right] + \mathbb{P}_q\left[\left\{\, \left|\overline{X}\right| \leq \alpha \,\right\} \cap M^c\right]$$

and

$$\mathbb{P}_q\left[\left\{\, \left|\overline{X}\right| \leq \alpha \,\right\} \cap M^c\right] \quad = \quad \frac{\mathbb{E}\left[\chi_{\left\{\, \left|\overline{X}\right|\leq\alpha \,\right\}\cap M^c}\mathrm{e}^{qX}\right]}{\mathbb{E}\left[\mathrm{e}^{qX}\right]}\,.$$

Now,

$$\mathbb{E}\left[\mathrm{e}^{qX}\right] \geq \mathbb{E}\left[\mathrm{e}^{\frac{s}{2}X}\right]^{\frac{2q}{s}}\,.$$

Let us further assume, by contradiction, that

$$\mathbb{E}\left[\mathrm{e}^{\frac{s}{2}X}\right] \geq \mathrm{e}^{-cL}$$

since if that is false then we are anyway finished with localization. Hence, as $q\in\left(\frac{s}{2},s\right)$ we have

$$\mathbb{E}\left[\mathrm{e}^{qX}\right] \geq \mathrm{e}^{-c\frac{2q}{s}L}\,.$$

For the numerator we have

$$\begin{aligned}\mathbb{E}\left[\chi_{\left\{\,\left|\overline{X}\right|\leq\alpha\,\right\}\cap M^{c}}\mathrm{e}^{qX}\right] &\leq \sqrt{\mathbb{E}\left[\mathrm{e}^{2qX}\right]\mathbb{P}\left[\left\{\,\left|\overline{X}\right|\leq\alpha\,\right\}\cap M^{c}\right]}\\ &\leq \sqrt{\mathbb{E}\left[\mathrm{e}^{2qX}\right]\mathbb{P}\left[M^{c}\right]}\\ &\leq C\sqrt{\mathbb{P}\left[M^{c}\right]}\end{aligned}$$

where in the last step we have invoked the a-priori bound Theorem 2.12. Combining everything together we find

$$\begin{aligned}\mathbb{P}_{q}\left[\left\{\,\left|\overline{X}\right|\leq\alpha\,\right\}\right] &\leq \mathrm{e}^{c\frac{2q}{s}L}C\sqrt{\mathbb{P}\left[M^{c}\right]}+\\ &\quad +R_{M}\sqrt{\prod_{\sigma\in\{\,\pm\,\}}\int_{\Gamma\in M}\mathrm{e}^{qX-E(\Gamma)}\chi_{\left\{\,\sigma\delta\inf_{M}F-\alpha\leq\overline{X}\,\right\}}\left(\Gamma\right)\mathrm{d}\Gamma}\,.\end{aligned}$$

It is readily seen that this estimate is of the form Lemma 2.23 if we can arrange that:

1. R_M is bounded in L.
2. $\mathrm{e}^{c\frac{2q}{s}L}C\sqrt{\mathbb{P}\left[M^{c}\right]}<\frac{1}{2}$.
3. $\inf_M F\geq\varphi L$ for some fraction $\varphi\in(0,1)$ independent of L.

To fulfill these, we need to ask that

$$\delta\varphi L-\alpha=2\alpha$$

which fixes δ as

$$\delta = \frac{3}{\varphi L}\alpha=\frac{3}{\varphi}\frac{1}{\sqrt{L}}\,.$$

2.11 Consequences of the fractional moment condition

2.11.1 Decay of the Fermi projection

Theorem 2.25. *For a given random operator $\omega\mapsto H_\omega$ for which*

$$\sup_{\varepsilon>0}\mathbb{E}\left[\left|G\left(x,y;E+\mathrm{i}\varepsilon\right)\right|^{s}\right] \leq C\mathrm{e}^{-\mu\|x-y\|} \qquad \left(x,y\in\mathbb{Z}^{d}\right)$$

we have

$$\mathbb{E}\left[\left|\left\langle\delta_{x},\chi_{(-\infty,E)}\left(H\right)\delta_{y}\right\rangle\right|\right] \leq C\mathrm{e}^{-\mu\|x-y\|} \qquad \left(x,y\in\mathbb{Z}^{d}\right)\,.$$

Proof. We know that almost surely E is not an eigenvalue of H (... TODO, explain why). Then we may use the contour formula

$$\chi_{(-\infty,E)}\left(H\right)=\frac{1}{2\pi\mathrm{i}}\oint R\left(z\right)\mathrm{d}z$$

where the contour is a rectangle with vertices on the complex plane given by

$$\left\{\,-\|H\|-\mathrm{i},E-\mathrm{i},E+\mathrm{i},-\|H\|+\mathrm{i}\,\right\}\,.$$

It passes through E vertically and otherwise passes through another vertical line below $-\|H\|$. Thanks to the Combes-Thomas estimate Theorem 1.18 we only need to obtain exponential decay estimates of the vertical line that passes through E:

$$\frac{1}{2\pi}\int_{\varepsilon=-1}^{1}R\left(E+\mathrm{i}\varepsilon\right)\mathrm{d}\varepsilon\,.$$

Taking the x, y matrix elements we get

$$\begin{aligned}\left|\left[\chi_{(-\infty,E)}(H)\right]_{xy}\right| &\leq \frac{1}{2\pi}\int_{\varepsilon=-1}^{1}|G(x,y;E+\mathrm{i}\varepsilon)|\,\mathrm{d}\varepsilon+\text{other legs}\\ &= \frac{1}{2\pi}\int_{\varepsilon=-1}^{1}|G(x,y;E+\mathrm{i}\varepsilon)|^{s}\,|G(x,y;E+\mathrm{i}\varepsilon)|^{1-s}\,\mathrm{d}\varepsilon+\cdots\\ &\leq \frac{1}{2\pi}\int_{\varepsilon=-1}^{1}|G(x,y;E+\mathrm{i}\varepsilon)|^{s}\,\frac{1}{|\varepsilon|^{1-s}}\mathrm{d}\varepsilon\end{aligned}$$

Now we take expectation and we find, using Fubini's theorem and the trivial estimate

$$|G(x,y;z)|\leq\|R(z)\|\leq(\operatorname{dist}(z,\sigma(H)))^{-1}$$

that

$$\begin{aligned}\mathbb{E}\left[\left|\left[\chi_{(-\infty,E)}(H)\right]_{xy}\right|\right] &\leq \int_{\varepsilon=-1}^{1}\mathbb{E}\left[|G(x,y;E+\mathrm{i}\varepsilon)|^{s}\right]\frac{1}{|\varepsilon|^{1-s}}\mathrm{d}\varepsilon\\ &\leq \int_{\varepsilon=-1}^{1}\frac{1}{|\varepsilon|^{1-s}}\mathrm{d}\varepsilon\sup_{\varepsilon\in[-1,1]}\mathbb{E}\left[|G(x,y;E+\mathrm{i}\varepsilon)|^{s}\right]\\ &= \frac{2}{s}\sup_{\varepsilon\in[-1,1]}\mathbb{E}\left[|G(x,y;E+\mathrm{i}\varepsilon)|^{s}\right].\end{aligned}$$

□

2.11.2 Decay of the measurable functional calculus

Theorem 2.26. *Let $\Delta\subseteq\mathbb{R}$ be an interval on which the second moment condition (2.3) holds for every energy $E\in\Delta$ for some random Hamiltonian $\omega\mapsto H_\omega$. Let $B_1(\Delta)$ be set of measurable functions $f:\mathbb{R}\to\mathbb{C}$ which obey: (1) $\|f\|_\infty\leq 1$, (2) There are two constants $C_1, C_2\in\mathbb{C}$ such that $f(t_i)=C_i$ for all $t_1\leq\inf\Delta$ and for all $t_2\geq\sup\Delta$. Then for any interval $\tilde{\Delta}\subsetneq\Delta$ there are constants $C,\mu\in(0,\infty)$ such that*

$$\mathbb{E}\left[\sup_{f\in B_1(\tilde{\Delta})}|\langle\delta_x, f(H)\delta_y\rangle|\right]\leq C\mathrm{e}^{-\mu\|x-y\|}\qquad\left(x,y\in\mathbb{Z}^d\right).$$

Proof. [TODO: finish this proof] Pick some interval $\hat{\Delta}$ such that $\tilde{\Delta}\subsetneq\hat{\Delta}\subsetneq\Delta$ and let $g:\mathbb{R}\to[0,1]$ be some smooth function with the following properties:

1. g restricted to $\hat{\Delta}$ is 1.
2. g restricted to Δ^c is 0.

and let us note that the function $f(1-g)$ is zero on $\hat{\Delta}$ and constant otherwise. As such the operator

$$f(H)(\mathbb{1}-g(H))$$

is a multiple of the identity below and above $\hat{\Delta}$ and hence its matrix elements are proportional to δ_{xy}. We thus need only concentrate on $h=fg$ which is supported within Δ, the interval where (2.3) is obeyed. To that end, let us write

$$\begin{aligned}h(E) &= \int_{\tilde{E}\in\Delta}h\left(\tilde{E}\right)\delta\left(E-\tilde{E}\right)\mathrm{d}\tilde{E}\\ &= \lim_{\eta\to0^+}\int_{\tilde{E}\in\Delta}h\left(\tilde{E}\right)\delta_\eta\left(E-\tilde{E}\right)\mathrm{d}\tilde{E}\\ &= \lim_{\eta\to0^+}\int_{\tilde{E}\in\Delta}h\left(\tilde{E}\right)\frac{1}{\pi}\mathbb{Im}\left\{\frac{1}{E-\tilde{E}-\mathrm{i}\eta}\right\}\mathrm{d}\tilde{E}.\end{aligned}$$

Hence, by the functional calculus, replacing E with H in the above equation yields a strong limit of operators whose matrix elements are given by

$$h\left(H\right)_{xy} \quad = \quad \lim_{\eta\to 0^+}\int_{\tilde{E}\in\Delta} h\left(\tilde{E}\right)\frac{1}{\pi}\,\mathbb{Im}\left\{G\left(x,y;\tilde{E}+\mathrm{i}\eta\right)\right\}\mathrm{d}\tilde{E}\,.$$

Actually we will use instead

$$\begin{aligned}\mathbb{Im}\left\{R\left(z\right)\right\} &= \frac{1}{2\mathrm{i}}\left[R\left(z\right)-R\left(z\right)^{*}\right]\\ &= \frac{1}{2\mathrm{i}}\left[R\left(z\right)-R\left(\overline{z}\right)\right]\\ &= \frac{1}{2\mathrm{i}}\left[R\left(z\right)\left(\overline{z}-z\right)R\left(\overline{z}\right)\right]\\ &= -\,\mathbb{Im}\left\{z\right\}R\left(z\right)R\left(\overline{z}\right)\end{aligned}$$

Taking now the matrix element

$$h\left(H\right)_{xy} \quad = \quad -\lim_{\eta\to 0^+}\int_{\tilde{E}\in\Delta} h\left(\tilde{E}\right)\frac{\eta}{\pi}\sum_{\tilde{x}\in\mathbb{Z}^d} G\left(x,\tilde{x},\tilde{E}+\mathrm{i}\eta\right)G\left(\tilde{x},y;\tilde{E}-\mathrm{i}\eta\right)\mathrm{d}\tilde{E}$$

So that

$$\begin{aligned}\mathbb{E}\left[\left|h\left(H\right)_{xy}\right|\right] &\leq \mathbb{E}\left[\lim_{\eta\to 0^+}\frac{\eta}{\pi}\int_{\tilde{E}\in\Delta}\sum_{\tilde{x}\in\mathbb{Z}^d}\left|G\left(x,\tilde{x},\tilde{E}+\mathrm{i}\eta\right)\right|\left|G\left(\tilde{x},y;\tilde{E}-\mathrm{i}\eta\right)\right|\mathrm{d}\tilde{E}\right]\\ &\leq \liminf_{\eta\to 0^+}\frac{\eta}{\pi}\mathbb{E}\left[\int_{\tilde{E}\in\Delta}\sum_{\tilde{x}\in\mathbb{Z}^d}\left|G\left(x,\tilde{x},\tilde{E}+\mathrm{i}\eta\right)\right|\left|G\left(\tilde{x},y;\tilde{E}-\mathrm{i}\eta\right)\right|\mathrm{d}\tilde{E}\right]\\ &\overset{\text{C.S.}}{\leq} \liminf_{\eta\to 0^+}\frac{\eta}{\pi}\mathbb{E}\left[\int_{\tilde{E}\in\Delta}\sum_{\tilde{x}\in\mathbb{Z}^d}\left|G\left(x,\tilde{x},\tilde{E}+\mathrm{i}\eta\right)\right|\left|G\left(\tilde{x},y;\tilde{E}-\mathrm{i}\eta\right)\right|\mathrm{d}\tilde{E}\right]\end{aligned}$$

□

2.11.3 Almost-sure consequences

One thing that will be useful for us in the sequel will be the following almost-sure consequence

Theorem 2.27. *Let $\{\,A_i\,\}_{i\in I}$ be a sequence of random operators (for I countable or uncountable) on $\ell^2\left(\mathbb{Z}^d\right)\otimes\mathbb{C}^N$ such that there exists some $C,\mu\in(0,\infty)$ with which*

$$\mathbb{E}\left[\sup_{i\in I}\left\|\left\langle\delta_x,A_i\delta_y\right\rangle\right\|\right] \quad\leq\quad C\mathrm{e}^{-\mu\|x-y\|}\qquad\left(x,y\in\mathbb{Z}^d\right)\,.$$

Then almost-surely, for any $\mu'\in(0,\mu)$ and any $a\in\ell^1\left(\mathbb{Z}^d\right)$ there exists some (random) $C_a<\infty$ such that

$$\sup_{i\in I}\left\|\left\langle\delta_x,A_i\delta_y\right\rangle\right\| \quad\leq\quad C_{a,\mu'}\frac{1}{\left|a\left(x\right)\right|}\exp\left(-\mu'\|x-y\|\right)\qquad\left(x,y\in\mathbb{Z}^d\right)\,.$$

Proof. Let $a \in \ell^1$ and $\mu' < \mu$. Then by the Fatou's lemma, the following expectation is bounded from above by

$$\begin{aligned}
& \mathbb{E}\left[\sum_{x,y\in\mathbb{Z}^d} \sup_{i\in I} \|\langle \delta_x, A_i \delta_y\rangle\| \mathrm{e}^{+\mu'\|x-y\|} |a(x)|\right] \\
= \quad & \mathbb{E}\left[\lim_{L\to\infty} \sum_{x,y\in B_L(0)\cap\mathbb{Z}^d} \sup_{i\in I} \|\langle \delta_x, A_i \delta_y\rangle\| \mathrm{e}^{+\mu'\|x-y\|} |a(x)|\right] \\
\overset{\text{Fatou}}{\leq} \quad & \liminf_{L\to\infty} \mathbb{E}\left[\sum_{x,y\in B_L(0)\cap\mathbb{Z}^d} \sup_{i\in I} \|\langle \delta_x, A_i \delta_y\rangle\| \mathrm{e}^{+\mu'\|x-y\|} |a(x)|\right] \\
= \quad & \liminf_{L\to\infty} \sum_{x,y\in B_L(0)\cap\mathbb{Z}^d} \mathbb{E}\left[\sup_{i\in I} \|\langle \delta_x, A_i \delta_y\rangle\|\right] \mathrm{e}^{+\mu'\|x-y\|} |a(x)| \\
\overset{\text{hypo}}{\leq} \quad & \liminf_{L\to\infty} \sum_{x,y\in B_L(0)\cap\mathbb{Z}^d} C \mathrm{e}^{-(\mu-\mu')\|x-y\|} |a(x)| \\
= \quad & C \sum_{x,y\in\mathbb{Z}^d} \mathrm{e}^{-(\mu-\mu')\|x-y\|} |a(x)| \\
\leq \quad & \tilde{C}(\mu-\mu', a) \, .
\end{aligned}$$

We find that the function

$$\omega \quad \mapsto \quad \sum_{x,y\in\mathbb{Z}^d} \sup_{i\in I} \|\langle \delta_x, A_i \delta_y\rangle\| \mathrm{e}^{+\mu'\|x-y\|} |a(x)|$$

is an integrable, non-negative function. As a result it must be finite for almost all ω. I.e., there is some random constant $\omega \mapsto D_\omega < \infty$ such that

$$\sum_{x,y\in\mathbb{Z}^d} \sup_{i\in I} \|\langle \delta_x, A_i \delta_y\rangle\| \mathrm{e}^{+\mu'\|x-y\|} |a(x)| \leq D_\omega$$

for almost-all ω. Thus, however, implies thanks to non-negativity, that for any $x, y \in \mathbb{Z}^d$,

$$\begin{aligned}
& \sup_{i\in I} \|\langle \delta_x, A_i \delta_y\rangle\| \mathrm{e}^{+\mu'\|x-y\|} |a(x)| \\
\leq \quad & \sum_{\tilde{x},\tilde{y}\in\mathbb{Z}^d} \sup_{i\in I} \|\langle \delta_{\tilde{x}}, A_i \delta_{\tilde{y}}\rangle\| \mathrm{e}^{+\mu'\|\tilde{x}-\tilde{y}\|} |a(\tilde{x})| \\
\leq \quad & D_\omega
\end{aligned}$$

and so

$$\sup_{i\in I} \|\langle \delta_x, A_i \delta_y\rangle\| \quad \leq \quad D_\omega \frac{1}{|a(x)|} \mathrm{e}^{-\mu'\|x-y\|} \, .$$

□

2.11.4 The SULE basis

Definition 2.28 (SULE basis)**.** A semi-uniformly localized basis for a vector subspace $V \subseteq \mathcal{H} \equiv \ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N$ is an ONB $\{\psi_n\}_{n\in\mathbb{N}}$ such that there is a sequence of "localization centers" $\{x_n\}_n \subseteq \mathbb{Z}^d$ such that for any $a \in \ell^1(\mathbb{Z}^d \to \mathbb{C})$ there is a $C_a \in (0, \infty)$ such that

$$\|\psi_n(x)\| \leq C_a \mathrm{e}^{-\mu\|x-x_n\|} \frac{1}{|a(x_n)|} \qquad (x \in \mathbb{Z}^d, n \in \mathbb{N}) \, . \tag{2.17}$$

This notion was originally defined in [dRJLS96].

Claim 2.29. The localization centers $\{ x_n \}_{n\in\mathbb{N}}$ of a SULE basis of some vector space $V \subseteq \mathbb{N}$ obey

$$\sum_{n\in\mathbb{N}} \frac{1}{(1+\|x_n\|)^{d+\varepsilon}} < \infty \qquad (\varepsilon > 0) \ . \tag{2.18}$$

Proof. Let $\varepsilon > 0$ be given, and decompose the sum using

$$1 = \sum_{k=1}^{\infty} \chi_{\{ k \}} (\|x_n\|)$$

to get

$$\begin{aligned}
\sum_{n\in\mathbb{N}} \frac{1}{(1+\|x_n\|)^{d+\varepsilon}} &= \sum_{n\in\mathbb{N}} \sum_{k=1}^{\infty} \chi_{\{ k \}} (\|x_n\|) \frac{1}{(1+\|x_n\|)^{d+\varepsilon}} \\
&= \sum_{n\in\mathbb{N}} \sum_{k=1}^{\infty} \chi_{\{ k \}} (\|x_n\|) \frac{1}{(1+k)^{d+\varepsilon}} \\
&= \sum_{k=1}^{\infty} |\{ n \in \mathbb{N} \mid \|x_n\| = k \}| \frac{1}{(1+k)^{d+\varepsilon}} \\
&= \sum_{k=1}^{\infty} \left(|\{ n \in \mathbb{N} \mid \|x_n\| \leq k \}| - |\{ n \in \mathbb{N} \mid \|x_n\| \leq k-1 \}|\right) \frac{1}{(1+k)^{d+\varepsilon}} \ .
\end{aligned}$$

Let us thus derive upper and lower bounds on

$$\mathbb{N} \ni L \quad \mapsto \quad |\{ n \in \mathbb{N} \mid \|x_n\| \leq L \}| \ .$$

For the upper bound, let us study, for any $L \in \mathbb{N}$ and $\delta > 0$,

$$\sum_{n\in\mathbb{N}:\|x-x_n\|\geq\delta(\|x_n\|+L)} |\varphi_n(x)|^2 \ .$$

Thanks to (2.17) we get

$$|\varphi_n(x)|^2 \leq C_a \mathrm{e}^{-\mu\|x-x_n\|} \frac{1}{|a(x_n)|} \qquad \left(x \in \mathbb{Z}^d, n \in \mathbb{N}\right) \ .$$

In particular taking $a(x_n) = \mathrm{e}^{-\xi\|x_n\|}$ for some $\xi > 0$ yields

$$|\varphi_n(x)|^2 \leq C_\xi \mathrm{e}^{-\mu\|x-x_n\|+\xi\|x_n\|} \qquad \left(x \in \mathbb{Z}^d, n \in \mathbb{N}\right) \ .$$

But now, if $\|x - x_n\| \geq \delta(\|x_n\| + L)$, then

$$\|x-x_n\| \geq \frac{1}{2}\|x-x_n\| + \frac{1}{2}\delta\|x_n\| + \frac{1}{2}\delta L$$

whence

$$\begin{aligned}
\sum_{n\in\mathbb{N}:\|x-x_n\|\geq\delta(\|x_n\|+L)} |\varphi_n(x)|^2 &\leq \sum_{n\in\mathbb{N}:\|x-x_n\|\geq\delta(\|x_n\|+L)} C_\xi^2 \mathrm{e}^{-2\mu\|x-x_n\|+2\xi\|x_n\|} \\
&= \sum_{n\in\mathbb{N}:\|x-x_n\|\geq\delta(\|x_n\|+L)} C_\xi^2 \mathrm{e}^{-\mu\|x-x_n\|-\delta\mu\|x_n\|-\delta\mu L+2\xi\|x_n\|}
\end{aligned}$$

Choosing $\xi = \frac{1}{2}\delta\mu$ yields

$$\begin{aligned}\sum_{n\in\mathbb{N}:\|x-x_n\|\geq\delta(\|x_n\|+L)} |\varphi_n(x)|^2 &\leq \sum_{n\in\mathbb{N}:\|x-x_n\|\geq\delta(\|x_n\|+L)} C^2_{\frac{1}{2}\delta\mu}\mathrm{e}^{-\mu\|x-x_n\|-\delta\mu L}\\ &\leq C^2_{\frac{1}{2}\delta\mu}\mathrm{e}^{-\delta\mu L}\sum_{n\in\mathbb{N}:\|x-x_n\|\geq\delta(\|x_n\|+L)}\mathrm{e}^{-\mu\|x-x_n\|}\\ &= C^2_{\frac{1}{2}\delta\mu}\mathrm{e}^{-\delta\mu L}\sum_{k\geq\delta(\|x_n\|+L)}\mathrm{e}^{-\mu k}\\ &\leq C^2_{\frac{1}{2}\delta\mu}\mathrm{e}^{-\delta\mu L}\sum_{k\geq\delta\|x_n\|}\mathrm{e}^{-\mu k}\\ &= C^2_{\frac{1}{2}\delta\mu}\mathrm{e}^{-\delta\mu L}\frac{\mathrm{e}^{\mu-\mu\delta\|x_n\|}}{\mathrm{e}^{\mu}-1}\\ &=: \widetilde{C^2_{\frac{1}{2}\delta\mu}}\mathrm{e}^{-\delta\mu L-\mu\delta\|x_n\|}.\end{aligned}$$

We conclude

$$\sum_{n\in\mathbb{N}:\|x-x_n\|\geq\delta(\|x_n\|+L)} |\varphi_n(x)|^2 \leq \widetilde{C^2_{\frac{1}{2}\delta\mu}}\mathrm{e}^{-\delta\mu L-\mu\delta\|x_n\|}. \tag{2.19}$$

Next, assume that $\|x_n\| \leq L$ and $\|x\| \geq (1+2\delta)L$. Then

$$\begin{aligned}\|x-x_n\| &\geq \|x\|-\|x_n\|\\ &\geq (1+2\delta)L-L\\ &= 2\delta L\\ &\geq \delta(L+\|x_n\|).\end{aligned}$$

This implies that for $\|x_n\| \leq L$ we get

$$\begin{aligned}\sum_{\|x\|\geq(1+2\delta)L} |\varphi_n(x)|^2 &\leq \sum_{\|x-x_n\|\geq\delta(L+\|x_n\|)} |\varphi_n(x)|^2\\ &\leq \widetilde{C^2_{\frac{1}{2}\delta\mu}}\mathrm{e}^{-\delta\mu L-\mu\delta\|x_n\|}\\ &\leq \widetilde{C^2_{\frac{1}{2}\delta\mu}}\mathrm{e}^{-\delta\mu L}.\end{aligned}$$

Next, since $\|\varphi_n\|^2 = 1$, we have

$$\sum_{x\in\mathbb{Z}^d}\|\varphi_n(x)\|^2 = 1$$

and hence

$$\begin{aligned}\sum_{\|x\|<(1+2\delta)L} |\varphi_n(x)|^2 &= 1-\sum_{\|x\|\geq(1+2\delta)L} |\varphi_n(x)|^2\\ &\geq 1-\widetilde{C^2_{\frac{1}{2}\delta\mu}}\mathrm{e}^{-\delta\mu L}.\end{aligned}$$

But also, $\sum_{n\in\mathbb{N}} \varphi_n \otimes \varphi_n^*$ is the self-adjoint projection onto V, so in particular,

$$\begin{aligned} \left\langle \delta_x, \sum_{n\in\mathbb{N}} \varphi_n \otimes \varphi_n^* \delta_x \right\rangle &\leq 1 \\ &\downarrow \\ \sum_{n\in\mathbb{N}} \left|\varphi_n\left(x\right)\right|^2 &\leq 1 \\ &\downarrow \\ \sum_{\|x\|<(1+2\delta)L} \sum_{n\in\mathbb{N}} \left|\varphi_n\left(x\right)\right|^2 &\leq \sum_{\|x\|<(1+2\delta)L} 1 = \left(2\left(1+2\delta\right)L+1\right)^d . \end{aligned}$$

This last equation yields

$$\begin{aligned} \left(2\left(1+2\delta\right)L+1\right)^d &\geq \sum_{\|x\|<(1+2\delta)L} \sum_{n\in\mathbb{N}} \left|\varphi_n\left(x\right)\right|^2 \\ &= \sum_{\|x\|<(1+2\delta)L} \sum_{n\in\mathbb{N}:\|x_n\|\leq L} \left|\varphi_n\left(x\right)\right|^2 \\ &= \sum_{n\in\mathbb{N}:\|x_n\|\leq L} \sum_{\|x\|<(1+2\delta)L} \left|\varphi_n\left(x\right)\right|^2 \\ &\geq \sum_{n\in\mathbb{N}:\|x_n\|\leq L} \left(1-\widetilde{C^2_{\frac{1}{2}\delta\mu}}\mathrm{e}^{-\delta\mu L}\right) \\ &= \left|\left\{\, n\in\mathbb{N} \mid \|x_n\|\leq L \,\right\}\right| \left(1-\widetilde{C^2_{\frac{1}{2}\delta\mu}}\mathrm{e}^{-\delta\mu L}\right) . \end{aligned}$$

In particular we find some $\tilde{C}\in(0,\infty)$ with which

$$\begin{aligned} \left|\left\{\, n\in\mathbb{N} \mid \|x_n\|\leq L \,\right\}\right| &\leq \left(2\left(1+2\delta\right)L+1\right)^d \left(1-\widetilde{C^2_{\frac{1}{2}\delta\mu}}\mathrm{e}^{-\delta\mu L}\right)^{-1} \\ &\lesssim \tilde{C}L^d \qquad \left(L\in\mathbb{N}\right) . \end{aligned}$$

We now turn to the lower bound. Using again the fact that $\sum_{n\in\mathbb{N}} \left|\varphi_n\left(x\right)\right|^2 \leq 1$ we sum up this inequality for $\|x\|\leq L$ to obtain

$$\left(2L+1\right)^d \geq$$

□

Theorem 2.30. *Let $\Delta\subseteq\mathbb{R}$ be an interval on which H has simple pure point spectrum and on which H exhibits localization in the sense that its bounded measurable functional calculus exhibits decay in the deterministic sense of Theorem 2.27 applied to Theorem 2.26. Then there exists a SULE basis to the space $\operatorname{im}\left(\chi_\Delta\left(H\right)\right)$ consisting of eigenfunctions of H.*

Proof. Since $\chi_{\{\,\lambda\,\}}$ is a bounded Borel function on Δ, we conclude that

$$\left\|\left(\chi_{\{\,\lambda\,\}}\left(H\right)\right)_{xy}\right\| \leq C_a \mathrm{e}^{-\mu\|x-y\|}\frac{1}{\left|a\left(y\right)\right|} \qquad \left(x,y\in\mathbb{Z}^d\right) .$$

Now, all eigenfunctions are of finite multiplicity, so

$$\operatorname{tr}\left(\chi_{\{\,\lambda\,\}}\left(H\right)\right) \quad < \quad \infty .$$

Hence

$$\mathbb{Z}^d \ni x \mapsto \left\|\left(\chi_{\{\,\lambda\,\}}\left(H\right)\right)_{xx}\right\| \leq \operatorname{tr}\left(\left(\chi_{\{\,\lambda\,\}}\left(H\right)\right)_{xx}\right) =: a_x .$$

is a summable sequence. Let $x_0\in\mathbb{Z}^d$ be the point where a_x attains its maximal value and let $v_0\in\mathbb{C}^N$ be $\|v_0\|=1$

with

$$\left\langle v_0, \left(\chi_{\{\lambda\}}(H)\right)_{x_0 x_0} v_0 \right\rangle = a_{x_0}\,.$$

Define then

$$\psi(x) := \frac{1}{\sqrt{a_{x_0}}}\left(\chi_{\{\lambda\}}(H)\right)_{x x_0} v_0\,.$$

Then

$$\begin{aligned}
(H\psi)_x &= \sum_y H_{xy}\frac{1}{\sqrt{a_{x_0}}}\left(\chi_{\{\lambda\}}(H)\right)_{y x_0} v_0 \\
&= \frac{1}{\sqrt{a_{x_0}}}\left(H\chi_{\{\lambda\}}(H)\right)_{x x_0} v_0 \\
&= \frac{1}{\sqrt{a_{x_0}}}\left(\lambda\chi_{\{\lambda\}}(H)\right)_{x x_0} v_0 \\
&= \lambda\psi(x)\,.
\end{aligned}$$

Hence, $H\psi = \lambda\psi$ and moreover,

$$\begin{aligned}
\|\psi\|^2 &= \sum_x \|\psi(x)\|^2 \\
&= \sum_x \left\| \frac{1}{\sqrt{a_{x_0}}}\left(\chi_{\{\lambda\}}(H)\right)_{x x_0} v_0 \right\|^2 \\
&= \frac{1}{a_{x_0}}\left\langle v_0, \chi_{\{\lambda\}}(H)_{x_0 x_0} v_0 \right\rangle \\
&\equiv 1\,.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\left\|\chi_{\{\lambda\}}(H)_{x x_0} v_0\right\| &\leq \max_{\|v\|=1}\left|\left\langle \delta_x\otimes v, \chi_{\{\lambda\}}(H)\,\delta_{x_0}\otimes v_0\right\rangle\right| \\
&\leq \max_{\|v\|=1}\left|\left\langle \chi_{\{\lambda\}}(H)\,\delta_x\otimes v, \chi_{\{\lambda\}}(H)\,\delta_{x_0}\otimes v_0\right\rangle\right| \\
&\leq \sqrt{a_x}\sqrt{a_{x_0}} \\
&\leq a_{x_0}\,.
\end{aligned}$$

Hence

$$\begin{aligned}
\|\psi(x)\| &\leq \sqrt{\left\| \frac{1}{\sqrt{a_{x_0}}}\left(\chi_{\{\lambda\}}(H)\right)_{x x_0} v_0 \right\|}\times \\
&\quad \times\sqrt{\left\| \frac{1}{\sqrt{a_{x_0}}}\left(\chi_{\{\lambda\}}(H)\right)_{x x_0} v_0 \right\|} \\
&\leq \sqrt{\frac{1}{\sqrt{a_{x_0}}}C_a \mathrm{e}^{-\mu\|x-x_0\|}\frac{1}{|a(x_0)|}}\times \\
&\quad \times\sqrt{a_{x_0}} \\
&\leq (a_{x_0})^{\frac{1}{4}}\sqrt{C_a \mathrm{e}^{-\mu\|x-x_0\|}\frac{1}{|a(x_0)|}}\,.
\end{aligned}$$

Now apply the process again to $\chi_{\{\lambda\}}(H) - \psi\otimes\psi^*$ whose rank is smaller by 1 compared with $\chi_{\{\lambda\}}(H)$ we obtain the result by induction. □

2.12 The physics argument for delocalization

[TODO]

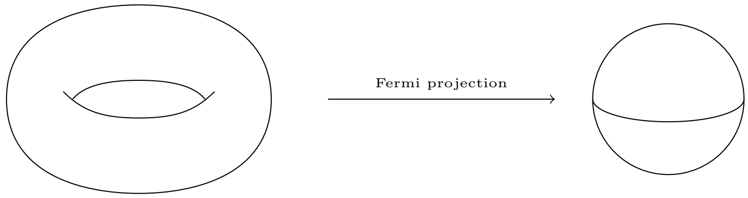

Figure 1: The topology of two-dimensional periodic two-level insulating systems.

3 Topology in condensed matter physics

In 1979 von Klitzing conducted an experiment [KDP80] of *the Hall effect* which is nowadays considered famous and won him the Nobel prize. Explaining this experiment has led condensed matter theoretical physicists down a rabbit hole which is by now known as "topology in condensed matter physics". We shall go on to describe the classical and quantum Hall effect in detail, but let us briefly describe the gist of the idea.

It turns out that there are certain *exotic materials* (e.g. 2D Gallium Arsenic) which exhibit *macroscopic* stability of certain physical observables (e.g., electric conductivity). This macroscopic stability is manifested in two ways:

1. It is quantized in appropriate physical units to a discrete *additive group* (e.g., $\mathbb{Z}$ or $\mathbb{Z}_2$). This is an example of a macroscopic quantum mechanical effect.

2. It is constant with respect to various experimental tweaks (e.g. doping, impurities, etc).

To explain this experimental phenomenon from the theoretical point of view, one is led to the following general mathematical program:

1. Define a space of quantum mechanical Hamiltonians $\mathcal{S}$ which describe the space of all models for materials.

2. Define an ambient topology $\operatorname{Open}(\mathcal{S})$ on this space.

3. Define a *continuous* map $f : \mathcal{S} \to \mathbb{G}$ where $\mathbb{G}$ is a discrete additive group. It is automatically locally constant.

4. To connect f to experimental physics, exhibit f as a physical observable associated to an element $H \in \mathcal{S}$.

5. *Bonus*: show that f is a *complete* invariant, i.e., show that f lifts to a bijection

$$\tilde{f} : \pi_0(\mathcal{S}) \quad \cong \quad \mathbb{G}$$

where $\pi_0(\mathcal{S})$ is the set of *path-connected components* of $\mathcal{S}$, defined via $\operatorname{Open}(\mathcal{S})$.

This is *one* way to think about the mathematical theory of topological insulators. Of course there are many additional facets to it. Another important aspect is the *bulk-edge correspondence*: the fact that the geometry of the sample (whether it has a boundary or not) describes different physical effects, different experimental observables, and different topological classification. Despite all of these a-priori differences, it turns out (this is the bulk-edge correspondence) that if one starts from a bulk system (defined on an infinite, boundary-less geometry) and truncates it to have a boundary, then these various aspects agree.

Example 3.1 (The topological classification of periodic systems leads to vector-bundle K-theory)**.** To illustrate our point above, let us briefly explain the situation in the context of periodic systems as in Definition 1.7. These systems are *not* realistic from the physics point of view, but their analysis reduces to classical algebraic topology and in that sense it is appealing. We have seen above in Proposition 1.27 that any periodic

$$H = H^* \in \mathcal{B}\left(\ell^2\left(\mathbb{Z}^d\right) \otimes \mathbb{C}^N\right)$$

stands in one-to-one correspondence with a *symbol*

$$h : \mathbb{T}^d \to \operatorname{Herm}_{N\times N}(\mathbb{C}) \cong \mathbb{R}^{N^2} .$$

via

$$h(k) = \sum_{x\in\mathbb{Z}^d} \mathrm{e}^{\mathrm{i}\langle k,x\rangle} H_{0,x} \qquad \left(k \in \mathbb{T}^d\right) .$$

Altland-Zirnbauer Symmetry				space dimension							
AZ	Θ	Ξ	Π	1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

Table 1: The Kitaev table of topological insulators. The entries indicate only the strong, i.e., top-dimensional, topological invariants.

Moreover, the Riemann-Lebesgue lemma Theorem 1.14 implies that h has a certain regularity due to the locality of H. In particular if H is exponentially local as in (1.2) then h is analytic. Let us suppose for convenience that H is as local in such a way so that h is continuous (if we insisted to proceed with h analytic we could also just use an approximation theorem, e.g., see [Lee03]. E.g. the Whitney approximation theorem says that for smooth manifolds X and Y (with $\partial Y = \varnothing$), any continuous map $X \to Y$ is (continuously) homotopic to a smooth map $X \to Y$). Hence we see that (vaguely) local periodic Hamilotonians stand in one-to-one correspondence with continuous maps

$$h : \mathbb{T}^d \quad \to \quad \mathrm{Herm}_{N\times N}(\mathbb{C}) \cong \mathbb{R}^{N^2} .$$

Since we are dealing with continuous maps there is a natural topology induced on the space of such maps, which is the compact open topology. This topology in our case is induced by the metric

$$\|f\|_\infty := \sup_{k\in\mathbb{T}^d} \|f(k)\|_{\mathrm{Mat}_{N\times N}(\mathbb{C})} .$$

Clearly with no further constraint we find that the homotopy classes of such maps are trivial:

$$\pi_0\,(\text{periodic local Hamiltonians}) \quad = \quad \left[\mathbb{T}^d \to \mathbb{R}^{N^2}\right] = \{\, 0 \,\} .$$

To find something interesting, let us further suppose that our systems are insulators, by employing the gap condition (we have seen in Remark 1.67 that if H is gapped at E_F then the zero-temperature conductivity is zero $\sigma_{ij}(E_F) = 0$). So let us fix $E_F = 0$ and always assume that $0 \notin \sigma(H)$, which means at the level of the symbo

$$0 \neq E_j(k) \qquad \left(j = 1, \dots, N;\, k \in \mathbb{T}^d\right)$$

where $E_j(k)$ are the energy eigenvalues of $h(k)$. This is still not enough because if we always have all the spectrum above zero or all the spectrum below zero then we don't expect we should be able to deform between these two scenarios without closing the gap. Hence we need to restrict to the case when there is spectrum both above and below zero, so the most interesting case is apparently if

$$E_1(k) \leq \cdots \leq E_m(k) < 0 < E_{m+1}(k) \leq \cdots \leq E_N(k) .$$

I.e., if the gap is precisely in between the mth and $m+1$th level for $m = 1, \dots, N-1$. We would then divide the analysis based on the value of m. We ultimately find our space of Hamiltonians stands in one-to-one correspondence with

$$h : \mathbb{T}^d \to \mathcal{S}_{N,m}$$

where

$$\mathcal{S}_{N,m} := \{\, M \in \mathrm{Herm}_{N\times N}(\mathbb{C}) \mid \lambda_m(M) < 0 < \lambda_{m+1}(M) \,\}$$

and where $\lambda_j(M)$ is the jth eigenvalue of the matrix M in ascending order. We identify that $\mathcal{S}_{N,m}$ has a deformation retraction which is the Grassmannian manifold: the space of m-dimensional vector subspaces within $\mathbb{C}^N$, denoted by $\mathrm{Gr}_m(\mathbb{C}^N)$. Indeed, this is obtained by considering the straight-line homotopy

$$[0,1] \ni t \mapsto (1-t)M + t\left(-\chi_{(-\infty,0)}(M) + \chi_{(0,\infty)}(M)\right) .$$

As such, we now find that the space of periodic local insulators at $E_F = 0$ which are gapped after m levels stands in one to one correspondence with the space of continuous maps

$$\mathbb{T}^d \to \mathrm{Gr}_m(\mathbb{C}^N) .$$

For example if $d = N = 2$ and $= 1$ we find

$$\mathbb{T}^2 \to \mathrm{Gr}_1(\mathbb{C}^2) \cong \mathbb{S}^2 .$$

This last homeomorphism proceeds by writing any rank-1 projection which is a two-by-two matrix using the Pauli matrices times a unit vector. We thus arrive at a picture as in Figure 1. It turns out that because $\mathbb{S}^2$ is simply-connected, the one-dimensional loops that generate $\mathbb{T}^2$ are unimportant and

$$\left[\mathbb{T}^2 \to \mathbb{S}^2\right] \cong \left[\mathbb{S}^2 \to \mathbb{S}^2\right] \equiv \pi_2\left(\mathbb{S}^2\right) \cong \mathbb{Z}$$

the last bijection being *the degree of the map*. In fact in we shall see that this degree has the physical interpretation of the Chern number. We also see that if $d = 1$ then we get for the case $m = 1$, $N = 2$ (with $\mathbb{T}^1 \equiv \mathbb{S}^1$)

$$\left[\mathbb{S}^1 \to \mathbb{S}^2\right] \equiv \pi_1\left(\mathbb{S}^2\right) \cong \{\,0\,\}$$

since the sphere is simply-connected. This somewhat explains why one-dimensional insulators (with no further symmetry) are trivial whereas two-dimensional ones exhibit the non-trivial integer quantum Hall effect.

To proceed further one should allow N to be arbitrarily large (these levels are unoccupied anyway). Doing so lands us with

$$\left[\mathbb{T}^d \to \mathrm{Gr}_m(\mathbb{C}^\infty)\right] \cong \mathrm{Vect}_m\left(\mathbb{T}^d\right)$$

the isomorphism classes of rank-m vector-bundles over the base space $\mathbb{T}^d$. This is because $\mathrm{Gr}_m(\mathbb{C}^\infty)$ is the classifying space for vector bundles. At this stage one may employ Atiyah's K-theory [Ati94], the classification scheme of vector-bundles. K-theory yields a stable and relative classification of $\mathrm{Vect}_m\left(\mathbb{T}^d\right)$ (apparently the space of periodic local insulators at $E_F = 0$ which are gapped after m levels) whose result is

$$K_0\left(\mathbb{T}^d\right) \quad \cong \quad \mathbb{Z}^{2^{d-1}} .$$

However, only some of these copies of $\mathbb{Z}$ are "top" dimensional and the rest are called weak (they explore only lower dimensions of the system). It turns out that if one counts only the top dimensional copies of $\mathbb{Z}$ one obtains

$$K_0\left(\mathbb{T}^d\right)_{\text{top dim.}} \cong K_0\left(\mathbb{S}^d\right) \cong \begin{cases} \mathbb{Z} & d \in 2\mathbb{N} \\ \{\,0\,\} & d \in 2\mathbb{N}+1 \end{cases} .$$

Furthermore, one might want to take symmetries into account (which would means working in an equivariant version of $\mathrm{Gr}_m(\mathbb{C}^N)$; e.g., time-reversal symmetry on Fermions implies a quaternionic structure on $\mathrm{Gr}_m(\mathbb{C}^N)$ which the maps under homotopy should respect). In this case one is led to real K-theory. All together people have worked out a whole table of topological insulators, which Kitaev organized [Kit09] in a periodic way patterned after K-theory and the Clifford algebras [ABS64]. It is depicted in Table 1 and we shall get back to it later on to explain the various symmetry classes in detail.

We begin the discussion from the important case of the Hall effect and later on we shall return to this abstract program.

3.1 The classical Hall effect

The classical Hall effect is the phenomenon that two-dimensional electrons in a constant perpendicular magnetic field exhibit a *transversal* current. To measure the current, one applies an electric field in one direction and measures a current in the transversal direction.

Let us derive this:

3.1.1 Classical motion in constant electric and magnetic fields

Classically, the equations of motion for the trajectory of an electron $\gamma : \mathbb{R} \to \mathbb{R}^2$ are given by

$$\ddot{\gamma} = E(\gamma) + \dot{\gamma} \wedge B(\gamma)$$

where $E : \mathbb{R}^2 \to \mathbb{R}$ is the electric field and $B : \mathbb{R}^2 \to \mathbb{R}^3$ is the magnetic field. For more generality, let us also add a friction term

$$\ddot{\gamma} = E(\gamma) + \dot{\gamma} \wedge B(\gamma) + r\dot{\gamma}$$

for some $r \in \mathbb{R}$ (related to the resistance of the material). Since we are running an experiment where we want a constant perpendicular magnetic field, we choose

$$B(x) = B_0 e_3 \qquad \left(x \in \mathbb{R}^2\right)$$

where $B_0 > 0$ is the magnetic field strength. Furthermore, the experiment is set up with the electric field in some direction within the plane. For simplicity we pick the easiest case scenario, which is a constant electric field too, so we just have

$$E(x) = E_0 e_1 \qquad \left(x \in \mathbb{R}^2\right) .$$

Since γ only has components within the plane, we can already find that

$$\begin{aligned} \dot{\gamma} \wedge B(\gamma) &= (\dot{\gamma}_1 e_1 + \dot{\gamma}_2 e_2) \wedge (B_0 e_3) \\ &= -B_0 \dot{\gamma}_1 e_2 + B_0 \dot{\gamma}_2 e_1 . \end{aligned}$$

We find the equations of motion given by

$$\begin{aligned} \ddot{\gamma}_1 &= E_0 + B_0 \dot{\gamma}_2 + r\dot{\gamma}_1 \\ \ddot{\gamma}_2 &= -B_0 \dot{\gamma}_1 + r\dot{\gamma}_2 . \end{aligned}$$

This may be written as

$$\ddot{\gamma} = B_0 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \dot{\gamma} + r\dot{\gamma} + \begin{bmatrix} E_0 \\ 0 \end{bmatrix} .$$

It will be algebraically easier to re-cast this equation as $\gamma : \mathbb{R} \to \mathbb{C}$ instead, so that in those coordinates we have

$$\ddot{\gamma} = (r - B_0 \mathrm{i})\, \dot{\gamma} + E_0$$

whose solution is

$$\gamma(t) = -\frac{E_0}{r - \mathrm{i}B_0} t + \mathrm{e}^{(r-\mathrm{i}B_0)t} C_1 + C_2 \qquad (t \in \mathbb{R}) .$$

where $C_1, C_2 \in \mathbb{C}$ are two constants. We write them in terms of the boundary conditions which yields

$$\gamma(t) = -\frac{E_0}{r - \mathrm{i}B_0} t + \left(\mathrm{e}^{(r-\mathrm{i}B_0)t} - 1\right) \left(\frac{E_0}{(r - \mathrm{i}B_0)^2} + \frac{1}{r - \mathrm{i}B_0} \dot{\gamma}(0) \right) + \gamma(0) \qquad (t \in \mathbb{R}) .$$

We conclude that:

1. If $r = E_0 = 0$, then we have simply circular motion:

$$\begin{aligned} \gamma(t) &= \mathrm{i}\frac{1}{B_0} \left(\mathrm{e}^{-\mathrm{i}B_0 t} - 1\right) \dot{\gamma}(0) + \gamma(0) \\ &= \mathrm{i}\frac{1}{B_0} \mathrm{e}^{-\mathrm{i}B_0 t} \dot{\gamma}(0) + \gamma(0) - \mathrm{i}\frac{1}{B_0} \dot{\gamma}(0) \end{aligned}$$

   the initial position of which is

$$\gamma(0) - \mathrm{i}\frac{1}{B_0} \dot{\gamma}(0)$$

   and the radius of motion is

$$\frac{|\dot{\gamma}(0)|}{B_0} .$$

2. If $r = 0$ and $E_0 \neq 0$ then there is an overall drift motion, on top of the circular motion in the direction of the negative vertical axis.

3. If $r = B_0 = 0$ we get the free Newton equations which yield

$$\gamma(t) = \gamma(0) + \dot{\gamma}(0)\, t + \frac{1}{2} E_0 t^2 .$$

3.1.2 The classical hall conductivity

Now we imagine that the electric field is applied and the system is allowed to relax. As this happens, more and more electrons accumulate on the two upper and lower vertical edges of the sample. Suppose it has width w. Then such an accumulation will continue until the external magnetic field is balanced with the voltage building up, at which point we will reach equilibrium, and the forces are equal:

$$-\left(r-\mathrm{i}B_0\right)\dot{\gamma}=E_0\,. \tag{3.1}$$

Moreover, the two-dimensional current density is

$$j \quad = \quad n\dot{\gamma}$$

where n is the charge density, and we have the basical Ohm's law

$$j=\sigma E$$

where σ is the conductivity matrix (for us a complex number) and so putting these two together we find

$$E=\frac{1}{\sigma}n\dot{\gamma}\,.$$

Comparing this with (3.1) we find

$$\frac{1}{\sigma}n=-\left(r-\mathrm{i}B_0\right)$$

or

$$\boxed{\sigma=\frac{n}{-\left(r-\mathrm{i}B_0\right)}}\,.$$

We conclude the following:

1. If $B_0=0$ then $\sigma=-\frac{n}{r}$ which makes sense, since r was related to friction, hence resistivity, and thus it is real. Moreover, it diverges as $r\to 0$, of course.
2. If $B_0\neq 0$ then σ does *not* diverge even as $r\to 0$. Indeed, it becomes purely imaginary (i.e. it corresponds to *transversal* conductivity. We define

$$\sigma_{\text{Hall}} \quad := \quad -\lim_{r\to 0}\mathbb{Im}\left\{\sigma\right\}=\frac{n}{B_0}\,.$$

   I.e., it is the *off-diagonal* matrix element of the conductivity matrix. We see moreover that the longitudinal conductivity (the diagonal matrix elements) is zero (of course) when $r=0$.

In conclusion, from this classical calculation we expect

$$\sigma_{\text{Hall}}$$

to behave linearly as n, the density of electrons, is increased.

3.1.3 Conductivity versus conductance in two-dimensions

TODO

3.2 The quantum Hall effect

Our goal now is to repeat the above calculation of the Hall conductivity within a quantum mechanical framework. We choose to work with non-interacting particles since that already yields some non-trivial results.

The easiest generalization to quantum mechanics of the model above (already with the limit $r=0$) leads one to the Hamiltonian on $L^2\left(\mathbb{R}^2\right)$ given by

$$H \quad = \quad \left(P-B_0A\left(X\right)\right)^2+E_0X_1$$

where $P\equiv\begin{bmatrix}P_1\\P_2\end{bmatrix}$ is the momentum operator, $X\equiv\begin{bmatrix}X_1\\X_2\end{bmatrix}$ is the position operator, and $A:\mathbb{R}^2\to\mathbb{R}^2$ is the magnetic vector potential which corresponds to a constant magnetic field via

$$\operatorname{curl}\left(A\right) \quad = \quad e_3\,.$$

There are two common choices for A: the symmetric gauge which is $A\left(x\right)=\frac{1}{2}e_3\wedge x$ and the Landau gauge $A\left(x\right)=x_1e_2$.

3.2.1 Explicit diagonalization of this Hamiltonian

We choose the Landau gauge $A(x) = x_1 e_2$ so that we have

$$\begin{aligned} H &= (P - B_0 X_1 e_2)^2 + E_0 X_1 \\ &= P_1^2 + (P_2 - B_0 X_1)^2 + E_0 X_1 \,. \end{aligned}$$

Since this Hamiltonian does not depend on X_2 it makes sense to perform partial Fourier transform along the 2-axis so as to obtain

$$\hat{H}(k_2) = P_1^2 + (k_2 - B_0 X_1)^2 + E_0 X_1 \,. \tag{3.2}$$

We complete the square on X_1 to get

$$\begin{aligned} \hat{H}(k_2) &= P_1^2 + \left(B_0 X_1 - k_2 + \frac{E_0}{2B_0}\right)^2 - \left(k_2 - \frac{E_0}{2B_0}\right)^2 + k_2^2 \\ &= P_1^2 + \left(B_0 X_1 - k_2 + \frac{E_0}{2B_0}\right)^2 + \frac{E_0}{B_0} k_2 - \frac{E_0^2}{4B_0^2} \\ &= P_1^2 + B_0^2 \left(X_1 - \frac{k_2}{B_0} + \frac{E_0}{2B_0^2}\right)^2 + \frac{E_0}{B_0} k_2 - \frac{E_0^2}{4B_0^2} \end{aligned}$$

This operator in turn may also be diagonalized: it is merely a 1D harmonic oscillator whose origin has the transformation

$$\tilde{X}_1 := B_0 X_1 - k_2 + \frac{E_0}{2B_0}$$

and which is also shifted in energy by

$$\frac{E_0}{B_0} k_2 - \frac{E_0^2}{4B_0^2} \,.$$

Since we are applying the current along the 1-axis, to calculate the Hall conductivity we should calculate the expectation value of the velocity in the 2-axis (perhaps per unit area), i.e.,

$$\langle V_2 \rangle_F := \lim_{L_1, L_2 \to \infty} \frac{1}{4 L_1 L_2} \operatorname{tr}\left(P_F \mathrm{i} [H, X_2] \chi_{[-L_1, L_1]}(X_1) \chi_{[-L_2, L_2]}(X_2)\right) \,.$$

Here, P_F is the Fermi projection

$$P_F := \chi_{(-\infty, E_F)}(H) \,.$$

Since we have moved to momentum space, we should perform the partial Fourier transform also in this calculation. In momentum space, we have

$$\begin{aligned} \left(\chi_{[-L_2, L_2]}(X_2)\right)(k, p) &= \int_{x_2, \tilde{x}_2 \in \mathbb{R}} \mathrm{e}^{\mathrm{i}(k x_2 - p \tilde{x}_2)} \chi_{[-L_2, L_2]}(x_2, \tilde{x}_2) \, \mathrm{d}x_2 \mathrm{d}\tilde{x}_2 \\ &= \int_{x_2 \in \mathbb{R}} \mathrm{e}^{\mathrm{i}(k-p) x_2} \chi_{[-L_2, L_2]}(x_2) \, \mathrm{d}x_2 \\ &= \frac{2 \sin(L_2 (k - p))}{k - p} \\ &\overset{L_2 \to \infty}{=} \delta(k - p) \text{ in distribution.} \end{aligned}$$

This then yields,

$$\begin{aligned} \langle V_2 \rangle_F &= \lim_{L_1, L_2 \to \infty} \frac{1}{4 L_1 L_2} \int_{k_2 \in \mathbb{R}} \mathrm{d}k_2 \operatorname{tr}_{L^2(\mathbb{R}_{x_1})} \left(\chi_{(-\infty, E_F)}\left(\hat{H}(k_2)\right) \left(\partial_{k_2} \hat{H}\right)(k_2) \chi_{[-L_1, L_1]}(X_1)\right) \\ &= \lim_{L_1, L_2 \to \infty} \frac{1}{2 L_1 L_2} \int_{k_2 \in \mathbb{R}} \mathrm{d}k_2 \operatorname{tr}_{L^2(\mathbb{R}_{x_1})} \left(\chi_{(-\infty, E_F)}\left(\hat{H}(k_2)\right) (k_2 - B_0 X_1) \chi_{[-L_1, L_1]}(X_1)\right) \\ &= \lim_{L_1, L_2 \to \infty} \frac{1}{2 L_1 L_2} \int_{k_2 \in \mathbb{R}} \mathrm{d}k_2 \sum_{j \in \mathbb{N}_{\geq 0} : E_j(k_2) \leq E_F} \left\langle \psi_j(k_2), (k_2 - B_0 X_1) \chi_{[-L_1, L_1]}(X_1) \psi_j(k_2) \right\rangle_{L^2(\mathbb{R}_{x_1})} \end{aligned}$$

where $E_j(k_2)$ and $\psi_j(k_2)$ are the eigen energies and eigen wave-functions of the k_2-dependent 1D harmonic oscillator given in Section 3.2.1. The energy levels are given by

$$E_j(k_2) = B_0(2j+1) + \frac{E_0}{B_0}k_2 - \frac{E_0^2}{4B_0^2} \qquad (j \in \mathbb{N}_{\geq 0})$$

and recall that $k_2 \in \mathbb{R}$. Hence

$$E_j(k_2) \leq E_F$$

means

$$\begin{aligned}
\langle V_2 \rangle_F &= 2\int_{k_2\in\mathbb{R}} \mathrm{d}k_2 \sum_{j\in\mathbb{N}_{\geq 0}} \chi_{\{\, E_j(k_2)\leq E_F \,\}} \langle \psi_j(k_2), (k_2 - B_0 X_1)\psi_j(k_2)\rangle_{L^2(\mathbb{R}_{x_1})} \\
&= 2\sum_{j\in\mathbb{N}_{\geq 0}} \int_{k_2\in\mathbb{R}} \mathrm{d}k_2 \chi_{\{\, E_j(k_2)\leq E_F \,\}} \langle \psi_j(k_2), (k_2 - B_0 X_1)\psi_j(k_2)\rangle_{L^2(\mathbb{R}_{x_1})} \\
&= 2\sum_{j\in\mathbb{N}_{\geq 0}} \int_{k_2 \leq \frac{B_0}{E_0}E_F + \frac{E_0}{4B_0} - \frac{B_0^2}{E_0}(2j+1)} \mathrm{d}k_2 \langle \psi_j(k_2), (k_2 - B_0 X_1)\psi_j(k_2)\rangle_{L^2(\mathbb{R}_{x_1})} .
\end{aligned}$$

Since all the states are normalized,

$$\langle \psi_j(k_2), k_2\psi_j(k_2)\rangle_{L^2(\mathbb{R}_{x_1})} = k_2$$

whereas by the centering we choose,

$$\langle \psi_j(k_2), B_0 X_1\psi_j(k_2)\rangle_{L^2(\mathbb{R}_{x_1})} = k_2 - \frac{E_0}{2B_0}$$

so that

$$\langle \psi_j(k_2), (k_2 - B_0 X_1)\psi_j(k_2)\rangle_{L^2(\mathbb{R}_{x_1})} = \frac{E_0}{2B_0} .$$

As such we find

$$\langle V_2 \rangle_F = \frac{E_0}{B_0}\int_{k_2\in\mathbb{R}} \mathrm{d}k_2 \sum_{j\in\mathbb{N}_{\geq 0}} \chi_{\{\, E_j(k_2)\leq E_F \,\}} .$$

This expression is now obviously infinite, but (this is the part of the argument that is hand-wavy) we recognize

$$\sum_{j\in\mathbb{N}_{\geq 0}} \chi_{\{\, E_j(k_2)\leq E_F \,\}} \mathrm{d}k_2$$

as the differential density of states, which (at least according to the calculation for the Landau Hamiltonian) should correspond to $\frac{B_0}{2\pi}$ for each Landau level included. Since it's not clear how to make sense of this for the present Hamiltonian we proceed instead using a perturbative argument. I.e., since in the experiment $B_0 \gg 1$, we treatment E_0 as a perturbation. As such, let us solve the case $E_0 = 0$ first, which corresponds to the famous Landau Hamiltonian.

3.2.2 The Landau Hamiltonian

We now study the Landau Hamiltonian, which is the operator on $L^2(\mathbb{R}^2)$ given by

$$H = (P - bA(X))^2 \tag{3.3}$$

with $A : \mathbb{R}^2 \to \mathbb{R}^2$ the magnetic vector potential corresponding to

$$B = \operatorname{curl}(A)(x) = e_3 .$$

There are two basic choices for the gauge of A:

1. Symmetric gauge $A(x) = \frac{1}{2}\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$.
2. Landau gauge $A(x) = x_1 e_2$.

Let us choose the symmetric gauge and diagonalize the Hamiltonian.

Let $B \in \mathbb{R}$ be given. Then we define $A(x) := \frac{1}{2} B e_3 \wedge x$ for all $x \in \mathbb{R}^2$ (symmetric gauge). One verifies that $\operatorname{curl}(A) = B e_3$ (i.e. constant magnetic field).

Then

$$\begin{aligned} H &= (P - A(X))^2 \\ &= \left(P - \frac{1}{2} B e_3 \wedge X\right)^2 \\ &\quad \text{(Recall the angular momentum } L \equiv X \wedge P) \\ &= P^2 + \frac{B^2}{4} X^2 - B L_3 \end{aligned}$$

Solution Define

$$\begin{aligned} Z &:= X_1 + \mathrm{i} X_2 \\ &\downarrow \\ Z^* &= X_1 - \mathrm{i} X_2 \end{aligned}$$

Note that Z is not self-adjoint but normal, as $[X_1, X_2] = 0$. Also note that

$$\begin{aligned} \mathbb{Re}\{Z\} &= \frac{1}{2}(X_1 + \mathrm{i}X_2 + X_1 - \mathrm{i}X_2) \\ &= X_1 \end{aligned}$$

and

$$\begin{aligned} \mathbb{Im}\{Z\} &= \frac{1}{2\mathrm{i}}(X_1 + \mathrm{i}X_2 - X_1 + \mathrm{i}X_2) \\ &= X_2 \end{aligned}$$

$$\begin{aligned} |Z|^2 &\equiv Z^* Z \\ &= (X_1 + \mathrm{i}X_2)(X_1 - \mathrm{i}X_2) \\ &= X_1^2 + X_2^2 \\ &\equiv X^2 \end{aligned}$$

We also define

$$D := \frac{\mathrm{i}}{2}(P_1 - \mathrm{i}P_2)$$

so that

$$-D^* = \frac{\mathrm{i}}{2}(P_1 + \mathrm{i}P_2)$$

Again D is not self-adjoint but normal as $[P_1, P_2] = 0$. We calculate

$$\begin{aligned} \mathbb{Re}\{D\} &\equiv \frac{1}{2}\left(\frac{\mathrm{i}}{2}(P_1 - \mathrm{i}P_2) - \frac{\mathrm{i}}{2}(P_1 + \mathrm{i}P_2)\right) \\ &= \frac{\mathrm{i}}{4}(P_1 - \mathrm{i}P_2 - P_1 - \mathrm{i}P_2) \\ &= \frac{1}{2} P_2 \end{aligned}$$

$$\begin{aligned} \mathbb{Im}\{D\} &\equiv \frac{1}{2\mathrm{i}}\left(\frac{\mathrm{i}}{2}(P_1 - \mathrm{i}P_2) + \frac{\mathrm{i}}{2}(P_1 + \mathrm{i}P_2)\right) \\ &= \frac{1}{2} P_1 \end{aligned}$$

$$
\begin{aligned}
|D|^2 &\equiv D^* D \\
&= -\frac{\mathrm{i}}{2}\left(P_1 + \mathrm{i}P_2\right)\frac{\mathrm{i}}{2}\left(P_1 - \mathrm{i}P_2\right) \\
&= \frac{1}{4}\left(P_1 + \mathrm{i}P_2\right)\left(P_1 - \mathrm{i}P_2\right) \\
&= \frac{1}{4}\left(P_1^2 + P_2^2\right) \\
&\equiv \frac{1}{4}P^2
\end{aligned}
$$

and finally

$$
\begin{aligned}
L_3 &\equiv X_1 P_2 - X_2 P_1 \\
&= 2\left(\mathbb{Re}\left\{Z\right\}\mathbb{Re}\left\{D\right\} - \mathbb{Im}\left\{Z\right\}\mathbb{Im}\left\{D\right\}\right) \\
&= 2\left(\frac{1}{2}\left(Z + Z^*\right)\frac{1}{2}\left(D + D^*\right) - \frac{1}{2\mathrm{i}}\left(Z - Z^*\right)\frac{1}{2\mathrm{i}}\left(D - D^*\right)\right) \\
&= \frac{1}{2}\left(\left(Z + Z^*\right)\left(D + D^*\right) + \left(Z - Z^*\right)\left(D - D^*\right)\right) \\
&= \frac{1}{2}\left(ZD + ZD^* + Z^*D + Z^*D^* + ZD - ZD^* - Z^*D + Z^*D^*\right) \\
&= \frac{1}{2}\left(ZD + Z^*D^* + ZD + Z^*D^*\right) \\
&= ZD + Z^*D^*
\end{aligned}
$$

We find that, using the fact that

$$
\begin{aligned}
\left[D, Z\right] &= \left[\frac{\mathrm{i}}{2}\left(P_1 - \mathrm{i}P_2\right), X_1 + \mathrm{i}X_2\right] \\
&= \frac{\mathrm{i}}{2}\left(\underbrace{\left[P_1, X_1\right]}_{=-\mathrm{i}} + \left[P_2, X_2\right]\right) \\
&= \mathbb{1}
\end{aligned}
$$

that

$$
\begin{aligned}
L_3 &= ZD + Z^*D^* \\
&= 2\,\mathbb{Re}\left\{DZ\right\} - DZ + ZD \\
&= 2\,\mathbb{Re}\left\{ZD\right\} + \left[Z, D\right] \\
&= 2\,\mathbb{Re}\left\{ZD\right\} - \mathbb{1}
\end{aligned}
$$

so that finally,

$$
\begin{aligned}
H &= P^2 + \frac{B^2}{4}X^2 - BL_3 \\
&= 4\left|D\right|^2 + \frac{B^2}{4}\left|Z\right|^2 - B\left(2\,\mathbb{Re}\left\{DZ\right\} - \mathbb{1}\right) \\
&= \frac{B^2}{4}\left|Z\right|^2 + 4\left|D^*\right|^2 - BZ^*D^* - BDZ + B\mathbb{1} \\
&= \left(\frac{B}{2}Z^* - 2D\right)\left(\frac{B}{2}Z - 2D^*\right) + B\mathbb{1} \\
&= \left|\frac{B}{2}Z - 2D^*\right|^2 + B\mathbb{1} \\
&= B\left(2\left|\left(\frac{1}{2}\sqrt{\frac{B}{2}}Z - \sqrt{\frac{2}{B}}D^*\right)\right|^2 + \mathbb{1}\right)
\end{aligned}
$$

We redefine $\sqrt{\frac{B}{2}}Z\mapsto Z$ and $\sqrt{\frac{2}{B}}D\mapsto D$ (the commutation relation $[D,Z]=\mathbb{1}$ is unchanged) to get that

$$\frac{1}{B}H \quad = \quad 2\left|\frac{1}{2}Z-D^{*}\right|^{2}+\mathbb{1}$$

Define the ladder operator $C:=\frac{1}{2}Z-D^{*}$. Then we find

$$\frac{1}{B}H \quad = \quad 2\left|C\right|^{2}+\mathbb{1}$$

Note that we have the canonical commutation relations:

$$\begin{aligned}[C^{*},C] &= \left[\frac{1}{2}Z^{*}-D,\frac{1}{2}Z-D^{*}\right]\\ &= \frac{1}{2}\left(\left[Z^{*},D^{*}\right]-\left[D,Z\right]\right)\\ &= -\mathbb{1}\end{aligned}$$

Indeed, we can map the C-star algebra generated by C to $\mathcal{B}\left(\ell^{2}\left(\mathbb{N}_{\geq 0}\right)\right)$ by $C\mapsto R^{*}\sqrt{X}$ where R is the unilateral right shift and X is the position operator. We can then calculate that $\left|C\right|^{2}\mapsto X$, the position operator. We thus find that

$$\sigma\left(2\left|C\right|^{2}+\mathbb{1}\right)=\sigma\left(2X+\mathbb{1}\right)=\left\{\,2n+1\mid n\in\mathbb{N}_{\geq 0}\,\right\}.$$

and so we have found the full spectrum of the Landau Hamiltonian.

Density of states We proceed to calculate the degeneracies.

Let us assume we have a ground state ψ_{0} of $\frac{1}{B}H$. Then

$$\begin{aligned}\left(2\left|C\right|^{2}+\mathbb{1}\right)\psi_{0} &= \psi_{0}\\ \left|C\right|^{2}\psi_{0} &= 0\\ &\left(C^{*}\text{ is invertible}\right)\\ C\psi_{0} &= 0\end{aligned}$$

Now let us calculate

$$\begin{aligned}C &\equiv \frac{1}{2}Z-D^{*}\\ &= -\exp\left(-\frac{1}{2}\left|Z\right|^{2}\right)\left(\exp\left(\frac{1}{2}\left|Z\right|^{2}\right)D^{*}-\frac{1}{2}Z\exp\left(\frac{1}{2}\left|Z\right|^{2}\right)\right)\\ &= -\exp\left(-\frac{1}{2}\left|Z\right|^{2}\right)\left(D^{*}\exp\left(\frac{1}{2}\left|Z\right|^{2}\right)-\underbrace{\left[D^{*},\exp\left(\frac{1}{2}\left|Z\right|^{2}\right)\right]}_{\left[D^{*},Z^{*}\right]\exp\left(\frac{1}{2}\left|Z\right|^{2}\right)\frac{1}{2}Z}-\frac{1}{2}Z\exp\left(\frac{1}{2}\left|Z\right|^{2}\right)\right)\\ &= -\exp\left(-\frac{1}{2}\left|Z\right|^{2}\right)D^{*}\exp\left(\frac{1}{2}\left|Z\right|^{2}\right)\end{aligned}$$

So that

$$-\exp\left(-\frac{1}{2}\left|Z\right|^{2}\right)D^{*}\exp\left(\frac{1}{2}\left|Z\right|^{2}\right)\psi_{0} \quad = \quad 0$$

Since the first exponential is invertible, we find

$$D^{*}\exp\left(\frac{1}{2}\left|Z\right|^{2}\right)\psi_{0} \quad = \quad 0$$

Define $\psi_{0}\left(z\right)=:\exp\left(-\frac{1}{2}\left|z\right|^{2}\right)\tilde{\psi_{0}}\left(z\right)$. Then

$$D^{*}\tilde{\psi_{0}}\left(z\right) \quad = \quad 0$$

But actually, $D^{*}\propto\partial_{\overline{z}}\propto\partial_{1}+\partial_{2}$, i.e. the Cauchy-Riemann equations–we get that $\tilde{\psi_{0}}$ is *any* analytic function. Since the polynomials are analytic and dense in that set, we span the first Landau level with, e.g., the orthogonal polynomials

$$\psi_{0m}(z) \quad := \quad \frac{z^{m}}{\sqrt{\pi m!}}\exp\left(-\frac{1}{z}|z|^{2}\right)$$

The next Landau levels are obtained by applying the creation operator to ψ_{0m}:

$$\psi_{km} \quad := \quad \frac{1}{\sqrt{k!}}\left(C^{*}\right)^{k}\psi_{0m}$$

and one verifies that indeed $\frac{H}{B}\psi_{km}=(2k+1)\,\psi_{km}$ via the canonical commutation relations.

The density of states at energy E is the number of states per unit area, that is, $\sum_{k,m=0}^{\infty}|\psi_{km}(z)|^{2}$. One can show that

$$\begin{aligned}\sum_{m=0}^{\infty}|\psi_{0m}(z)|^{2} &= \sum_{m=0}^{\infty}|\psi_{0m}(0)|^{2}\\ &= \sum_{m=0}^{\infty}\left|\frac{0^{m}}{\sqrt{\pi m!}}\right|^{2}\\ &= \pi^{-1}\end{aligned}$$

Note that $L_{3}\psi_{0m}=m\psi_{0m}$ using the commutation relations. Now $[L_{3},C^{*}]=-C^{*}$ so

$$L_{3}\psi_{km} \quad = \quad (m-k)\,\psi_{km}$$

Since by rotational invariance $L_{3}\psi_{km}=0$, $\psi_{km}(0)$ may only be non-zero when $m=k$. We can calculate that $\psi_{kk}(0)=\frac{1}{\sqrt{\pi}}(-1)^{k}$ so that similarly

$$\sum_{m=0}^{\infty}|\psi_{km}(z)|^{2} \quad = \quad \frac{1}{\pi}$$

Density of states per area per energy:

$$\rho(E) \quad = \quad \sum_{k=0}^{\infty}\frac{1}{\pi}\delta(E-E_{k})$$

$$\begin{aligned}\rho(E) &= \text{trace per unit area}\left(\chi_{<0}(H-E)\right)\\ &= \lim_{\Lambda\to\mathbb{C}}\frac{1}{|\Lambda|}\int_{\Lambda}\chi_{<0}(H-E)(z,z)\,\mathrm{d}z\\ &= \lim_{\Lambda\to\mathbb{C}}\frac{1}{|\Lambda|}\int_{\Lambda}\sum_{k,m=0}^{\infty}|\psi_{km}(z)|^{2}\chi_{<0}(2k+1-E)\,\mathrm{d}z\end{aligned}$$

We find that each Landau level is infinitely degenerate with $\frac{1}{\pi}$ eigenstates per unit area.

and so

$$\text{trace per unit area}\left(\text{projection onto lowest } k \text{ L.L}\right) \quad = \quad B\frac{k}{\pi}$$

Alternate solution to explain degeneracies We go back to

$$\begin{aligned}H &= P^{2}+\frac{B^{2}}{4}X^{2}-BL_{3}\\ &= P^{2}+\omega^{2}X^{2}-BL_{3}\\ &= 2\left(\frac{1}{2}P^{2}+\frac{1}{2}\omega^{2}X^{2}-\frac{1}{2}BL_{3}\right)\end{aligned}$$

We saw above that the precise ratio between ω and B was important for the degeneracy of the Landau levels: $\frac{B}{\omega}=2$. Let us solve the general problem where $\frac{B}{\omega}$ may be different than 2 and see what kind of degeneracies we get.

$$P^2+\omega^2 X^2 = \sum_{j=1}^{2} P_j^2 + \omega^2 X_j^2$$

Let us define the ladder operator

$$A_j \quad := \quad \alpha\left(X_j + \mathrm{i}\beta P_j\right)$$

and calculate

$$\begin{aligned} |A_j|^2 &\equiv A_j^* A_j \\ &= \alpha^2\left(X_j - \mathrm{i}\beta P_j\right)\left(X_j + \mathrm{i}\beta P_j\right) \\ &= \alpha^2\left(X_j^2 + \beta^2 P_j^2 + \mathrm{i}\beta\left[X_j, P_j\right]\right) \\ & \quad \left(\left[X_j, P_j\right] = \mathrm{i}\right) \\ &= \alpha^2\left(X_j^2 + \beta^2 P_j^2 - \beta\right) \\ &= \alpha^2 X_j^2 + \alpha^2\beta^2 P_j^2 - \alpha^2\beta \end{aligned}$$

So that

$$\omega\left(|A_j|^2 + \gamma\right) \quad = \quad \omega\alpha^2 X_j^2 + \omega\alpha^2\beta^2 P_j^2 - \omega\alpha^2\beta + \omega\gamma$$

Let us pick α such that $\omega\alpha^2 = \frac{1}{2}\omega^2$, that is, $\alpha = \sqrt{\frac{\omega}{2}}$ and β such that $\omega\alpha^2\beta^2 = \frac{1}{2}$, so $\beta = \frac{1}{\omega}$. We thus find $-\omega\alpha^2\beta + \omega\gamma = -\frac{1}{2}\omega^2\frac{1}{\omega} + \omega\gamma = -\frac{\omega}{2} + \omega\gamma$. Hence if we pick $\gamma := \frac{1}{2}$ we find

$$\omega\left(|A_j|^2 + \frac{1}{2}\mathbb{1}\right) \quad = \quad \frac{1}{2}P_j^2 + \frac{1}{2}\omega^2 X_j^2$$

and so

$$A_j \quad \equiv \quad \sqrt{\frac{\omega}{2}}\left(X_j + \mathrm{i}\frac{1}{\omega}P_j\right)$$

We verify the commutation relations

$$\begin{aligned} \left[A_j, A_j^*\right] &= \left[\sqrt{\frac{\omega}{2}}\left(X_j + \mathrm{i}\frac{1}{\omega}P_j\right), \sqrt{\frac{\omega}{2}}\left(X_j - \mathrm{i}\frac{1}{\omega}P_j\right)\right] \\ &= \frac{\omega}{2}\left(\frac{-\mathrm{i}}{\omega}\left[X_j, P_j\right] + \frac{\mathrm{i}}{\omega}\left[P_j, X_j\right]\right) \\ &= 1 \end{aligned}$$

We note that

$$\begin{aligned} \sqrt{\frac{2}{\omega}}\,\mathbb{Re}\left\{A_j\right\} &= \sqrt{\frac{2}{\omega}}\frac{1}{2}\left(A_j + A_j^*\right) \\ &= \frac{1}{2}\sqrt{\frac{2}{\omega}}\sqrt{\frac{\omega}{2}}\left(X_j + \mathrm{i}\frac{1}{\omega}P_j\right) + \frac{1}{2}\sqrt{\frac{2}{\omega}}\sqrt{\frac{\omega}{2}}\left(X_j - \mathrm{i}\frac{1}{\omega}P_j\right) \\ &= X_j \end{aligned}$$

and

$$\begin{aligned} \sqrt{2\omega}\,\mathbb{Im}\left\{A_j\right\} &= \sqrt{2\omega}\frac{1}{2\mathrm{i}}\left(A_j - A_j^*\right) \\ &= \frac{\sqrt{2\omega}}{2\mathrm{i}}\left(\sqrt{\frac{\omega}{2}}\left(X_j + \mathrm{i}\frac{1}{\omega}P_j\right) - \sqrt{\frac{\omega}{2}}\left(X_j - \mathrm{i}\frac{1}{\omega}P_j\right)\right) \\ &= P_j \end{aligned}$$

We thus find

$$
\begin{aligned}
L_3 &\equiv (X\wedge P)_3 \\
&= \varepsilon_{3ij}X_jP_j \\
&= X_1P_2 - X_2P_1 \\
&= \sqrt{\frac{2}{\omega}}\,\mathbb{Re}\left\{A_1\right\}\sqrt{2\omega}\,\mathbb{Im}\left\{A_2\right\} - \sqrt{\frac{2}{\omega}}\,\mathbb{Re}\left\{A_2\right\}\sqrt{2\omega}\,\mathbb{Im}\left\{A_1\right\} \\
&= 2\left(\mathbb{Re}\left\{A_1\right\}\mathbb{Im}\left\{A_2\right\} - \mathbb{Re}\left\{A_2\right\}\mathbb{Im}\left\{A_1\right\}\right) \\
&= 2\left(\frac{1}{4\mathrm{i}}\left(A_1+A_1^*\right)\left(A_2-A_2^*\right) - \frac{1}{4\mathrm{i}}\left(A_2+A_2^*\right)\left(A_1-A_1^*\right)\right) \\
&= \frac{1}{2\mathrm{i}}\left(A_1A_2 - A_1A_2^* + A_1^*A_2 - A_1^*A_2^* - A_2A_1 + A_2A_1^* - A_2^*A_1 + A_2^*A_1^*\right) \\
&= \frac{1}{\mathrm{i}}\left(-A_1A_2^* + A_1^*A_2\right) \\
&= 2\,\mathbb{Im}\left\{A_1^*A_2\right\}
\end{aligned}
$$

So that the full Hamiltonian is given by

$$
\begin{aligned}
H &= 2\left(\frac{1}{2}P^2 + \frac{1}{2}\omega^2X^2 - \frac{1}{2}BL_3\right) \\
&= \omega\left(2\left|A_1\right|^2 + \mathbb{1}\right) + \omega\left(2\left|A_2\right|^2 + \mathbb{1}\right) - 2B\,\mathbb{Im}\left\{A_1^*A_2\right\}
\end{aligned}
$$

Note that without the Zeeman term there is no infinite degeneracy.

The Zeeman term is a bit painful to handle so let us rotate to another basis where it's easier to see the spectrum.

$$
B_1 := \frac{1}{\sqrt{2}}\left(A_1 + \mathrm{i}A_2\right)
$$

$$
B_2 := \frac{1}{\sqrt{2}}\left(A_1 - \mathrm{i}A_2\right)
$$

Then

$$
\begin{aligned}
\left|B_1\right|^2 &= \frac{1}{2}\left(A_1+\mathrm{i}A_2\right)^*\left(A_1+\mathrm{i}A_2\right) \\
&= \frac{1}{2}\left(A_1^* - \mathrm{i}A_2^*\right)\left(A_1+\mathrm{i}A_2\right) \\
&= \frac{1}{2}\left(\left|A_1\right|^2 + \mathrm{i}\left(A_1^*A_2 - A_2^*A_1\right) + \left|A_2\right|^2\right) \\
&= \frac{1}{2}\left(\left|A_1\right|^2 + \left|A_2\right|^2 + 2\,\mathbb{Im}\left\{A_2^*A_1\right\}\right)
\end{aligned}
$$

$$
\begin{aligned}
\left|B_2\right|^2 &= \frac{1}{2}\left(A_1^* + \mathrm{i}A_2^*\right)\left(A_1-\mathrm{i}A_2\right) \\
&= \frac{1}{2}\left(\left|A_1\right|^2 + \left|A_2\right|^2 + 2\,\mathbb{Im}\left\{A_1^*A_2\right\}\right) \\
&= \frac{1}{2}\left(\left|A_1\right|^2 + \left|A_2\right|^2 - 2\,\mathbb{Im}\left\{A_2^*A_1\right\}\right)
\end{aligned}
$$

We find that

$$
\left|B_1\right|^2 + \left|B_2\right|^2 = \left|A_1\right|^2 + \left|A_2\right|^2
$$

yet

$$
\left|B_1\right|^2 - \left|B_2\right|^2 = 2\,\mathbb{Im}\left\{A_1^*A_2\right\}
$$

The moral of the story is that

$$\begin{aligned} H &= \omega\left(2\left|A_1\right|^2+\mathbb{1}\right)+\omega\left(2\left|A_2\right|^2+\mathbb{1}\right)-2B\,\mathbb{Im}\left\{A_1^*A_2\right\} \\ &= 2\omega\left(\left|A_1\right|^2+\left|A_2\right|^2\right)+2\omega\mathbb{1}-2B\,\mathbb{Im}\left\{A_1^*A_2\right\} \\ &= 2\omega\left(\left|B_1\right|^2+\left|B_2\right|^2+\mathbb{1}\right)-B\left(\left|B_1\right|^2-\left|B_2\right|^2\right) \end{aligned}$$

and here lies the miracle: if $B=2\omega$, then

$$H = B\left(2\left|B_2\right|^2+\mathbb{1}\right)$$

and since H does not depend on $\left|B_1\right|^2$, we get infinite degeneracy for each level of $\left|B_2\right|^2$.

The case $B=2\omega$ only happens when we have pure magnetic field and no perturbing harmonic potential. Otherwise, if we *do* have additional harmonic potential, i.e., if the whole Hamiltonian is of the form

$$\begin{aligned} H &= P^2+\frac{B^2}{4}X^2-BL_3+\frac{1}{2}\Omega^2X^2 \\ &= P^2+\left(\frac{B^2}{4}+\frac{1}{2}\Omega^2\right)X^2-BL_3 \\ &= P^2+\left(\sqrt{\frac{B^2}{4}+\frac{1}{2}\Omega^2}\right)^2X^2-BL_3 \end{aligned}$$

Then asking for $B=2\omega$ means (solving for Ω)

$$\begin{aligned} B &= 2\sqrt{\frac{B^2}{4}+\frac{1}{2}\Omega^2} \\ \frac{B^2}{4} &= \frac{B^2}{4}+\frac{1}{2}\Omega^2 \\ \Omega &= 0 \end{aligned}$$

I.e. any non-zero value of Ω will lift the infinite Landau degeneracy.

In conclusion the energy levels of a 2D harmonic oscillator with frequency Ω in uniform magnetic field B are given by

$$E_{n_1,n_2} = 2\sqrt{\frac{B^2}{4}+\frac{1}{2}\Omega^2}\left(n_1+n_2+1\right)-B\left(n_1-n_2\right) \qquad \left(n_1,n_2\in\mathbb{N}_{\geq 0}\right)$$

The first level equals

$$E_{0,0} = 2\sqrt{\frac{B^2}{4}+\frac{1}{2}\Omega^2}$$

the second level equals

$$E_{1,0} = 4\sqrt{\frac{B^2}{4}+\frac{1}{2}\Omega^2}-B$$

$$E_{0,1} = 4\sqrt{\frac{B^2}{4}+\frac{1}{2}\Omega^2}+B$$

Hence the first gap size is

$$\begin{aligned} E_{1,0}-E_{0,0} &= 4\sqrt{\frac{B^2}{4}+\frac{1}{2}\Omega^2}-B-2\sqrt{\frac{B^2}{4}+\frac{1}{2}\Omega^2} \\ &= 2\sqrt{\frac{B^2}{4}+\frac{1}{2}\Omega^2}-B \\ &= \sqrt{B^2+2\Omega^2}-B \\ &\sim \sqrt{\lambda^2+2\lambda^2}-\lambda \\ &= \lambda\left(\sqrt{3}-1\right) \end{aligned}$$

3.3 A double-commutator formula for the Hall conductivity

We now go back to Theorem 1.70, which says that

$$\sigma_{\mathrm{Hall}} = \mathrm{i}\mathrm{tr}\left(P\left[\left[\Lambda_1, P\right], \left[\Lambda_2, P\right]\right]\right)$$

where we mean the Hall conductivity at zero temperature at Fermi energy E_F (so that $P \equiv \chi_{(-\infty,E_F)}(H)$). Let us derive that formula.

In order to do so, we start from the expression for the linear response (1.18) which says that

$$\sigma_{\mathrm{Hall}} = -\mathrm{i}\int_{-\infty}^{0}\mathrm{tr}\left(\mathrm{e}^{-\mathrm{i}tH}B\mathrm{e}^{\mathrm{i}tH}\left[A,P\right]\right)f\left(t\right)\mathrm{d}t$$

where B is the observable and A is the perturbation. For us, the observable is the velocity along the 1-axis and the perturbation is the electric voltage across the 2-axis.

Our model for an electric field can be either a constant one, in which case we would have

$$E\left(x\right) = E_0 e_2$$

or instead we can consider a delta function

$$E\left(x\right) = E_0\delta\left(x_2\right)e_2\,.$$

The integral of this (the voltage) is given by

$$A = -E_0\underbrace{\Theta\left(x_2\right)}_{=:\Lambda_2}$$

Moreover, the observable is going to be the amount of charge passed from left to right, per unit time, i.e., the time derivative of that. So we can take that observable as

$$B := \mathrm{i}\left[H, \Lambda_1\right]\,.$$

Hence all together we have

$$\sigma_{\mathrm{Hall}} = \lim_{\varepsilon\to 0^+}\int_{-\infty}^{0}\mathrm{tr}\left(\mathrm{e}^{-\mathrm{i}tH}\mathrm{i}\left[H,\Lambda_1\right]\mathrm{e}^{\mathrm{i}tH}\mathrm{i}\left[\Lambda_2,P\right]\right)\mathrm{e}^{\varepsilon t}\mathrm{d}t\,.$$

Observe that

$$\partial_t\left(\mathrm{e}^{-\mathrm{i}tH}\Lambda_1\mathrm{e}^{\mathrm{i}tH}-\Lambda_1\right) = \mathrm{e}^{-\mathrm{i}tH}\mathrm{i}\left[H,\Lambda_1\right]\mathrm{e}^{\mathrm{i}tH}$$

using that and integration by parts, and the fact that

$$t \mapsto \mathrm{e}^{-\mathrm{i}tH}\Lambda_1\mathrm{e}^{\mathrm{i}tH}-\Lambda_1$$

is zero at $t = 0$ (so there is no boundary term) we get

$$\sigma_{\mathrm{Hall}} = \lim_{\varepsilon\to 0^+}\mathrm{i}\varepsilon\mathrm{tr}\int_{-\infty}^{0}\mathrm{e}^{\varepsilon t}\left(\mathrm{e}^{-\mathrm{i}tH}\Lambda_1\mathrm{e}^{\mathrm{i}tH}-\Lambda_1\right)\left[\Lambda_2,P\right]\mathrm{d}t\,.$$

Next, we claim that

> *Claim* 3.2. We have
>
> $$\left[\Lambda_2,P\right] = P^{\perp}\left[\Lambda_2,P\right]P + P\left[\Lambda_2,P\right]P^{\perp}\,.$$

so we find

$$\begin{aligned}\sigma_{\mathrm{Hall}} &= \lim_{\varepsilon\to 0^+}\mathrm{i}\varepsilon\mathrm{tr}\int_{-\infty}^{0}\mathrm{e}^{\varepsilon t}\left(\Lambda_1\left(t\right)-\Lambda_1\right)\left(P^{\perp}\left[\Lambda_2,P\right]P+P\left[\Lambda_2,P\right]P^{\perp}\right)\mathrm{d}t\\ &= \lim_{\varepsilon\to 0^+}\mathrm{i}\varepsilon\mathrm{tr}\int_{-\infty}^{0}\mathrm{e}^{\varepsilon t}\left(P^{\perp}\left(\Lambda_1\left(t\right)-\Lambda_1\right)P+P\left(\Lambda_1\left(t\right)-\Lambda_1\right)P^{\perp}\right)\left[\Lambda_2,P\right]\mathrm{d}t\,.\end{aligned}$$

Next,

> *Claim* 3.3. We have
>
> $$\lim_{\varepsilon\to 0^+}\mathrm{i}\varepsilon\mathrm{tr}\int_{-\infty}^{0}\mathrm{e}^{\varepsilon t}\left(P^{\perp}\Lambda_1\left(t\right)P+P\Lambda_1\left(t\right)P^{\perp}\right)\left[\Lambda_2,P\right]\mathrm{d}t = 0\,.$$

Proof. Each term is zero separately. We write using the projection-valued measure that

$$P = \int_{\lambda=-\infty}^{\mu} \mathrm{d}P(\lambda)$$

so we get

$$\begin{aligned}
\lim_{\varepsilon\to0^+} \mathrm{i}\varepsilon\mathrm{tr}\int_{-\infty}^{0} \mathrm{e}^{\varepsilon t} P\Lambda_1(t)P^\perp[\Lambda_2,P]\,\mathrm{d}t &= \lim_{\varepsilon\to0^+} \mathrm{i}\varepsilon\mathrm{tr}\int_{-\infty}^{0}\mathrm{e}^{\varepsilon t}\int_{\lambda_1=-\infty}^{\mu}\mathrm{d}P(\lambda_1)\Lambda_1(t)\int_{\lambda_2=\mu}^{\infty}\mathrm{d}P(\lambda_2)[\Lambda_2,P]\,\mathrm{d}t \\
&= \lim_{\varepsilon\to0^+}\mathrm{tr}\int_{\lambda_1=-\infty}^{\mu}\int_{\lambda_2=\mu}^{\infty}\int_{\lambda_1=-\infty}^{\mu}\mathrm{d}P(\lambda_1)\Lambda_1\mathrm{d}P(\lambda_2)[\Lambda_2,P]\,\mathrm{i}\varepsilon\int_{-\infty}^{0}\mathrm{e}^{\varepsilon t-\mathrm{i}(\lambda_2-\lambda_1)t}\mathrm{d}t \\
&= \lim_{\varepsilon\to0^+}\mathrm{tr}\int_{\lambda_1=-\infty}^{\mu}\int_{\lambda_2=\mu}^{\infty}\int_{\lambda_1=-\infty}^{\mu}\mathrm{d}P(\lambda_1)\Lambda_1\mathrm{d}P(\lambda_2)[\Lambda_2,P]\underbrace{\frac{-\varepsilon}{\lambda_2-\lambda_1+\mathrm{i}\varepsilon}}_{\varepsilon\to0^+\text{ if }\lambda_2>\lambda_1}.
\end{aligned}$$

□

So we find

$$\sigma_{\mathrm{Hall}} = -\mathrm{itr}\left(\left(P\Lambda_1P^\perp+P^\perp\Lambda_1P\right)[\Lambda_2,P]\right)$$

Now we claim that

$$P\Lambda_1P^\perp+P^\perp\Lambda_1P = [[\Lambda_1,P],P]$$

and moreover,

$$\mathrm{tr}([A,B]C) = -\mathrm{tr}(B[A,C])$$

from which we finally find

$$\sigma_{\mathrm{Hall}} = \mathrm{itr}(P[[\Lambda_1,P],[\Lambda_2,P]])\,.$$

Now that we have derived this formula, it might be a good idea to argue why at all it makes sense, i.e., we want to show that

Claim 3.4. If P is a local operator (as in (1.2)) and Λ_1,Λ_2 are the two projections on $\ell^2\left(\mathbb{Z}^2\right)$ onto the right and upper half planes respectively, then

$$[\Lambda_1,P][\Lambda_2,P]\in\mathcal{J}_1(\mathcal{H})\,.$$

As a result, clearly that means (by the ideal property of $\mathcal{J}_1(\mathcal{H})$, see [Sha24]) that $P[[\Lambda_1,P],[\Lambda_2,P]]\in\mathcal{J}_1(\mathcal{H})$ so that the formula makes sense.

Proof. First we derive the fact that a commutator with such a projection exhibits decay also in the diagonal direction:

$$\begin{aligned}
\left\|[\Lambda_j,P]_{xy}\right\| &= |\Lambda_j(x)-\Lambda_j(y)|\,\|P_{xy}\| \\
&\leq |\Lambda_j(x)-\Lambda_j(y)|\,C\mathrm{e}^{-\mu\|x-y\|}\,.
\end{aligned}$$

Now the expression

$$|\Lambda_j(x)-\Lambda_j(y)| = \begin{cases}1 & (x_j\leq0\wedge y_j>0)\vee(x_j>0\wedge y_j\leq0)\\ 0 & \text{else}\end{cases}.$$

Since we have

$$\|x-y\|\geq\frac{1}{\sqrt{d}}\|x-y\|_1\equiv\frac{1}{\sqrt{d}}(|x_1-y_1|+|x_2-y_2|)$$

then we find that under the constraint of $|\Lambda_j(x)-\Lambda_j(y)|$ we get

$$\begin{aligned}|x_j-y_j| &= \begin{cases}x_j-y_j & x_j>0\wedge y_j\leq 0\\ y_j-x_j & x_j\leq 0\wedge y_j>0\end{cases}\\ &= |x_j|+|y_j|\,.\end{aligned}$$

As such, we find

$$\begin{aligned}\left\|[\Lambda_j,P]_{xy}\right\| &\leq |\Lambda_j(x)-\Lambda_j(y)|\,C\mathrm{e}^{-\mu\|x-y\|}\\ &\leq |\Lambda_j(x)-\Lambda_j(y)|\,C\mathrm{e}^{-\frac{1}{2}\mu\|x-y\|}\mathrm{e}^{-\frac{1}{2}\frac{\mu}{\sqrt{d}}(|x_j|+|y_j|)}\\ &\leq C\mathrm{e}^{-\frac{1}{2}\mu\|x-y\|}\mathrm{e}^{-\frac{1}{2}\frac{\mu}{\sqrt{d}}(|x_j|+|y_j|)}\,.\end{aligned}$$

Next, we use the estimate on the trace-class norm as

$$\|A\|_1\leq\sum_{x,y\in\mathbb{Z}^d}\|A_{xy}\|\,.$$

Indeed, this is true since (using the definition of the trace of an operator as in [Sha24])

$$\begin{aligned}\|A\|_1 &\equiv \operatorname{tr}(|A|)\\ &= \sum_n\langle\varphi_n,|A|\,\varphi_n\rangle\\ &\overset{\text{Cauchy-Schwarz}}{\leq} \sum_n\||A|\,\varphi_n\|\\ &= \sum_n\|A\varphi_n\|\\ &\overset{\|\psi\|\leq\sum_n|\langle\varphi_n,\psi\rangle|}{\leq} \sum_{n,m}|\langle\varphi_n,A\varphi_m\rangle|\,.\end{aligned}$$

As a result, we get

$$\begin{aligned}\|[\Lambda_1,P]\,[\Lambda_2,P]\|_1 &\leq \sum_{x,y\in\mathbb{Z}^d}\left\|([\Lambda_1,P]\,[\Lambda_2,P])_{xy}\right\|\\ &\leq \sum_{x,y,z\in\mathbb{Z}^d}\|[\Lambda_1,P]_{xz}\|\left\|[\Lambda_2,P]_{zy}\right\|\\ &\leq C^2\sum_{x,y,z\in\mathbb{Z}^d}\mathrm{e}^{-\frac{1}{2}\mu(\|x-z\|+\|z-y\|)}\mathrm{e}^{-\frac{1}{2}\frac{\mu}{\sqrt{d}}(|x_1|+|z_1|)}\mathrm{e}^{-\frac{1}{2}\frac{\mu}{\sqrt{d}}(|z_2|+|y_2|)}\\ &\leq C^2\sum_{x,y,z\in\mathbb{Z}^d}\mathrm{e}^{-\frac{1}{2}\mu(\|x-z\|+\|z-y\|)}\mathrm{e}^{-\frac{1}{2}\frac{\mu}{\sqrt{d}}(|z_1|+|z_2|)}\\ &= C^2\sum_{x,y,z\in\mathbb{Z}^d}\mathrm{e}^{-\frac{1}{2}\mu(\|x\|+\|y\|)}\mathrm{e}^{-\frac{1}{2}\frac{\mu}{\sqrt{d}}(|z_1|+|z_2|)}\\ &\leq C^2\left(D_{\frac{1}{2}\mu,d}\right)^2\left(D_{\frac{1}{2}\frac{\mu}{\sqrt{d}},1}\right)^2\\ &< \infty\,.\end{aligned}$$

where in the penultimate line we have used the definition (1.7)

$$D_{\nu,d}:=\sum_{z\in\mathbb{Z}^d}\mathrm{e}^{-\nu\|z\|}\,.$$

Anyway we see we obtain a finite sum so that we indeed derived the trace-class property. □

Next, we want to justify that the expression for σ_{Hall} is actually insensitive to the choice of Λ_1 and Λ_2. In particular they *do not* have to be step functions, they can be finite rank or even trace-class perturbations of step functions.

Claim 3.5. If Λ_j and $\tilde{\Lambda}_j$ differ on a finite number of sites, then

$$\begin{aligned}\operatorname{tr}\left(P\left[\left[\Lambda_1,P\right],\left[\Lambda_2,P\right]\right]\right) &= \operatorname{tr}\left(P\left[\left[\tilde{\Lambda}_1,P\right],\left[\Lambda_2,P\right]\right]\right)\\ &= \operatorname{tr}\left(P\left[\left[\Lambda_1,P\right],\left[\tilde{\Lambda}_2,P\right]\right]\right)\,.\end{aligned}$$

Proof. We show the proof changing only one of the axes. We have

$$\begin{aligned}P\left[\left[\Lambda_1,P\right],\left[\Lambda_2,P\right]\right] &= P\left[\Lambda_1,P\right]\left[\Lambda_2,P\right]-P\left[\Lambda_2,P\right]\left[\Lambda_1,P\right]\\ &= P\Lambda_1P\Lambda_2P-P\Lambda_1\Lambda_2P-P\Lambda_1P\Lambda_2+P\Lambda_1P\Lambda_2-\\ &\quad -P\Lambda_2P\Lambda_1P+P\Lambda_2\Lambda_1P+P\Lambda_2P\Lambda_1-P\Lambda_2P\Lambda_1\\ &= P\Lambda_1P\Lambda_2P-P\Lambda_2P\Lambda_1P\\ &= \left[P\Lambda_1P,P\Lambda_2P\right]\,.\end{aligned} \tag{3.4}$$

Then consider

$$\begin{aligned}\Delta &:= \operatorname{tr}\left(P\left[\left[\Lambda_1,P\right],\left[\Lambda_2,P\right]\right]\right)-\operatorname{tr}\left(P\left[\left[\tilde{\Lambda}_1,P\right],\left[\Lambda_2,P\right]\right]\right)\\ &= \operatorname{tr}\left(\left[P\left(\Lambda_1-\tilde{\Lambda}_1\right)P,P\Lambda_2P\right]\right)\,.\end{aligned}$$

Now, by hypothesis, $\Lambda_1-\tilde{\Lambda}_1$ is a finite rank operator, so in particular it is trace-class so that we could open the commutator and use cyclicity for free, to get

$$\begin{aligned}\Delta &= \operatorname{tr}\left(\left[P\left(\Lambda_1-\tilde{\Lambda}_1\right)P,P\Lambda_2P\right]\right)\\ &= \operatorname{tr}\left(P\left(\Lambda_1-\tilde{\Lambda}_1\right)P\Lambda_2P-P\Lambda_2P\left(\Lambda_1-\tilde{\Lambda}_1\right)P\right)\\ &= \operatorname{tr}\left(P\left(\Lambda_1-\tilde{\Lambda}_1\right)P\Lambda_2P\right)-\operatorname{tr}\left(P\Lambda_2P\left(\Lambda_1-\tilde{\Lambda}_1\right)P\right)\\ &= \operatorname{tr}\left(P\left(\Lambda_1-\tilde{\Lambda}_1\right)P\Lambda_2P\right)-\operatorname{tr}\left(P\left(\Lambda_1-\tilde{\Lambda}_1\right)P\Lambda_2P\right)\\ &= 0\,.\end{aligned}$$

□

3.4 Integrality of the Hall conductivity via the Kitaev formula

Theorem 3.6. *We have*

$$\sigma_{Hall}=\frac{1}{2\pi}\operatorname{index}\left(\Lambda_1\mathrm{e}^{-2\pi\mathrm{i}\Lambda_2P\Lambda_2}\Lambda_1+\Lambda_1^{\perp}\right) \tag{3.5}$$

so in particular

$$2\pi\sigma_{Hall}\in\mathbb{Z}\,.$$

In particular the RHS of (3.5) is called "the Kitaev index" taken from

Proof. Inserting Section 3.3 into the definition of σ_{Hall} we get

$$\begin{aligned}
\frac{1}{\mathrm{i}}\sigma_{\text{Hall}} &= \operatorname{tr}\left([P\Lambda_1 P, P\Lambda_2 P]\right) \\
&= \left(\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\alpha\right)\operatorname{tr}\left([P\Lambda_1 P, P\Lambda_2 P]\right) \\
&= \frac{1}{2\pi}\int_0^{2\pi}\operatorname{tr}\left(\mathrm{e}^{-\mathrm{i}\alpha P\Lambda_2 P}\left[P\Lambda_1 P, P\Lambda_2 P\right]\mathrm{e}^{\mathrm{i}\alpha P\Lambda_2 P}\right)\mathrm{d}\alpha \\
&= \frac{1}{2\pi\mathrm{i}}\int_0^{2\pi}\operatorname{tr}\left(\partial_\alpha \mathrm{e}^{-\mathrm{i}\alpha P\Lambda_2 P}P\Lambda_1 P\mathrm{e}^{\mathrm{i}\alpha P\Lambda_2 P}\right)\mathrm{d}\alpha \\
&= \frac{1}{2\pi\mathrm{i}}\operatorname{tr}\int_0^{2\pi}\partial_\alpha \mathrm{e}^{-\mathrm{i}\alpha P\Lambda_2 P}P\Lambda_1 P\mathrm{e}^{\mathrm{i}\alpha P\Lambda_2 P}\mathrm{d}\alpha \\
&= \frac{1}{2\pi\mathrm{i}}\operatorname{tr}\left(\mathrm{e}^{-\mathrm{i}2\pi P\Lambda_2 P}P\Lambda_1 P\mathrm{e}^{\mathrm{i}2\pi P\Lambda_2 P} - P\Lambda_1 P\right) \\
&= \frac{1}{2\pi\mathrm{i}}\operatorname{tr}\left(\mathrm{e}^{-\mathrm{i}2\pi P\Lambda_2 P}\underbrace{\left[P\Lambda_1 P, \mathrm{e}^{\mathrm{i}2\pi P\Lambda_2 P}\right]}_{P\left[\Lambda_1, \mathrm{e}^{\mathrm{i}2\pi P\Lambda_2 P}\right]P}\right) \\
&= \frac{1}{2\pi\mathrm{i}}\operatorname{tr}\left(P\mathrm{e}^{-\mathrm{i}2\pi P\Lambda_2 P}P\left[\Lambda_1, \mathrm{e}^{\mathrm{i}2\pi P\Lambda_2 P}\right]\right)\,.
\end{aligned}$$

Next, note that

$$\begin{aligned}
\mathrm{e}^{-\mathrm{i}2\pi P\Lambda_2 P} &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\mathrm{i}2\pi P\Lambda_2 P\right)^n \\
&= \mathbb{1} + \sum_{n=1}^{\infty}\frac{1}{n!}\left(-\mathrm{i}2\pi P\Lambda_2 P\right)^n \\
&= P + P^\perp + P\sum_{n=1}^{\infty}\frac{1}{n!}\left(-\mathrm{i}2\pi P\Lambda_2 P\right)^n P \\
&= P + P^\perp + P\left(\mathrm{e}^{-\mathrm{i}2\pi P\Lambda_2 P} - \mathbb{1}\right)P \\
&= P^\perp + P\mathrm{e}^{-\mathrm{i}2\pi P\Lambda_2 P}P\,.
\end{aligned}$$

However,

$$\begin{aligned}
\operatorname{tr}\left(P^\perp\left[\Lambda_1, \mathrm{e}^{\mathrm{i}2\pi P\Lambda_2 P}\right]\right) &= \operatorname{tr}\left(P^\perp\left[\Lambda_1, \mathrm{e}^{\mathrm{i}2\pi P\Lambda_2 P}\right]P^\perp\right) \\
&= \operatorname{tr}\left(P^\perp\Lambda_1\mathrm{e}^{\mathrm{i}2\pi P\Lambda_2 P}P^\perp\right) - \operatorname{tr}\left(P^\perp\mathrm{e}^{\mathrm{i}2\pi P\Lambda_2 P}\Lambda_1 P^\perp\right) \\
&= 0 - 0\,.
\end{aligned}$$

Hence,

$$\sigma_{\text{Hall}} = \frac{1}{2\pi}\operatorname{tr}\left(\mathrm{e}^{-\mathrm{i}2\pi P\Lambda_2 P}\left[\Lambda_1, \mathrm{e}^{\mathrm{i}2\pi P\Lambda_2 P}\right]\right)\,.$$

Now, this expression is of the form

$$\operatorname{tr}\left(U^*\left[Q, U\right]\right)$$

where U is a unitary and Q is a projection, so we can finish using the lemma Lemma 3.7 right below. □

The following lemma is taken from [ASS94, Prop 2.4]:

Lemma 3.7 (Index of pair in the easy case)**.** *If U is a unitary and Q is a projection such that $[Q, U] \in \mathcal{J}_1$ then*

$$\operatorname{tr}\left(U^*\left[U, Q\right]\right) = \operatorname{index}\left(QUQ + Q^\perp\right)\,.$$

Proof. Since $\mathcal{J}_1(\mathcal{H}) \subseteq \mathcal{K}(\mathcal{H})$ (see [Sha24]) we may employ **????** right below to find that $QUQ+Q^\perp$ is Fredholm so the right hand side makes sense at all. Next, using Fedosov's formula [Sha24, Theorem 9.78]:

$$\begin{aligned} \operatorname{index}\left(QUQ+Q^\perp\right) &= \operatorname{tr}\left(\left(QUQ+Q^\perp\right)\left(QU^*Q+Q^\perp\right)-\left(QU^*Q+Q^\perp\right)\left(QUQ+Q^\perp\right)\right) \\ &= \operatorname{tr}\left(QUQU^*Q-Q\right)-\operatorname{tr}\left(QU^*QUQ-Q\right) . \end{aligned}$$

Let us denote the projection $R := U^*QU$ so we get

$$\begin{aligned} \operatorname{index}\left(QUQ+Q^\perp\right) &= \operatorname{tr}\left(U^*\left(QUQU^*Q-Q\right)U\right)-\operatorname{tr}\left(QRQ-Q\right) \\ &= \operatorname{tr}\left(RQR-R\right)-\operatorname{tr}\left(QRQ-Q\right) . \end{aligned}$$

Actually, note that

$$\begin{aligned} Q\left(Q-R\right)^2 &= Q\left(Q-R\right)\left(Q-R\right)=\left(Q-QR\right)\left(Q-R\right) \\ &= Q-QR-QRQ+QR \\ &= Q-QRQ \\ &= Q-QRQ-RQ+RQ \\ &= \left(Q-R\right)\left(Q-RQ\right) \\ &= \left(Q-R\right)\left(Q-R\right)Q \\ &= \left(Q-R\right)^2 Q . \end{aligned}$$

As a result, $\left[Q,(Q-R)^2\right]=0$. Similarly, also $\left[R,(Q-R)^2\right]=0$. As such,

$$\begin{aligned} Q-QRQ &= \left(Q-R\right)^2 Q \\ R-RQR &= \left(Q-R\right)^2 R \end{aligned}$$

and taking the difference of the two, we find

$$Q-QRQ-\left(R-RQR\right)=\left(Q-R\right)^2\left(Q-R\right)=\left(Q-R\right)^3 .$$

But $Q-R=Q-U^*QU=U^*[U,Q]$. Hence, we find that

$$\operatorname{tr}\left(RQR-R\right)-\operatorname{tr}\left(QRQ-Q\right) = \operatorname{tr}\left(\left(Q-R\right)^3\right) .$$

Next, we claim that

$$\operatorname{tr}\left(\left(Q-R\right)^3\right) = \operatorname{tr}\left(Q-R\right) .$$

Indeed,

$$\begin{aligned} \left(Q-R\right)^3 &= Q-R-QRQ+RQR \\ &= Q-R-[QR,RQ] \\ &= Q-R-[QR,[R,Q-R]] . \end{aligned}$$

Since $Q-R=Q-U^*QU=U^*[Q,U]$ is trace-class by assumption, the second term is a total commutator with a trace-class operator which therefore vanishes, and so we get the result. □

Lemma 3.8. *If Q is a projection and U is unitary such that $[Q,U]$ is compact then $QUQ+Q^\perp$ is Fredholm.*

Proof. Using Atkinson's theorem [Sha24, Theorem 9.51], to show that $QUQ+Q^\perp$ is Fredholm it is enough to show that it has a parametrix. Indeed, $QU^*Q+Q^\perp$ is the desired parametrix:

$$\begin{aligned}\mathbb{1}-\left(QU^*Q+Q^\perp\right)\left(QUQ+Q^\perp\right) &= Q-QU^*QUQ\\ &= Q\left(\mathbb{1}-U^*QU\right)Q\\ &= Q\left(U^*U-U^*QU\right)Q\\ &= QU^*\left(\mathbb{1}-Q\right)UQ\\ &= QU^*Q^\perp UQ\\ &= QU^*Q^\perp\left[U,Q\right]\\ &\in \mathcal{K}\left(\mathcal{H}\right).\end{aligned}$$

and similarly for $\mathbb{1}-\left(QUQ+Q^\perp\right)\left(QU^*Q+Q^\perp\right)$. We get that $QUQ+Q^\perp$ indeed has a parametrix so it is Fredholm. ☐

3.5 The Laughlin index

On $\ell^2\left(\mathbb{Z}^2\right)$, define the operator

$$U=\exp\left(\mathrm{i}\arg\left(X_1+\mathrm{i}X_2\right)\right)$$

called the Laughlin flux insertion. This allows us yet another expression for the Hall conductivity as a Fredholm index, alternative to (3.5).

Theorem 3.9. *We also have*

$$\sigma_{Hall} \;=\; \frac{1}{2\pi}\mathrm{index}\left(PUP+P^\perp\right)$$

where U is the Laughlin flux insertion.

Proof. The first order of business is to show that $PUP+P^\perp\in\mathcal{F}$. To that end, we show that $[P,U]\in\mathcal{K}$, since, then we could employ **????**. To that end, we use Lemma 3.11 just below, applying the locality of P and the properties of $U=f\left(X\right)$ with f the polar part of a complex number (see Lemma 3.10 just below) we find that $[P,U]$ is indeed Schatten-3, hence compact, so that $PUP+P^\perp$ is Fredholm.

Next, we wish to perform a norm-continuous deformation from the Fredholm operator

$$PUP+P^\perp\to\Lambda_1\mathrm{e}^{-2\pi\mathrm{i}\Lambda_2P\Lambda_2}\Lambda_1+\Lambda_1^\perp\,.$$

For convenience, we introduce some notation:

$$\mathbb{P}U\equiv PUP+P^\perp\,,\qquad \mathbb{A}_1W\equiv\Lambda_1W\Lambda_1+\Lambda_1^\perp\,.$$

Let us start by deforming U. We have

$$U\equiv f\left(X\right)$$

where $f\left(z\right)\equiv\frac{z}{|z|}$ for all $z\in\mathbb{C}\setminus\{\,0\,\}$. Replacing $f\left(X\right)$ with $f\left(X-a\right)$ for any $a\in\mathbb{C}\setminus\mathbb{Z}^2$ is a norm continuous perturbation since

$$\left\|f\left(X\right)-g\left(X\right)\right\| \;\leq\; \sup_{x\in\mathbb{Z}^2}\left|f\left(x\right)-g\left(x\right)\right|\,.$$

Hence we have

$$\mathrm{index}\left(\mathbb{P}U\right)=\mathrm{index}\left(\mathbb{P}f\left(X-a\right)\right)\qquad\left(a\in\mathbb{C}\setminus\mathbb{Z}^2\right)\,.$$

Next, let $\varphi:\mathbb{S}^1\to\mathbb{S}^1$ be any continuous function of winding 1. Then

$$\mathrm{index}\left(\mathbb{P}f\left(X-a\right)\right)=\mathrm{index}\left(\mathbb{P}\mathrm{e}^{\mathrm{i}\varphi\left(\arg\left(X-a\right)\right)}\right)\,.$$

In particular, choose, for some $\nu < \frac{\pi}{2}$

$$\varphi(\theta) = \begin{cases} -\pi & \theta \leq -\nu \\ \text{linearly} & \theta \in [-\nu, \nu] \\ \pi & \theta \geq \nu \end{cases} .$$

If $a \in \mathbb{R} \setminus \mathbb{Z}$ is positive and large, let us show that

$$\text{index}\left(\mathbb{P}e^{i\varphi(\arg(X-a))}\right) = \text{index}\left(\mathbb{A}_1\mathbb{P}e^{i\varphi(\arg(X-a))}\right) .$$

To that end we want to show that

$$\begin{aligned} \mathbb{A}_1\mathbb{P}e^{i\varphi(\arg(X-a))} - \mathbb{P}e^{i\varphi(\arg(X-a))} &= \Lambda_1\mathbb{P}e^{i\varphi(\arg(X-a))}\Lambda_1 + \Lambda_1^\perp - \\ &\quad -\Lambda_1\mathbb{P}e^{i\varphi(\arg(X-a))}\Lambda_1 - \Lambda_1^\perp\mathbb{P}e^{i\varphi(\arg(X-a))}\Lambda_1 - \Lambda_1\mathbb{P}e^{i\varphi(\arg(X-a))}\Lambda_1^\perp - \Lambda_1^\perp\mathbb{P}e^{i\varphi(\arg(X-a))}\Lambda_1^\perp \\ &= \Lambda_1^\perp\left(\mathbb{1} - \mathbb{P}e^{i\varphi(\arg(X-a))}\right)\Lambda_1^\perp - \Lambda_1^\perp\mathbb{P}e^{i\varphi(\arg(X-a))}\Lambda_1 - \Lambda_1\mathbb{P}e^{i\varphi(\arg(X-a))}\Lambda_1^\perp .\end{aligned}$$

First we analyze

$$\begin{aligned} \Lambda_1^\perp\left(\mathbb{1} - \mathbb{P}e^{i\varphi(\arg(X-a))}\right)\Lambda_1^\perp &= \Lambda_1^\perp\left(P - Pe^{i\varphi(\arg(X-a))}P\right)\Lambda_1^\perp \\ &= \Lambda_1^\perp\left(P\left(e^{i\varphi(\arg(X-a))} - \mathbb{1}\right) - P\left[e^{i\varphi(\arg(X-a))}, P\right]\right)\Lambda_1^\perp .\end{aligned}$$

But the commutator is compact by Lemma 3.11 and by construction,

$$\left(e^{i\varphi(\arg(X-a))} - \mathbb{1}\right)\Lambda_1^\perp = 0 .$$

Similarly, the cross terms $\Lambda_1^\perp\mathbb{P}e^{i\varphi(\arg(X-a))}\Lambda_1$ are compact.

Next, we claim that we can add another flux on the left, that is,

$$\text{index}\left(\mathbb{A}_1\mathbb{P}e^{i\varphi(\arg(X-a))}\right) = \text{index}\left(\mathbb{A}_1\mathbb{P}\left(e^{i\varphi(\arg(X-a))}e^{-i\varphi(\arg(X+a))}\right)\right) .$$

We again show the difference of the two operators is compact:

$$\begin{aligned} \mathbb{A}_1\mathbb{P}e^{i\varphi(\arg(X-a))} - \mathbb{A}_1\mathbb{P}\left(e^{i\varphi(\arg(X-a))}e^{-i\varphi(\arg(X+a))}\right) &= \Lambda_1Pe^{i\varphi(\arg(X-a))}\left(\mathbb{1} - e^{-i\varphi(\arg(X+a))}\right)P\Lambda_1 \\ &= \Lambda_1Pe^{i\varphi(\arg(X-a))}\left[\left(\mathbb{1} - e^{-i\varphi(\arg(X+a))}\right), P\right]\Lambda_1 + \\ &\quad +\Lambda_1Pe^{i\varphi(\arg(X-a))}P\left(\mathbb{1} - e^{-i\varphi(\arg(X+a))}\right)\Lambda_1 \end{aligned}$$

but

$$\left(\mathbb{1} - e^{-i\varphi(\arg(X+a))}\right)\Lambda_1 = 0$$

by construction. In summary, so far we have found that

$$\text{index}(\mathbb{P}U) = \text{index}\left(\mathbb{A}_1\mathbb{P}\left(e^{i\varphi(\arg(X-a))}e^{-i\varphi(\arg(X+a))}\right)\right)$$

where $e^{i\varphi(\arg(X-a))}e^{-i\varphi(\arg(X+a))}$ is a function which winds about a in some rightward cone, and then winds about $-a$ with some leftward cone. We write it out explicitly as

$$e^{i\varphi(\arg(X-a))}e^{-i\varphi(\arg(X+a))} = \exp(i\xi(X))$$

with ξ as in **??**.

Next, we claim

$$\text{index}\left(\mathbb{A}_1\mathbb{P}\left(e^{i\xi(X)}\right)\right) = \text{index}\left(\mathbb{A}_1e^{iP\xi(X)P}\right) .$$

We show the difference is compact:

$$\mathbb{A}_1\mathbb{P}\left(\mathrm{e}^{\mathrm{i}\xi(X)}\right)-\mathbb{A}_1\mathrm{e}^{\mathrm{i}P\xi(X)P} = \Lambda_1\left(P\mathrm{e}^{\mathrm{i}\xi(X)}P+P^{\perp}-\mathrm{e}^{\mathrm{i}P\xi(X)P}\right)\Lambda_1$$
$$= \Lambda_1\left(P\mathrm{e}^{\mathrm{i}\xi(X)}P-P\mathrm{e}^{\mathrm{i}P\xi(X)P}P\right)\Lambda_1\,.$$

Now,

$$P\mathrm{e}^{\mathrm{i}\xi(X)}P-P\mathrm{e}^{\mathrm{i}P\xi(X)P}P = \sum_{n\geq 2}\frac{\mathrm{i}^n}{n!}\left(P\xi\left(X\right)^n P-\left(P\xi\left(X\right)P\right)^n\right)\,.$$

Note that $P\xi\left(X\right)^n P-\left(P\xi\left(X\right)P\right)^n$ always contains a factor of $P\xi\left(X\right)P^{\perp}$, which is always compact, since ξ obeys Lemma 3.10.

Finally, we want to deform $\xi\left(X\right)$ to $-2\pi\Lambda_2$, i.e., we claim that

$$\operatorname{index}\left(\mathbb{A}_1\mathrm{e}^{\mathrm{i}P\xi(X)P}\right)=\operatorname{index}\left(\mathbb{A}_1\mathrm{e}^{-\mathrm{i}2\pi P\Lambda_2 P}\right)\,.$$

This is done continuous by closing the cones, i.e., taking $\nu\to 0$. □

Lemma 3.10. *If $f:\mathbb{C}\setminus\{\,0\,\}\to\mathbb{S}^1$ is defined as the polar part, i.e.,*

$$f\left(z\right):=\frac{z}{|z|}\equiv\exp\left(\mathrm{i}\arg\left(z\right)\right)$$

then there exists some $D<\infty$ such that

$$|f\left(z\right)-f\left(w\right)|\leq D\frac{|z-w|}{1+|z|}\qquad\left(z,w\in\mathbb{C}\cap\mathbb{Z}^2\right)\,.$$

Proof. Write $z=r\mathrm{e}^{\mathrm{i}\theta}$ and $w=\rho\mathrm{e}^{\mathrm{i}\varphi}$. Then

$$\begin{aligned}|z-w|^2 &= |z|^2+|w|^2-2\,\mathbb{R}\mathrm{e}\left\{\overline{z}w\right\}\\ &= r^2+\rho^2-2r\rho\cos\left(\theta-\varphi\right)\\ &= r^2+\rho^2-2r\rho+2r\rho\left[1-\cos\left(\theta-\varphi\right)\right]\\ &= \left(r-\rho\right)^2+4r\rho\left(\sin\left(\frac{\theta-\varphi}{2}\right)\right)^2\,.\end{aligned}$$

Thus,

$$|z-w|\geq 2\sqrt{r\rho}\sin\left(\frac{\theta-\varphi}{2}\right)\,.$$

Now, according to Jordan's inequality, if $\frac{\theta-\varphi}{2}\in\left[0,\frac{\pi}{2}\right]$ then

$$\sin\left(\frac{\theta-\varphi}{2}\right)\geq\frac{\theta-\varphi}{\pi}\,.$$

□

Lemma 3.11. *Let P be local as in (1.2) and such that $\|P\|\leq 1$ and $f\in\ell^\infty\left(\mathbb{Z}^2\right)$ be such that there exists some $D<\infty$ with which*

$$|f(x)-f(y)| \leq D\frac{\|x-y\|}{1+\|x\|}\,. \tag{3.6}$$

Then $[P,f(X)]$ is Schatten-3. In particular it is compact.

Proof. We have $[P, f(X)]_{xy} = P_{xy}(f(x) - f(y))$ and using Lemma 3.12 just below, we have

$$\|[P, f(X)]\|_3 \leq \sum_{b\in\mathbb{Z}^2} \left(\sum_{x\in\mathbb{Z}^2} \|P_{x+b,x}\|^3 |f(x) - f(x+b)|^3 \right)^{1/3} .$$

Now we have

$$\|P_{x+b,x}\|^3 \leq C^3 \mathrm{e}^{-3\mu\|b\|}$$

so that together with (3.6) we have the estimate

$$\begin{aligned} \|[P, f(X)]\|_3 &\leq \sum_{b\in\mathbb{Z}^2} \left(\sum_{x\in\mathbb{Z}^2} C^3 \mathrm{e}^{-3\mu\|b\|} D^3 \frac{\|b\|^3}{(1+\|x\|)^3} \right)^{\frac{1}{3}} \\ &= CD \sum_{b\in\mathbb{Z}^2} \mathrm{e}^{-\mu\|b\|} \|b\| \left(\sum_{x\in\mathbb{Z}^2} \frac{1}{(1+\|x\|)^3} \right)^{\frac{1}{3}} \\ &< \infty . \end{aligned}$$

□

Lemma 3.12. *For any operator* $A \in \mathcal{B}\left(\ell^2\left(\mathbb{Z}^d\right) \otimes \mathbb{C}^N\right)$, *with*

$$A_{xy} \equiv \langle \delta_x, A\delta_y \rangle \in \mathrm{Mat}_{N\times N}(\mathbb{C})$$

and $\{ \delta_x \}_{x\in\mathbb{Z}^d}$ *the position basis, we have the estimate*

$$\|A\|_p \leq \sum_{k\in\mathbb{Z}^d} \left(\sum_{x\in\mathbb{Z}^d} |A_{x+k,x}|^p \right)^{\frac{1}{p}} .$$

where $\|A\|_p \equiv (\mathrm{tr}(|A|^p))^{\frac{1}{p}}$ *is the Schatten-p norm.*

Proof. Let us decompose A to its diagonals as

$$A = \sum_{k\in\mathbb{Z}^d} A^{(k)}$$

defined via $\left(A^{(k)}\right)_{xy} \equiv A_{xy}\delta_{x-y,k}$ for all $k \in \mathbb{Z}^d$. Since $\|\cdot\|_p$ is a norm, applying the triangle inequality we find

$$\|A\|_p \leq \sum_{k\in\mathbb{Z}^d} \left\|A^{(k)}\right\|_p .$$

But now,

$$\begin{aligned} \left\|A^{(k)}\right\|_p &= \left(\mathrm{tr}\left(\left|A^{(k)}\right|^p\right)\right)^{\frac{1}{p}} \\ &= \left(\mathrm{tr}\left(\left(\left|A^{(k)}\right|^2\right)^{\frac{p}{2}}\right)\right)^{\frac{1}{p}} \\ &= \left(\left\|\left|A^{(k)}\right|^2\right\|_{\frac{p}{2}}^{\frac{p}{2}}\right)^{\frac{1}{p}} \\ &= \sqrt{\left\|\left|A^{(k)}\right|^2\right\|_{\frac{p}{2}}} . \end{aligned}$$

But note that

$$\begin{aligned}
\left(\left|A^{(k)}\right|^2\right)_{xy} &\equiv \left(\left(A^{(k)}\right)^* A^{(k)}\right)_{xy} \\
&= \sum_{z\in\mathbb{Z}^d}\left(\left(A^{(k)}\right)^*\right)_{xz}\left(A^{(k)}\right)_{zy} \\
&= \sum_{z\in\mathbb{Z}^d}\left(A_{zx}\delta_{z-x,k}\right)^* A_{zy}\delta_{z-y,k} \\
&= \delta_{x,y}\sum_{z\in\mathbb{Z}^d}\left(A_{zx}\delta_{z-x,k}\right)^* A_{zy}\delta_{z-y,k} \\
&= \delta_{x,y}\left|A_{x+k,x}\right|^2 .
\end{aligned}$$

Since $\left|A^{(k)}\right|^2$ is a-posteriori a diagonal operator, it is easy to calculate its Schatten-$\frac{p}{2}$ norm, since it is easy to take its powers. Indeed,

$$\left[\left(\left|A^{(k)}\right|^2\right)^{\frac{p}{2}}\right]_{xy} = \delta_{x,y}\left|A_{x+k,x}\right|^p$$

and so

$$\begin{aligned}
\left\|\left|A^{(k)}\right|^2\right\|_{\frac{p}{2}}^{\frac{p}{2}} &= \operatorname{tr}\left(\left(\left|A^{(k)}\right|^2\right)^{\frac{p}{2}}\right) \\
&= \sum_{x\in\mathbb{Z}^d}\left[\left(\left|A^{(k)}\right|^2\right)^{\frac{p}{2}}\right]_{xx} \\
&= \sum_{x\in\mathbb{Z}^d}\left|A_{x+k,x}\right|^p .
\end{aligned}$$

Collecting everything together we find

$$\begin{aligned}
\|A\|_p &\leq \sum_{k\in\mathbb{Z}^d}\sqrt{\left(\sum_{x\in\mathbb{Z}^d}\left|A_{x+k,x}\right|^p\right)^{\frac{2}{p}}} \\
&\leq \sum_{k\in\mathbb{Z}^d}\left(\sum_{x\in\mathbb{Z}^d}\left|A_{x+k,x}\right|^p\right)^{\frac{1}{p}} .
\end{aligned}$$

□

Corollary 3.13 (Additivity of the index)**.** *If $P\perp Q$ and both $[P,U],[Q,U]\in\mathcal{K}$ then*

$$\operatorname{index}\left((P+Q)\,U\,(P+Q)+(P+Q)^\perp\right) = \operatorname{index}(\mathbb{P}U)+\operatorname{index}(\mathbb{Q}U) .$$

Proof. We have

$$\begin{aligned}
\operatorname{index}\left((P+Q)\,U\,(P+Q)+(P+Q)^\perp\right) &= \operatorname{index}\left(PUP+QUQ+PUQ+QUP+\mathbb{1}-(P+Q)\right) \\
&= \operatorname{index}\left(PUP+QUQ+[P,U]\,Q+[Q,U]\,P+\mathbb{1}-(P+Q)\right) \\
&= \operatorname{index}\left(PUP+QUQ+\mathbb{1}-(P+Q)\right) .
\end{aligned}$$

But by the logarithmic property of the Fredholm index,

$$\begin{aligned}\text{index}\,(\mathbb{P}U)+\text{index}\,(\mathbb{Q}U) &= \text{index}\,((\mathbb{P}U)\,(\mathbb{Q}U))\\ &= \text{index}\,((PUP+\mathbb{1}-P)\,(QUQ+\mathbb{1}-Q))\\ &= \text{index}\,(PUP+QUQ+\mathbb{1}-Q-P)\;.\end{aligned}$$

□

3.6 Calculation of Hall conductivity of a Landau level

Theorem 3.14. *If E_F does not intersect any Landau level, then the Landau Hamiltonian (3.3) has Hall conductivity equal to*

$$\sigma_{Hall}\,(E_F)=-\frac{1}{2\pi}m\,(E_F)$$

where $m\,(E_F)$ is the number of Landau levels below E_F.

Proof. Clearly we need the condition that

$$E_F \quad\notin\quad B_0\,(2\mathbb{N}+1)$$

to be in a spectral gap and thus have well-defined Hall conductivity. Thanks to Corollary 3.13 we may treat each Landau level separately. Now, since each Landau level n is spanned by angular momenta $l\geq -n$, it is isomorphic to

$$\ell^2\,(\mathbb{Z}_{\geq -n})\;.$$

Moreover, on it, the Laughlin flux insertion U acts as a unilateral right shift. Indeed,

$$\mathrm{e}^{\mathrm{i}\arg(X)}\cong\mathrm{e}^{\mathrm{i}\theta}$$

with l the angular momentum, is precisely a shift $l\mapsto l+1$ of the angular momentum l, since the real space polar angle θ is the conjugate variable to the angular momentum l. So if P is a projection onto just one Landau level,

$$PUP+P^\perp\cong R$$

where R is the unilateral right shift operator on $\ell^2\,(\mathbb{Z}_{\geq -n})$ and hence its Fredholm index is -1. □

3.7 Constancy of the Hall conductance within the strongly disordered regime–the plateus

Thanks to Corollary 3.13 we know that if

$$P_\mu \quad:=\quad \chi_{(-\infty,\mu)}\,(H)$$

then

$$\text{index}\,(\mathbb{P}_{\mu+\varepsilon}U) \quad=\quad \text{index}\,(\mathbb{P}_\mu U)+\text{index}\left(\chi_{[\mu,\mu+\varepsilon)}\,(H)\,U\chi_{[\mu,\mu+\varepsilon)}\,(H)+\chi_{[\mu,\mu+\varepsilon)}\,(H)^\perp\right)$$

assuming that both μ and $\mu+\varepsilon$ are in a gap of H. Actually, we have

Theorem 3.15. *If H has only Anderson localized states within $[\mu,\mu+\varepsilon)$ then*

$$\text{index}\left(\chi_{[\mu,\mu+\varepsilon)}\,(H)\,U\chi_{[\mu,\mu+\varepsilon)}\,(H)+\chi_{[\mu,\mu+\varepsilon)}\,(H)^\perp\right)=0\,.$$

Proof. For convenience set $Q:=\chi_{[\mu,\mu+\varepsilon)}\,(H)$. Using Theorem 2.30 we find that

$$Q=\sum_n\psi_n\otimes\psi_n^*$$

with $\{ \psi_n \}_n$ a SULE ONB as in Definition 2.28. In particular, we have

$$\|\psi_n(x)\| \leq C_a \mathrm{e}^{-\mu\|x-x_n\|} \frac{1}{|a(x_n)|} \qquad \left(x \in \mathbb{Z}^d; n \in \mathbb{N}\right) .$$

Let us define an operator $V : \operatorname{im}(Q) \to \operatorname{im}(Q)$ via

$$V : \psi_n \mapsto \mathrm{e}^{\mathrm{i}\arg(x_n)} \psi_n .$$

Since ψ_n is localized near x_n, it is to be expected that $U\psi_n \approx V\psi_n$. This is indeed the case, in the sense that

$$\mathbb{Q}U - \mathbb{Q}V \in \mathcal{K} . \tag{3.7}$$

However, $\mathbb{Q}V = V \oplus \mathbb{1}_{\operatorname{im}(Q^\perp)}$ and it is hence invertible, so its Fredholm index is zero. Let us thus prove (3.7). Let us define $B := (U - V)Q$. Then it suffices to show

$$\sum_k \left(\sum_x |B_{x,x+k}|^p \right)^{\frac{1}{p}} < \infty$$

for some $p \in \mathbb{N}$ thanks to Lemma 3.12. We have, with $f(x) := \mathrm{e}^{\mathrm{i}\arg(x)}$,

$$B_{xy} = \sum_{n\in\mathbb{N}} (f(x) - f(x_n)) \psi_n(x) \overline{\psi_n(y)} .$$

Hence

$$\begin{aligned} |B_{xy}|^p &\leq \left(\sum_{n\in\mathbb{N}} |f(x) - f(x_n)| |\psi_n(x)| |\psi_n(y)| \right)^p \\ &\leq \sum_n |f(x) - f(x_n)|^p |\psi_n(x)| |\psi_n(y)| \end{aligned}$$

where we have used in the last step Hoelder's inequality in the form

$$\left(\sum_j a_j b_j c_j \right)^p \leq \left(\sum_j a_j^p b_j c_j \right) \left(\sum_j b_j^2 \right)^{\frac{p-1}{2}} \left(\sum_j c_j^2 \right)^{\frac{p-1}{2}}$$

as well as the fact that $\sum_n |\psi_n(x)|^2 \leq 1$, i.e., $\sum_n \psi_n \otimes \psi_n^* = Q \leq \mathbb{1}$. But f obeys (3.6) so

$$|B_{x,x+y}|^p \leq D^p C_a^2 \sum_n \frac{\|x - x_n\|^p}{(1 + \|x\|)^{\frac{p}{2}} (1 + \|x_n\|)^{\frac{p}{2}}} \frac{1}{|a(x_n)|^2} \mathrm{e}^{-\mu\|x-x_n\| - \mu\|x-y-x_n\|} .$$

This last expression is readily seen to be decaying in both $\|x\|$ and $\|y\|$ thanks to applications of the triangle inequality as well as (2.18). □

3.8 The edge system and its index

A major feature of topological insulators is the *bulk-edge correspondence*. In order to discuss it we must discuss "edge systems". So far, all systems we have considered were spectrally gapped (at the Fermi energy E_F) and geometrically, their configuration space contained no edge. That is, they were defined on the entirety of either $\mathbb{R}^d$ or $\mathbb{Z}^d$. It turns out that a different mechanism of electric conductivity takes place if the system is truncated to have a boundary. There is a discussion to be had about what are admissible geometries of truncation to take, but the easiest choice is to take the half-space, which for us means the replacement

$$\mathbb{R}^d \to \mathbb{R}^{d-1} \times [0, \infty)$$

in the continuum or

$$\mathbb{Z}^d \to \mathbb{Z}^{d-1} \times \mathbb{N}$$

on the lattice. Since we are talking about Hall systems, we have $d=2$ and hence we now want to consider "the edge" Hilbert space as

$$\widehat{\mathcal{H}} := \ell^2(\mathbb{Z}\times\mathbb{N})\otimes\mathbb{C}^N\,.$$

There is clearly a partial isometry that takes us from the edge Hilbert space to the bulk Hilbert space $\mathcal{H}\equiv\ell^2(\mathbb{Z}^2)\otimes\mathbb{C}^N$ given by extending a wave-function by zero:

$$\begin{aligned} J:\widehat{\mathcal{H}} &\to \mathcal{H}\\ \hat{\psi} &\mapsto \left(\mathbb{Z}^2\ni x\mapsto\begin{cases}\hat{\psi}(x) & x\in\mathbb{Z}\times\mathbb{N}\\ 0 & \text{else}\end{cases}\right)\,.\end{aligned}$$

One easily verifies that

$$|J|^2\equiv J^*J=\mathbb{1}_{\widehat{\mathcal{H}}}$$

yet

$$|J^*|^2\equiv JJ^*=\text{ projection onto}\operatorname{span}\left(\{\,\delta_x\,\}_{x\in\mathbb{Z}\times\mathbb{N}}\right)\equiv\Lambda_2\,.$$

It is then natural to make the

Definition 3.16 (The Dirichlet edge Hamiltonian to a given bulk one)**.** Given any bulk Hamiltonian $H\in\mathcal{B}(\mathcal{H})$, the associated Dirichlet edge Hamiltonian is given by J^*HJ.

It goes without saying that any edge Hamiltonian we want to consider exhibits locality in the sense of (1.2), and that the Dirichlet truncation preserves that property.

Claim 3.17. If H is local (as in (1.2)) then J^*HJ is too.

Proof. We have

$$\begin{aligned}\left\|(J^*HJ)_{xy}\right\| &\equiv \begin{cases}\|H_{xy}\| & x\in\mathbb{Z}\times\mathbb{N}\wedge y\in\mathbb{Z}\times\mathbb{N}\\ 0 & \text{else}\end{cases}\\ &\leq C\mathrm{e}^{-\mu\|x-y\|}\qquad (x,y\in\mathbb{Z}\times\mathbb{N})\,.\end{aligned}$$

□

In principle one may also want to consider more general boundary conditions. A reasonable constraint on such definitions is that they be local too and decay away from the truncation, i.e.,

Definition 3.18. $\hat{H}\in\mathcal{B}\left(\widehat{\mathcal{H}}\right)$ is a general truncation of a bulk Hamiltonian $H\in\mathcal{B}(\mathcal{H})$ iff

$$\hat{H}=J^*HJ+H_{\mathrm{BC}}$$

where $H_{\mathrm{BC}}=(H_{\mathrm{BC}})^*$ is a *local* operator which also obeys

$$\left\|(H_{\mathrm{BC}})_{xy}\right\|\leq C\mathrm{e}^{-\nu(x_2+y_2)}\qquad (x,y\in\mathbb{Z}\times\mathbb{N})\,.$$

As we mentioned, as a rule, edge systems are *not* insulators: they exhibit *spontaneous* currents which correspond, roughly speaking, to electrons bouncing along the edge of the sample. However, edge systems have a special property: they are to be envisioned as *truncations of spectrally gapped* bulk Hamiltonians. This yields two possible definitions of edge Hamiltonians which "descend" from bulk Hamiltonians, as it were:

Definition 3.19 (bulk-spectral-gap for edge Hamiltonian, alt 1)**.** The edge Hamiltonian $\hat{H}\in\mathcal{B}\left(\widehat{\mathcal{H}}\right)$ has a bulk-gap on the interval $\Delta\subseteq\mathbb{R}$ iff there exists some operator $K\in\mathcal{B}(\mathcal{H})$ such that: (1) K has a spectral gap on Δ and (2) If we truncate K onto the edge via J^*KJ, the it converges to the given edge Hamiltonian $\hat{H}$ as we go into the bulk, in the sense that

$$\left\|\left(J^*KJ-\hat{H}\right)_{xy}\right\|\leq C\mathrm{e}^{-\mu\|x-y\|}\mathrm{e}^{-\nu(x_2+y_2)}\qquad (x,y\in\mathbb{Z}\times\mathbb{N})\,. \tag{3.8}$$

There is a more elegant alternative which corresponds to the following fact. If we take a bulk Hamiltonian H which has a gap on Δ and truncate it to $\hat{H}$, then we expect that even if the gap closes, the states which are in it are "localized near the edge", which means, that if $g:\mathbb{R}\to\mathbb{C}$ is any smooth function supported only within Δ, then we expect

$$\left\|g\left(\hat{H}\right)_{xy}\right\|\leq C\mathrm{e}^{-\mu\|x-y\|}\mathrm{e}^{-\nu(|x_2|+|y_2|)}\qquad(x,y\in\mathbb{Z}\times\mathbb{N})\ .$$

This motivates

Definition 3.20 (bulk-spectral-gap for edge Hamiltonian, alt 2)**.** The edge Hamiltonian $\hat{H}\in\mathcal{B}\left(\hat{\mathcal{H}}\right)$ has a bulk-gap on the interval $\Delta\subseteq\mathbb{R}$ iff for any smooth function $g:\mathbb{R}\to\mathbb{C}$ supported within Δ, we have

$$\left\|g\left(\hat{H}\right)_{xy}\right\|\leq C\mathrm{e}^{-\mu\|x-y\|}\mathrm{e}^{-\nu(x_2+y_2)}\qquad(x,y\in\mathbb{Z}\times\mathbb{N})\ .$$

Lemma 3.21. *The first definition implies the second definition.*

Proof. Let us assume that $\hat{H}$ obeys Definition 3.19. That means there exists some bulk Hamiltonian K gapped on Δ such that the two obey (3.8). One may write the smooth function of the Hamiltonian $\hat{H}$ via the smooth functional calculus (1.9) as

$$f\left(\hat{H}\right)=\frac{1}{2\pi}\int_{z\in\mathbb{C}}\left(\partial_{\overline{z}}\tilde{f}\right)(z)\left(\hat{H}-z\mathbb{1}\right)^{-1}\mathrm{d}z$$

where $\tilde{f}:\mathbb{C}\to\mathbb{C}$ is a quasi-analytic extension of f which is supported in some strip about the real axis. I.e., we have, for all $N\in\mathbb{N}$,

$$\left|\left(\partial_{\overline{z}}\tilde{f}\right)(z)\right|\leq C_N\left|\mathbb{Im}\left\{z\right\}\right|^N\qquad(z\in\mathbb{C}\text{ with }\left|\mathbb{Im}\left\{z\right\}\right|\text{ small})\ .$$

Moreover,

$$J\hat{H}J^*:\mathcal{H}\to\mathcal{H}$$

is a bulk operator given by matrix elements

$$\left(J\hat{H}J^*\right)_{xy}=\begin{cases}\hat{H}_{xy} & x_2,y_2>0\\ 0 & \text{else}\end{cases}\ .$$

Moreover, we note that

$$g\left(\hat{H}\right)_{xy}=g\left(J\hat{H}J^*\right)_{xy}\qquad(x,y\in\mathbb{Z}\times\mathbb{N})$$

since $g(A\oplus B)=g(A)\oplus g(B)$ and $J\hat{H}J^*\cong 0\oplus\hat{H}$. Now since by definition $g(K)=0$ as $\sigma(K)\cap\Delta=\varnothing$ and $\operatorname{supp}(g)\subseteq\Delta$, we have, and for $x_2,y_2>0$ we get

$$\begin{aligned}g\left(\hat{H}\right)_{xy} &= g\left(J\hat{H}J^*\right)_{xy}\\ &= g\left(J\hat{H}J^*\right)_{xy}-g(K)_{xy}\\ &= \frac{1}{2\pi}\int_{z\in\mathbb{C}}\left(\partial_{\overline{z}}\tilde{g}\right)(z)\left(\left[\left(J\hat{H}J^*-z\mathbb{1}\right)^{-1}\right]_{xy}-\left[(K-z\mathbb{1})^{-1}\right]_{xy}\right)\mathrm{d}z\\ &= \frac{1}{2\pi}\int_{z\in\mathbb{C}}\left(\partial_{\overline{z}}\tilde{g}\right)(z)\left(\sum_{\tilde{x},\tilde{y}\in\mathbb{Z}^2}\left[\left(J\hat{H}J^*-z\mathbb{1}\right)^{-1}\right]_{x\tilde{x}}\left(K-J\hat{H}J^*\right)_{\tilde{x}\tilde{y}}\left[(K-z\mathbb{1})^{-1}\right]_{\tilde{y}y}\right)\mathrm{d}z\,.\end{aligned}$$

Now, we have

$$\left(K-J\hat{H}J^*\right)_{\tilde{x}\tilde{y}} = K_{\tilde{x}\tilde{y}}-\left(J\hat{H}J^*\right)_{\tilde{x}\tilde{y}} = \begin{cases}\left(J^*KJ-\hat{H}\right)_{\tilde{x}\tilde{y}} & \tilde{x},\tilde{y}\in\mathbb{Z}\times\mathbb{N}\\ K_{\tilde{x}\tilde{y}} & \text{else}\end{cases}.$$

Next, we also have

$$\left[\left(J\hat{H}J^*-z\mathbb{1}\right)^{-1}\right]_{x\tilde{x}} = \begin{cases}\left[\left(\hat{H}-z\mathbb{1}\right)^{-1}\right]_{x\tilde{x}} & x,\tilde{x}\in\mathbb{Z}\times\mathbb{N}\\ -\frac{1}{z}\delta_{x\tilde{x}} & x,\tilde{x}\in\mathbb{Z}\times\mathbb{Z}_{<0}\\ 0 & \text{else}\end{cases}.$$

Collecting these together we find, using the fact that $x_2,y_2>0$,

$$\begin{aligned}
(\text{bracket}) &= \sum_{\tilde{x},\tilde{y}\in\mathbb{Z}^2}\left[\left(J\hat{H}J^*-z\mathbb{1}\right)^{-1}\right]_{x\tilde{x}}\left(K-J\hat{H}J^*\right)_{\tilde{x}\tilde{y}}\left[(K-z\mathbb{1})^{-1}\right]_{\tilde{y}y}\\
&= \sum_{\tilde{x},\tilde{y}\in\mathbb{Z}\times\mathbb{N}}\left[\left(J\hat{H}J^*-z\mathbb{1}\right)^{-1}\right]_{x\tilde{x}}\left(K-J\hat{H}J^*\right)_{\tilde{x}\tilde{y}}\left[(K-z\mathbb{1})^{-1}\right]_{\tilde{y}y}+\\
&\quad+\sum_{\tilde{x}\in\mathbb{Z}\times\mathbb{N}}\sum_{\tilde{y}\in\mathbb{Z}\times\mathbb{Z}_{<0}}\left[\left(J\hat{H}J^*-z\mathbb{1}\right)^{-1}\right]_{x\tilde{x}}K_{\tilde{x}\tilde{y}}\left[(K-z\mathbb{1})^{-1}\right]_{\tilde{y}y}\\
&= \sum_{\tilde{x},\tilde{y}\in\mathbb{Z}\times\mathbb{N}}\left[\left(\hat{H}-z\mathbb{1}\right)^{-1}\right]_{x\tilde{x}}\left(K-J\hat{H}J^*\right)_{\tilde{x}\tilde{y}}\left[(K-z\mathbb{1})^{-1}\right]_{\tilde{y}y}+\\
&\quad+\sum_{\tilde{x}\in\mathbb{Z}\times\mathbb{N}}\sum_{\tilde{y}\in\mathbb{Z}\times\mathbb{Z}_{<0}}\left[\left(\hat{H}-z\mathbb{1}\right)^{-1}\right]_{x\tilde{x}}K_{\tilde{x}\tilde{y}}\left[(K-z\mathbb{1})^{-1}\right]_{\tilde{y}y}.
\end{aligned}$$

Applying now the Combes-Thomas estimate on the resolvents and using locality of K as well as (3.8) we find, with $\eta:=\mathbb{Im}\{z\}$,

$$\begin{aligned}
\|(\text{bracket})\| &\leq \sum_{\tilde{x},\tilde{y}\in\mathbb{Z}\times\mathbb{N}}\frac{2}{|\eta|}\mathrm{e}^{-\mu|\eta|\|x-\tilde{x}\|}C\mathrm{e}^{-\mu\|\tilde{x}-\tilde{y}\|}\mathrm{e}^{-\nu(|\tilde{x}_2|+|\tilde{y}_2|)}\frac{2}{|\eta|}\mathrm{e}^{-\mu|\eta|\|\tilde{y}-y\|}+\\
&\quad+\sum_{\tilde{x}\in\mathbb{Z}\times\mathbb{N}}\sum_{\tilde{y}\in\mathbb{Z}\times\mathbb{Z}_{<0}}\frac{2}{|\eta|}\mathrm{e}^{-\mu|\eta|\|x-\tilde{x}\|}C\mathrm{e}^{-\mu\|\tilde{x}-\tilde{y}\|}\frac{2}{|\eta|}\mathrm{e}^{-\mu|\eta|\|\tilde{y}-y\|}.
\end{aligned}$$

We claim that if A is local and B is local *and* decays into the bulk, then AB decays into the bulk. This is a simple application of the triangle inequality. So the first line already obeys the desired decay into the bulk. Let us therefore focus on the second line. We have $\tilde{x}_2>0$ and $\tilde{y}_2\leq 0$, and so

$$\begin{aligned}
\|\tilde{x}-\tilde{y}\| &\geq \frac{1}{\sqrt{2}}\left(|\tilde{x}_1-\tilde{y}_1|+|\tilde{x}_2-\tilde{y}_2|\right)\\
&= \frac{1}{\sqrt{2}}\left(|\tilde{x}_1-\tilde{y}_1|+\tilde{x}_2+|\tilde{y}_2|\right)\\
&= \frac{1}{\sqrt{2}}\left(|\tilde{x}_1-\tilde{y}_1|+|\tilde{x}_2|+|\tilde{y}_2|\right).
\end{aligned}$$

As a result, by downgrading the rates of decay we can rewrite the second line in the same way as the first line. □

Claim 3.22. The edge Hall conductivity is given by

$$\widehat{\sigma_{\text{Hall}}} = \mathrm{i}\operatorname{tr}\left(g'\left(\hat{H}\right)\left[\hat{H},\Lambda_1\right]\right)$$

where g is a smooth function approximating $\chi_{(-\infty,E_F)}$. In particular the expression in the trace is trace-class if $\hat{H}$ has a bulk-gap on $\Delta \ni E_F$.

Proof. It is clear that $g'\left(\hat{H}\right)\left[\hat{H},\Lambda_1\right]$ is trace-class. Since $\hat{H}$ is local, $\left[\hat{H},\Lambda_1\right]$ decays in the 1-axis, and since g' is supported only within Δ and hence Definition 3.20 is obeyed.

Next we may ask why that formula is the edge Hall conductivity. On an intuitive level, $\mathrm{i}\left[\Lambda_1,\hat{H}\right]$ may be understood as the rate (in time) at which charge accumulates from the left to the right of space, i.e., $\Delta\dot{Q}$. This is then calculated with an expectation with the density matrix $g'\left(\hat{H}\right) \approx \chi_\Delta\left(\hat{H}\right)$, i.e., all states in the bulk's gap. Note that we may not use $\chi_\Delta\left(\hat{H}\right)$ directly because that would not yield a trace-class expression (since χ_Δ is not continuous). □

Theorem 3.23 (Kellendonk, Richter, Schulz-Baldes [SBKR99])**.** *The edge Hall conductivity is quantized, via the formula*

$$\widehat{\sigma_{Hall}} = \frac{1}{2\pi}\operatorname{index}\left(\mathbb{\Lambda}_1 \mathrm{e}^{-2\pi\mathrm{i}g\left(\hat{H}\right)}\right) .$$

Proof. First we ask why is $\mathbb{\Lambda}_1 \mathrm{e}^{-2\pi\mathrm{i}g\left(\hat{H}\right)}$ Fredholm at all? We want to apply Lemma 3.7 so we'd like to verify that

$$\left[\Lambda_1,\mathrm{e}^{-2\pi\mathrm{i}g\left(\hat{H}\right)}\right] = \left[\Lambda_1,\mathrm{e}^{-2\pi\mathrm{i}g\left(\hat{H}\right)}-\mathbb{1}\right] \in \mathcal{J}_1\left(\hat{\mathcal{H}}\right) .$$

But this is indeed the case if we can establish that

$$\mathrm{e}^{-2\pi\mathrm{i}g\left(\hat{H}\right)}-\mathbb{1}$$

which is automatically local by the smooth functional calculus Section 1.3.2, also exhibits decay in the 2-axis. To that end we use Lemma 3.24 right below, to focus on showing that

$$\left(g^2-g\right)\left(\hat{H}\right)$$

has decay into the bulk. But this is basically Definition 3.20. As such we learn that $\mathbb{\Lambda}_1 \mathrm{e}^{-2\pi\mathrm{i}g\left(\hat{H}\right)}$ is indeed Fredholm, and moreover, with $U := \mathrm{e}^{-2\pi\mathrm{i}g\left(\hat{H}\right)}$ we have

$$\operatorname{index}\left(\mathbb{\Lambda}_1 U\right) = \operatorname{tr}\left(U^*\left[\Lambda_1,U\right]\right) .$$

Moreover, we also have

$$\operatorname{tr}\left(\left(U^*\right)^n\left[\Lambda_1,U^n\right]\right) = n\operatorname{tr}\left(U^*\left[\Lambda_1,U\right]\right) \qquad \left(n\in\mathbb{Z}\setminus\{\,0\,\}\right) .$$

Indeed, use

$$\left[\Lambda,U^n\right] \quad = \quad \left[\Lambda,U\right]U^{n-1}+\cdots+U^{n-1}\left[\Lambda,U\right]$$

and the cyclicity of the trace.

By the same argument, we may replace U^* by U^* to get

$$\operatorname{tr}\left(\left(\left(U^*\right)^n-\mathbb{1}\right)\left[\Lambda_1,U^n\right]\right) = n\operatorname{tr}\left(\left(U^*-\mathbb{1}\right)\left[\Lambda_1,U\right]\right) \qquad \left(n\in\mathbb{Z}\setminus\{\,0\,\}\right) .$$

Next, by Duhamel, we have

$$\begin{aligned}\operatorname{tr}\left(\left(\left(U^*\right)^n-\mathbb{1}\right)\left[\Lambda_1,U^n\right]\right) &= n\operatorname{tr}\left(\left(\left(U^*\right)^n-\mathbb{1}\right)\int_{s=0}^1 U^{ns}\mathrm{i}\left[\Lambda_1,-2\pi\mathrm{i}g\left(\hat{H}\right)\right]U^{n(1-s)}\mathrm{d}s\right)\\ &= 2\pi n\operatorname{tr}\left(\left(\mathbb{1}-U^n\right)\left[\Lambda_1,g\left(\hat{H}\right)\right]\right)\,.\end{aligned}$$

Using the smooth functional calculus, we have

$$\begin{aligned}\left[\Lambda_1,g\left(\hat{H}\right)\right] &= \left[\Lambda_1,\frac{1}{2\pi}\int_{z\in\mathbb{C}}\left(\partial_{\overline{z}}\tilde{g}\right)(z)\left(\hat{H}-z\mathbb{1}\right)^{-1}\mathrm{d}z\right]\\ &= \frac{1}{2\pi}\int_{z\in\mathbb{C}}\left(\partial_{\overline{z}}\tilde{g}\right)(z)\left[\Lambda_1,\left(\hat{H}-z\mathbb{1}\right)^{-1}\right]\mathrm{d}z\\ &= -\frac{1}{2\pi}\int_{z\in\mathbb{C}}\left(\partial_{\overline{z}}\tilde{g}\right)(z)\left(\hat{H}-z\mathbb{1}\right)^{-1}\left[\Lambda_1,\hat{H}\right]\left(\hat{H}-z\mathbb{1}\right)^{-1}\mathrm{d}z\end{aligned}$$

But now we may use the cyclicity of the trace (recall $\hat{H}$ and U commute) to get the derivative of g, so

$$\operatorname{index}\left(\mathbb{A}_1 U\right)=2\pi\operatorname{tr}\left(\left(U^n-\mathbb{1}\right)g'\left(\hat{H}\right)\left[\Lambda_1,\hat{H}\right]\right)\,.$$

Let now $\varphi:[0,1]\to\mathbb{R}$ be any differentiable function such that $\varphi(0)=\varphi(1)=0$. Its Fourier series may be written as

$$\varphi(x)=\sum_{n\in\mathbb{Z}}\mathrm{e}^{-2\pi\mathrm{i}nx}a_n$$

with

$$a_n\equiv\int_{x=0}^1\mathrm{e}^{+2\pi\mathrm{i}nx}\varphi(x)\,\mathrm{d}x\,.$$

Then

$$0=\varphi(0)=\sum_{n\in\mathbb{Z}}a_n\Longrightarrow a_0=-\sum_{n\neq0}a_n\,.$$

We thus find

$$\begin{aligned}a_0\operatorname{index}\left(\mathbb{A}_1U\right) &= \left(-\sum_{n\neq0}a_n\right)\operatorname{index}\left(\mathbb{A}_1U\right)\\ &= \left(-\sum_{n\neq0}a_n\operatorname{index}\left(\mathbb{A}_1U\right)\right)\\ &= -\sum_{n\neq0}a_n2\pi\operatorname{tr}\left(\left(U^n-\mathbb{1}\right)g'\left(\hat{H}\right)\left[\Lambda_1,\hat{H}\right]\right)\\ &= -\sum_{n\in\mathbb{Z}}a_n2\pi\operatorname{tr}\left(\left(U^n-\mathbb{1}\right)g'\left(\hat{H}\right)\left[\Lambda_1,\hat{H}\right]\right)\\ &\overset{\sum_{n\in\mathbb{Z}}a_n=0}{=} -2\pi\operatorname{tr}\left(\left(\sum_{n\in\mathbb{Z}}a_nU^n\right)g'\left(\hat{H}\right)\left[\Lambda_1,\hat{H}\right]\right)\\ &= -2\pi\operatorname{tr}\left(\varphi\left(g\left(\hat{H}\right)\right)g'\left(\hat{H}\right)\left[\Lambda_1,\hat{H}\right]\right)\end{aligned}$$

But since φ was arbitrary, we may replace it it with a sequence converging to $\chi_{[0,1]}$ pointwise so that

$$a_0=\int_0^1\varphi(x)\,\mathrm{d}x\to\int_0^1\chi_{[0,1]}(x)\,\mathrm{d}x=1$$

and since $\operatorname{im}(g)\subseteq[0,1]$, $\varphi\circ g\to1$. □

Lemma 3.24. *If A is almost a projection in the sense that $A^2 - A$ is small then so is $\mathrm{e}^{-2\pi\mathrm{i}A} - \mathbb{1}$, in the sense that*

$$\mathrm{e}^{-2\pi\mathrm{i}A} - \mathbb{1} = \left(A^2 - A\right) h\left(A\right)$$

for some holomorphic h.

Proof. We rewrite

$$\mathrm{e}^{-2\pi\mathrm{i}A} - \mathbb{1} \quad = \quad \sum_{n=1}^{\infty} \frac{\left(-2\pi\mathrm{i}A\right)^n}{n!}\,.$$

But we also have $\mathrm{e}^{-2\pi\mathrm{i}} = 1$, i.e., $\sum_{n=1}^{\infty} \frac{(-2\pi\mathrm{i})^n}{n!} = 0$. Hence

$$\begin{aligned}
\mathrm{e}^{-2\pi\mathrm{i}A} - \mathbb{1} \quad &= \quad \sum_{n=1}^{\infty} \frac{\left(-2\pi\mathrm{i}A\right)^n}{n!} - 0 \\
&= \quad \sum_{n=1}^{\infty} \frac{\left(-2\pi\mathrm{i}A\right)^n}{n!} - \left(\sum_{n=1}^{\infty} \frac{\left(-2\pi\mathrm{i}\right)^n}{n!}\right) A \\
&= \quad \sum_{n=1}^{\infty} \frac{\left(-2\pi\mathrm{i}\right)^n}{n!} \left(A^n - A\right) \\
&= \quad \left(\sum_{n=2}^{\infty} \frac{1}{n!} \left(-2\pi\mathrm{i}\right)^n \sum_{k=0}^{n-2} A^k\right) \left(A^2 - A\right)\,.
\end{aligned}$$

□

3.9 The bulk-edge correspondence

The bulk-edge correspondence refers to the fact that calculating the bulk topological index, or truncating a bulk Hamiltonian and then calculating an edge topological index, should yield the same number. This was first proven for the IQHE by Hatsugai in the early 90s [Hat93]. Later on it was proven in more generality by Kellendonk-Richter-Schulz–Baldes [SBKR99] and by Elbau-Graf [EG02]. Finally it was proven for strongly-disordered systems (mobility gapped systems) by Elgart-Graf-Schenker [EGS05]. Here we present a simplified proof using Fredholm theory, which appeared in [FSS+20].

Definition 3.25 (Local operators which decay into the bulk)**.** We say that a local operator $A \in \mathcal{B}\left(\hat{\mathcal{H}}\right)$ decays into the bulk iff there exists some $\mu, \nu > 0$ such that for any $N \in \mathbb{N}$ there is some $C_N < \infty$ such that

$$\|A_{xy}\| \leq C_N \left(1 + \mu\|x - y\|\right)^{-N} \left(1 + \nu\left(x_2 + y_2\right)\right)^{-N} \qquad \left(x, y \in \mathbb{Z} \times \mathbb{N}\right)\,.$$

We denote all such operators as LOC_2. Note that this is an ideal within local operators [TODO: prove this, from [ST19] Section 3].

Definition 3.26 (quasi-projections)**.** We say that a local operator is a quasi-projection, denoted by $A \in \mathcal{P}_2$ iff

$$A^2 - A \in \mathrm{LOC}_2\,.$$

Claim 3.27*.* If $A, B \in \mathcal{P}_2$ such that $A - B \in \mathrm{LOC}_2$ then

$$\mathrm{index}\left(\Lambda_1 \mathrm{e}^{-2\pi\mathrm{i}A}\right) = \mathrm{index}\left(\Lambda_1 \mathrm{e}^{-2\pi\mathrm{i}B}\right)\,.$$

Proof. Define the homotopy $[0,1] \ni t \mapsto t(A-B)+B$. Then

$$(t(A-B)+B)^2 - (t(A-B)+B) = t^2(A-B)^2 + t((A-B)B + B(A-B)) + B^2 - B - t(A-B)\,.$$

Each term on the right hand side is separately in LOC_2 so that at any $t \in [0,1]$ we have

$$t(A-B)+B \in \mathscr{P}_2\,.$$

But now,

$$[0,1] \ni t \mapsto \mathbb{A}_1 \mathrm{e}^{-2\pi\mathrm{i}(t(A-B)+B)}$$

is a norm continuous family of Fredholm operators and hence its index is constant. □

Theorem 3.28 (The bulk-edge correspondence)**.** *Let H be a local (as in (1.2)) operator on $\ell^2(\mathbb{Z}^2)\otimes\mathbb{C}^N$ and gapped on some interval $\Delta \subseteq \mathbb{R}$. Let $\hat{H}$ be a local operator on $\ell^2(\mathbb{Z}\times\mathbb{N})\otimes\mathbb{C}^N$. Assume further that $\hat{H}$ has a bulk-gap on Δ as in Definition 3.20. Finally assume that H and $\hat{H}$ are compatible at infinity, in the sense that*

$$\left\|\left(\hat{H} - J^* H J\right)_{xy}\right\| \leq C\mathrm{e}^{-\mu\|x-y\|}\mathrm{e}^{-\nu(x_2+y_2)} \qquad (x,y\in\mathbb{Z}\times\mathbb{N})\,.$$

Then the bulk Hall conductivity associated with H equals the edge Hall conductivity associated with $\hat{H}$:

$$\sigma_{Hall}(H) = \widehat{\sigma_{Hall}}\left(\hat{H}\right)\,.$$

Proof. Using Theorem 3.9 and Theorem 3.6, we know that

$$\sigma_{\mathrm{Hall}}(H) = \frac{1}{2\pi}\mathrm{index}(\mathbb{P}U) = \frac{1}{2\pi}\mathrm{index}\left(\mathbb{A}_1\mathrm{e}^{-2\pi\mathrm{i}P\Lambda_2 P}\right)$$

where $P \equiv \chi_{(-\infty,E_F)}(H)$ is the Fermi projection (with $E_F \in \Delta$). Unsurprisingly, it is the latter expression that is most convenient to start from. For the edge, in Theorem 3.23 we saw that

$$\widehat{\sigma_{\mathrm{Hall}}}\left(\hat{H}\right) = \frac{1}{2\pi}\mathrm{index}\left(\mathbb{A}_1\mathrm{e}^{-2\pi\mathrm{i}g\left(\hat{H}\right)}\right)$$

where g is a smooth version of $\chi_{(-\infty,E_F)}$ with the constraint that $\mathrm{supp}(g') \subseteq \Delta$. Hence we are trying to show that

$$\mathrm{index}_{\ell^2(\mathscr{H})}\left(\mathbb{A}_1\mathrm{e}^{-2\pi\mathrm{i}P\Lambda_2 P}\right) = \mathrm{index}_{\ell^2(\hat{\mathscr{H}})}\left(\mathbb{A}_1\mathrm{e}^{-2\pi\mathrm{i}g\left(\hat{H}\right)}\right)$$

where we write the subscripts to emphasize the indices are calculated on different Hilbert spaces.

Step 1: Since $\sigma(H)\cap\Delta = \varnothing$ and $\mathrm{supp}\left(\chi_{(-\infty,E_F)} - g\right) \subseteq \Delta$, we have

$$P = g(H)$$

and hence

$$\mathrm{index}_{\ell^2(\mathscr{H})}\left(\mathbb{A}_1\mathrm{e}^{-2\pi\mathrm{i}P\Lambda_2 P}\right) = \mathrm{index}_{\ell^2(\mathscr{H})}\left(\mathbb{A}_1\mathrm{e}^{-2\pi\mathrm{i}g(H)\Lambda_2 g(H)}\right)\,.$$

Step 2: Replace $g(H)\Lambda_2 g(H)$ with $\Lambda_2 g(H)\Lambda_2$. To do so, we use Claim 3.27 above, so that we need to verify that

$$g(H)\Lambda_2 g(H)\,,\Lambda_2 g(H)\Lambda_2, g(H)\Lambda_2 g(H) - \Lambda_2 g(H)\Lambda_2 \in \mathscr{P}_2\,.$$

Then (with $P\equiv g\left(H\right)$)

$$\begin{aligned}\left(P\Lambda_2 P\right)^2-P\Lambda_2 P &= P\Lambda_2 P\Lambda_2 P-P\Lambda_2 P\\ &= P\left(\Lambda_2 P\Lambda_2-\Lambda_2\right)P\\ &= P\Lambda_2\left(P-\mathbb{1}\right)\Lambda_2 P\\ &= -P\Lambda_2 P^\perp\Lambda_2 P\\ &= -P\Lambda_2 P^\perp\left[\Lambda_2,P\right]\\ &\in \mathrm{LOC}_2\,.\end{aligned}$$

Similarly,

$$\begin{aligned}\left(\Lambda_2 P\Lambda_2\right)^2-\Lambda_2 P\Lambda_2 &= \Lambda_2 P\Lambda_2 P\Lambda_2-\Lambda_2 P\Lambda_2\\ &= \Lambda_2\left(P\Lambda_2 P-P\right)\Lambda_2\\ &= \Lambda_2 P\left(\Lambda_2-\mathbb{1}\right)P\Lambda_2\\ &= -\Lambda_2 P\Lambda_2^\perp P\Lambda_2\\ &= -\left[\Lambda_2,P\right]\Lambda_2^\perp P\Lambda_2\\ &\in \mathrm{LOC}_2\,.\end{aligned}$$

Finally, the difference also decays into the bulk, since

$$\begin{aligned}P\Lambda_2 P-\Lambda_2 P\Lambda_2 &= P\left[\Lambda_2,P\right]+P\Lambda_2-\Lambda_2 P\Lambda_2\\ &= P\left[\Lambda_2,P\right]+\Lambda_2^\perp P\Lambda_2\\ &= P\left[\Lambda_2,P\right]+\Lambda_2^\perp\left[P,\Lambda_2\right]\\ &\in \mathrm{LOC}_2\,.\end{aligned}$$

So Claim 3.27 implies we have

$$\mathrm{index}_{\ell^2(\mathcal{H})}\left(\mathbb{A}_1 \mathrm{e}^{-2\pi\mathrm{i}g(H)\Lambda_2 g(H)}\right)=\mathrm{index}_{\ell^2(\mathcal{H})}\left(\mathbb{A}_1 \mathrm{e}^{-2\pi\mathrm{i}\Lambda_2 g(H)\Lambda_2}\right)\,.$$

Step 3: Replace $\Lambda_2 g\left(H\right)\Lambda_2$ with $\Lambda_2 g\left(\Lambda_2 H\Lambda_2\right)\Lambda_2$. For convenience let us denote $G:=g\left(\Lambda_2 H\Lambda_2\right)$. Note that since $\Lambda_2 H\Lambda_2$ maybe have spectrum on Δ, G is *not* necessarily a projection. This is OK. We have already seen that $\Lambda_2 P\Lambda_2\in\mathscr{P}_2$. This is also true for $\Lambda_2 Q\Lambda_2$:

$$\begin{aligned}\left(\Lambda_2 Q\Lambda_2\right)^2-\Lambda_2 Q\Lambda_2 &= \Lambda_2 Q\Lambda_2 Q\Lambda_2-\Lambda_2 Q\Lambda_2\\ &= \Lambda_2\left(Q\Lambda_2 Q-Q\right)\Lambda_2\\ &= \Lambda_2\left(Q\left[\Lambda_2,Q\right]+Q^2\Lambda_2-Q\right)\Lambda_2\\ &= \Lambda_2 Q\left[\Lambda_2,Q\right]\Lambda_2+\Lambda_2\left(Q^2-Q\right)\Lambda_2\,.\end{aligned}$$

Now, $\left[\Lambda_2,Q\right]\in\mathrm{LOC}_2$ since $\Lambda_2 H\Lambda_2$ is local and g is smooth. Moreover,

$$\begin{aligned}Q^2-Q &= g\left(\Lambda_2 H\Lambda_2\right)g\left(\Lambda_2 H\Lambda_2\right)-g\left(\Lambda_2 H\Lambda_2\right)\\ &= \left(g^2-g\right)\left(\Lambda_2 H\Lambda_2\right)\,.\end{aligned}$$

However, the function g obeys $\operatorname{supp}\left(g^2-g\right)\subseteq\Delta$ and so by Lemma 3.21 we learn that

$$Q^2-Q\in\mathrm{LOC}_2\,.$$

Finally,

$$\Lambda_2 P\Lambda_2-\Lambda_2 G\Lambda_2=\Lambda_2\left(g\left(H\right)-g\left(\Lambda_2 H\Lambda_2\right)\right)\Lambda_2\,.$$

Now, if we write

$$H=\begin{bmatrix}\Lambda_2^\perp H\Lambda_2^\perp & \Lambda_2^\perp H\Lambda_2\\ \Lambda_2 H\Lambda_2^\perp & \Lambda_2 H\Lambda_2\end{bmatrix}$$

and note that

$$\Lambda_2 g\left(\Lambda_2 H \Lambda_2\right)\Lambda_2 = \Lambda_2 g\left(\begin{bmatrix}\Lambda_2^\perp H \Lambda_2^\perp & 0\\ 0 & \Lambda_2 H \Lambda_2\end{bmatrix}\right)\Lambda_2$$

then we learn that

$$\Lambda_2 P \Lambda_2 - \Lambda_2 G \Lambda_2 = \Lambda_2\left(g\left(\begin{bmatrix}\Lambda_2^\perp H \Lambda_2^\perp & \Lambda_2^\perp H \Lambda_2\\ \Lambda_2 H \Lambda_2^\perp & \Lambda_2 H \Lambda_2\end{bmatrix}\right) - g\left(\begin{bmatrix}\Lambda_2^\perp H \Lambda_2^\perp & 0\\ 0 & \Lambda_2 H \Lambda_2\end{bmatrix}\right)\right)\Lambda_2\,.$$

Now we invoke Section 1.3.4 to get

$$\begin{aligned} g\left(\begin{bmatrix}\Lambda_2^\perp H \Lambda_2^\perp & \Lambda_2^\perp H \Lambda_2\\ \Lambda_2 H \Lambda_2^\perp & \Lambda_2 H \Lambda_2\end{bmatrix}\right) - g\left(\begin{bmatrix}\Lambda_2^\perp H \Lambda_2^\perp & 0\\ 0 & \Lambda_2 H \Lambda_2\end{bmatrix}\right) &= \frac{1}{2\pi}\int_{z\in\mathbb{C}}\left(\partial_{\overline{z}}\tilde{g}\right)(z)\left(\left(\begin{bmatrix}\Lambda_2^\perp H \Lambda_2^\perp & \Lambda_2^\perp H \Lambda_2\\ \Lambda_2 H \Lambda_2^\perp & \Lambda_2 H \Lambda_2\end{bmatrix} - z\mathbb{1}\right)^{-1} - \left(\begin{bmatrix}\Lambda_2^\perp H \Lambda_2^\perp \\ 0\end{bmatrix}\right.\right. \\ &= \frac{1}{2\pi}\int_{z\in\mathbb{C}}\left(\partial_{\overline{z}}\tilde{g}\right)(z)\left(\left(\begin{bmatrix}\Lambda_2^\perp H \Lambda_2^\perp & \Lambda_2^\perp H \Lambda_2\\ \Lambda_2 H \Lambda_2^\perp & \Lambda_2 H \Lambda_2\end{bmatrix} - z\mathbb{1}\right)^{-1}\left(\begin{bmatrix}0 \\ -\Lambda_2 H \Lambda_2^\perp\end{bmatrix}\right.\right. \end{aligned}$$

But now, the middle term contains $\Lambda_2 H \Lambda_2^\perp = \left[\Lambda_2, H\right]\Lambda_2^\perp \in \mathrm{LOC}_2$ so that by the ideal property this decays into the bulk too.

Step 4: Replace

$$\mathrm{index}_{\ell^2(\mathcal{H})}\left(\mathbb{A}_1 \mathrm{e}^{-2\pi \mathrm{i}\Lambda_2 g(\Lambda_2 H \Lambda_2)\Lambda_2}\right) = \mathrm{index}_{\ell^2\left(\hat{\mathcal{H}}\right)}\left(\mathbb{A}_1 \mathrm{e}^{-2\pi \mathrm{i} g(J^* H J)}\right)\,.$$

Actually we will make a statement that does not involve a homotopy, but directly relating the kernels as

$$\ker_{\ell^2(\mathcal{H})}\left(\mathbb{A}_1 \mathrm{e}^{\mp 2\pi \mathrm{i}\Lambda_2 g(\Lambda_2 H \Lambda_2)\Lambda_2}\right) = \ker_{\ell^2\left(\hat{\mathcal{H}}\right)}\left(\mathbb{A}_1 \mathrm{e}^{\mp 2\pi \mathrm{i} g(J^* H J)}\right)\,.$$

Let us define

$$F := J\left(\mathbb{A}_1 \mathrm{e}^{\mp 2\pi \mathrm{i} g(J^* H J)}\right)J^* + \Lambda_2^\perp$$

as an operator on $\ell^2(\mathcal{H})$. We claim it is Fredholm and has the same kernel on $\ell^2(\mathcal{H})$ as $\mathbb{A}_1 \mathrm{e}^{\mp 2\pi \mathrm{i} g(J^* H J)}$ does on $\ell^2\left(\hat{\mathcal{H}}\right)$. To see it is Fredholm, let us define its parametrix

$$G := J\left(\mathbb{A}_1 \mathrm{e}^{\pm 2\pi \mathrm{i} g(J^* H J)}\right)J^* + \Lambda_2^\perp\,.$$

Then

$$\begin{aligned} \mathbb{1} - FG &= \mathbb{1} - \left(J\left(\mathbb{A}_1 \mathrm{e}^{\mp 2\pi \mathrm{i} g(J^* H J)}\right)J^* + \Lambda_2^\perp\right)\left(J\left(\mathbb{A}_1 \mathrm{e}^{\pm 2\pi \mathrm{i} g(J^* H J)}\right)J^* + \Lambda_2^\perp\right) \\ &= \Lambda_2 - J\left(\mathbb{A}_1 \mathrm{e}^{\mp 2\pi \mathrm{i} g(J^* H J)}\right)\underbrace{J^* J}_{=\mathbb{1}}\left(\mathbb{A}_1 \mathrm{e}^{\pm 2\pi \mathrm{i} g(J^* H J)}\right)J^* \\ &= \Lambda_2 - J\left(\mathbb{A}_1 \mathrm{e}^{\mp 2\pi \mathrm{i} g(J^* H J)}\right)\left(\mathbb{A}_1 \mathrm{e}^{\pm 2\pi \mathrm{i} g(J^* H J)}\right)J^* \\ &= J\left(\mathbb{1} - \left(\mathbb{A}_1 \mathrm{e}^{\mp 2\pi \mathrm{i} g(J^* H J)}\right)\left(\mathbb{A}_1 \mathrm{e}^{\pm 2\pi \mathrm{i} g(J^* H J)}\right)\right)J^* \\ &= J\left(\Lambda_1 - \Lambda_1 \mathrm{e}^{\mp 2\pi \mathrm{i} g(J^* H J)}\Lambda_1 \mathrm{e}^{\pm 2\pi \mathrm{i} g(J^* H J)}\Lambda_1\right)J^* \\ &= J\Lambda_1 \mathrm{e}^{\mp 2\pi \mathrm{i} g(J^* H J)}\Lambda_1^\perp \mathrm{e}^{\pm 2\pi \mathrm{i} g(J^* H J)}\Lambda_1 J^* \\ &= \underbrace{J\left[\Lambda_1, \mathrm{e}^{\mp 2\pi \mathrm{i} g(J^* H J)}\right]}_{\in\mathcal{K}}\Lambda_1^\perp \mathrm{e}^{\pm 2\pi \mathrm{i} g(J^* H J)}\Lambda_1 J^* \\ &\in \mathcal{K}\,. \end{aligned}$$

To see that $\mathbb{A}_1 \mathrm{e}^{\mp 2\pi \mathrm{i} g(J^* H J)}$ has the same kernel as F, let us setup a bijection

$$\begin{aligned} \eta : \ker\left(\mathbb{A}_1 \mathrm{e}^{\mp 2\pi \mathrm{i} g(J^* H J)}\right) &\to \ker(F) \\ \psi &\mapsto J\psi\,. \end{aligned}$$

First, this is well-defined since

$$\begin{aligned} FJ\psi &= \left(J\left(\mathbb{A}_1 \mathrm{e}^{\mp 2\pi \mathrm{i} g(J^*HJ)}\right)J^* + \Lambda_2^\perp\right)J\psi \\ &= J\left(\mathbb{A}_1 \mathrm{e}^{\mp 2\pi \mathrm{i} g(J^*HJ)}\right)\underbrace{J^*J}_{=\mathbb{1}}\psi + \underbrace{\Lambda_2^\perp J}_{=0}\psi \\ &= \underbrace{J\left(\mathbb{A}_1 \mathrm{e}^{\mp 2\pi \mathrm{i} g(J^*HJ)}\right)\psi}_{=0} \\ &= 0\,. \end{aligned}$$

It is a bijection since it has an inverse $\eta^{-1} : \ker(F) \to \ker\left(\mathbb{A}_1 \mathrm{e}^{\mp 2\pi \mathrm{i} g(J^*HJ)}\right)$ given by

$$\eta^{-1}(\psi) := J^*\psi\,.$$

The inverse is well-defined since

$$\begin{aligned} \left(\mathbb{A}_1 \mathrm{e}^{\mp 2\pi \mathrm{i} g(J^*HJ)}\right)J^*\psi &= \left(\left(\mathbb{A}_1 \mathrm{e}^{\mp 2\pi \mathrm{i} g(J^*HJ)}\right)J^* + \Lambda_2^\perp\right)\psi \\ &= \underbrace{J^*J}_{=\mathbb{1}}\left(\left(\mathbb{A}_1 \mathrm{e}^{\mp 2\pi \mathrm{i} g(J^*HJ)}\right)J^* + \Lambda_2^\perp\right)\psi \\ &= J^*\underbrace{F\psi}_{=0} \\ &= 0\,. \end{aligned}$$

We verify that

$$\eta^{-1}\circ\eta = \mathbb{1}\,.$$

Indeed, $J^*J = \mathbb{1}_{\hat{\mathcal{H}}}$ and conversely, $\eta\circ\eta^{-1} = \Lambda_2$ which acts trivially on

$$\ker\left(J\left(\mathbb{A}_1 \mathrm{e}^{\mp 2\pi \mathrm{i} g(J^*HJ)}\right)J^* + \Lambda_2^\perp\right)\,.$$

As a result, η is indeed a linear bijection and so the two kernels are isomorphic. But now we have, using the fact that $[J,\Lambda_1] = [J^*,\Lambda_1] = 0$, we have the operator as

$$\begin{aligned} J\left(\mathbb{A}_1 \mathrm{e}^{\mp 2\pi \mathrm{i} g(J^*HJ)}\right)J^* + \Lambda_2^\perp &= J\left(\Lambda_1 \mathrm{e}^{\mp 2\pi \mathrm{i} g(J^*HJ)}\Lambda_1 + \Lambda_1^\perp\right)J^* + \Lambda_2^\perp \\ &= \Lambda_1 J \mathrm{e}^{\mp 2\pi \mathrm{i} g(J^*HJ)}J^*\Lambda_1 + \Lambda_1^\perp\underbrace{JJ^*}_{=\Lambda_2} + \Lambda_2^\perp \\ &= \Lambda_1 J \mathrm{e}^{\mp 2\pi \mathrm{i} g(J^*HJ)}J^*\Lambda_1 + \Lambda_1^\perp\Lambda_2 + \Lambda_2^\perp\,. \end{aligned}$$

Next, note that $g(J^*HJ) = J^*g(\Lambda_2 H\Lambda_2)J$. Indeed,

$$\begin{aligned} g(J^*HJ) &= \frac{1}{2\pi}\int_{z\in\mathbb{C}}(\partial_{\overline{z}}\tilde{g})(z)\left(J^*HJ - z\mathbb{1}_{\hat{\mathcal{H}}}\right)^{-1}\mathrm{d}z \\ &= \frac{1}{2\pi}\int_{z\in\mathbb{C}}(\partial_{\overline{z}}\tilde{g})(z)\,J^*\left(\Lambda_2 H\Lambda_2 - z\mathbb{1}_{\mathcal{H}}\right)^{-1}J\mathrm{d}z \\ &= J^*g(\Lambda_2 H\Lambda_2)\,J\,. \end{aligned}$$

So we get that our kernel equals

$$\ker\left(\Lambda_1 J \mathrm{e}^{\mp 2\pi \mathrm{i} J^* g(\Lambda_2 H\Lambda_2)J}J^*\Lambda_1 + \Lambda_1^\perp\Lambda_2 + \Lambda_2^\perp\right)\,.$$

But if we expand the exponential we get

$$\begin{aligned} J\mathrm{e}^{\mp 2\pi \mathrm{i} J^* g(\Lambda_2 H \Lambda_2) J} J^* &= J \sum_{n=0}^{\infty} \frac{1}{n!} \left(\mp 2\pi \mathrm{i} J^* g\left(\Lambda_2 H \Lambda_2\right) J\right)^n J^* \\ &= J J^* + \sum_{n=1}^{\infty} \frac{1}{n!} J \left(\mp 2\pi \mathrm{i} J^* g\left(\Lambda_2 H \Lambda_2\right) J\right)^n J^* \\ &= \Lambda_2 + \Lambda_2 \sum_{n=1}^{\infty} \frac{1}{n!} \left(\mp 2\pi \mathrm{i} \Lambda_2 g\left(\Lambda_2 H \Lambda_2\right) \Lambda_2\right)^n \Lambda_2 \\ &= \Lambda_2 + \Lambda_2 \left(\mathrm{e}^{\mp 2\pi \mathrm{i} \Lambda_2 g(\Lambda_2 H \Lambda_2) \Lambda_2} - \mathbb{1}\right) \Lambda_2 \\ &= \mathrm{e}^{\mp 2\pi \mathrm{i} \Lambda_2 g(\Lambda_2 H \Lambda_2) \Lambda_2} - \Lambda_2^\perp . \end{aligned}$$

Finally, we use

$$\Lambda_1^\perp = -\Lambda_1 \Lambda_2^\perp + \Lambda_1^\perp \Lambda_2 + \Lambda_2^\perp$$

to get

$$\ker_{\mathcal{H}} \left(\Lambda_1 \mathrm{e}^{\mp 2\pi \mathrm{i} \Lambda_2 g(\Lambda_2 H \Lambda_2) \Lambda_2} \Lambda_1 + \Lambda_1^\perp\right)$$

which is what we wanted.

Step 5: Replace $g\left(J^* H J\right)$ with $g\left(\hat{H}\right)$. This amounts to the same lemma Section 1.3.4 because we assume that $J^* H J - \hat{H} \in \mathrm{LOC}_2$. We are thus finished. □

3.10 Chiral 1D systems

The following section might have some overlap with [Sha24] but we include it here for convenience of the reader.

A Fredholm theory

In this section, we temporarily forget about the spatial structure of our Hilbert space. Thus, $\mathcal{H}$ is *any* separable Hilbert space, which schematically is usually denoted as $\mathbb{C}^\infty$ as above. If we want a concrete choice it could be $\ell^2(\mathbb{N})$ with ONB $\{e_j\}_{j=1}^\infty$. On $\mathcal{H}$ we consider $\mathcal{B}(\mathcal{H})$, the C-star algebra of bounded linear operators $\mathcal{H} \to \mathcal{H}$ together with the operator norm

$$\|A\| \equiv \sup\left(\{ \|A\varphi\| \mid \varphi \in \mathcal{H} : \|\varphi\| = 1 \}\right) .$$

Definition A.1. An operator $A \in \mathcal{B}(\mathcal{H})$ is called *Fredholm* iff $\ker A$ and $\operatorname{coker} A$ are finite dimensional. These are the two vector spaces defined as

$$\ker A \equiv \{ \varphi \in \mathcal{H} \mid A\varphi = 0 \}$$

and

$$\operatorname{coker} A \equiv \mathcal{H}/\operatorname{im} A$$

with $\operatorname{im} A \equiv \{ \varphi \in \mathcal{H} \mid \exists \psi : A\psi = \varphi \}$. Recall that the quotient vector space is defined as:

$$\varphi \sim \psi \iff \varphi - \psi \in \operatorname{im} A$$

and

$$\begin{aligned} [\varphi] &:= \{ \psi \in \mathcal{H} \mid \varphi \sim \psi \} \\ \operatorname{coker} A &= \{ [\varphi] \mid \varphi \in \mathcal{H} \} . \end{aligned}$$

For every Fredholm operator, we define the index, an integer associated with that operator, as

$$\operatorname{index}(A) := \dim\ker A - \dim\operatorname{coker} A \in \mathbb{Z} .$$

We denote the space of all Fredholm operators as $\mathcal{F}(\mathcal{H})$.

The kernel of an operator $\ker A$ should not be confused with its *integral kernel*

$$A_{xy} \equiv \langle e_x, A e_y \rangle$$

in the context of PDEs, i.e., its matrix elements in the context of quantum mechanics.

Intuitively speaking, the kernel measures how much an operator deviates from being injective, whereas the cokernel measures how much an operator deviates from being surjective. Hence, Fredholm operators are those that are injective and surjective (and hence invertible) up to some finite dimensional "defect", and the index of the operator measures the severity of this defect, so to speak (in a signed way). The fact that one has a difference of two numbers instead of just one number should be associated with the Grothendiek construction of a group out of a semigroup, or with the construction of $\mathbb{Z}$ out of $\mathbb{N}$ as pairs of naturals which should be identified with the difference.

The following result allows us to speak only of kernels of operators instead of the more mysterious cokernel:

Lemma A.2. *$\operatorname{coker} A$ is finite-dimensional iff $\operatorname{im} A \in \operatorname{Closed}(\mathcal{H})$ and $\ker A^*$ is finite dimensional.*

Proof. Assume that $\operatorname{im} A \in \operatorname{Closed}(\mathcal{H})$. Then via Claim B.1,

$$\begin{aligned} (\ker(A^*))^\perp &= \left((\operatorname{im} A)^\perp\right)^\perp \\ &= \overline{\operatorname{im} A} && \text{(Via Claim B.4)} \\ &= \operatorname{im} A . && \text{(By hypothesis)} \end{aligned}$$

Now, we always have

$$\begin{aligned}\mathscr{H} &= (\ker A)\oplus\left((\ker A)^{\perp}\right) = (\ker A^*)\oplus\left((\ker A^*)^{\perp}\right)\\ &= (\ker A^*)\oplus \mathrm{im}A\,.\end{aligned}$$

Hence

$$\begin{aligned}\mathrm{coker}A &\equiv \mathscr{H}/\mathrm{im}A\\ &\cong (\mathrm{im}A)^{\perp}\\ &= \ker A^*\,.\end{aligned}$$

Hence if $\dim\ker A^*$ is finite, so is $\dim\mathrm{coker}A$.

Conversely, assume that $\dim\mathrm{coker}A$ is finite. We want to show that $\mathrm{im}A\in\mathrm{Closed}\,(\mathscr{H})$.

Define a map

$$\begin{aligned}\eta:(\mathscr{H}/\ker A)\oplus(\mathrm{im}A)^{\perp} &\to \mathscr{H}\\ ([\varphi],\psi) &\mapsto A\varphi+\psi\,.\end{aligned}$$

It is easy to verify that η is a bounded linear bijection (it is in verifying that η is bounded that we used the fact $\mathrm{coker}A\cong(\mathrm{im}A)^{\perp}$ is finite dimensional). Hence

$$\mathrm{im}A\cong\eta\left((\mathscr{H}/\ker A)\oplus\{\,0\,\}\right)\in\mathrm{Closed}\,(\mathscr{H})$$

where the last statement is due to the open mapping them which says that the inverse of η is also continuous, i.e., η is a closed map and hence maps closed sets to closed sets. □

See also [AA02, Corollary 2.17].

The Fredholm index is continuous, as well shall see, but only half of it is *not*:

Lemma A.3. $\dim\ker:\mathscr{F}(\mathscr{H})\to\mathbb{N}_{\geq 0}$ *is upper semicontinuous.*

Proof. Decompose $\mathscr{H}=\ker(A)\oplus\ker(A)^{\perp}\cong\ker(A^*)\oplus\ker(A^*)^{\perp}$. Since $\mathrm{im}\,(A)\cong\ker(A^*)^{\perp}$, we have

$$A = \begin{bmatrix}a & 0\\ 0 & 0\end{bmatrix}$$

for some isomorphism $a:\ker(A)^{\perp}\to\mathrm{im}\,(A)$. Taking any norm perturbation B of size at most $\left\|a^{-1}\right\|^{-1}$ will mean that $A+B$ is injective on $\ker(A)^{\perp}$ and hence $\dim\ker A+B\leq\dim\ker A$. □

To characterizer the the image of an operator being closed, we have

Lemma A.4. *For $A\in\mathscr{B}(\mathscr{H})$, the following are equivalent:*

1. $\mathrm{im}A\in\mathrm{Closed}\,(\mathscr{H})$.
2. $0\notin\sigma\left(|A|^2\right)$ *or zero is an isolated point of* $\sigma\left(|A|^2\right)$.
3. $\exists\varepsilon>0$ *such that*

$$\|A\varphi\| \geq \varepsilon\|\varphi\| \qquad \left(\varphi\in(\ker A)^{\perp}\right)\,.$$

Proof. ((1)=>(3)): Assume that $\mathrm{im}\,(A)\in\mathrm{Closed}\,(\mathscr{H})$. Then $\tilde{A}:\ker(A)^{\perp}\to\mathrm{im}\,(A)$ is a bijection. Since $\mathrm{im}\,(A)\in\mathrm{Closed}\,(\mathscr{H})$, $\mathrm{im}\,(A)$ is a complete metric space, which implies that $\tilde{A}^{-1}:\mathrm{im}\,(A)\to\ker(A)^{\perp}$ is bounded by the "bounded inverse theorem" [RS80]. I.e., $\left\|\tilde{A}^{-1}\right\|<\infty$, which is tantamount to saying that $\exists c<\infty$ such that

$\left\|\tilde{A}^{-1}\varphi\right\| < c\|\varphi\|$ for all $\varphi \in \operatorname{im}(A)$. Now if $\psi \in \ker(A)^\perp$, then $A\psi \in \operatorname{im}(A)$, and so $\tilde{A}^{-1}A\psi \equiv \psi$. Hence

$$\begin{aligned} \|\psi\| &\leq c\|A\psi\| \\ &\updownarrow \\ \frac{1}{c}\|\psi\| &\leq \|A\psi\| . \end{aligned}$$

((3)=>(1)): Let $\{ \varphi_n \}_n \subseteq \operatorname{im}(A)$ such that $\lim_n \varphi_n = \psi$ for some $\psi \in \mathcal{H}$. Our goal is to show that $\psi \in \operatorname{im}(A)$. If $\psi = 0$ we are finished so assume $\psi \neq 0$. Since $\{ \varphi_n \} \subseteq \operatorname{im}(A)$, $\exists \{ \eta_n \}_n \subseteq \mathcal{H}$ such that $A\eta_n = \varphi_n$. We assume WLOG that $\eta_n \in \ker(A)^\perp$ (if this is false for all η_n then $\varphi_n = 0$, so if necessary take a subsequence of such η_n). Hence, $\|A\eta_n\| \geq \varepsilon\|\eta_n\|$ by hypothesis. We claim $\{ \eta_n \}_n$ is Cauchy. Indeed, $\|\eta_n - \eta_m\| \leq \frac{1}{\varepsilon}\|A(\eta_n - \eta_m)\| = \frac{1}{\varepsilon}\|\varphi_n - \varphi_m\|$. But $\{ \varphi_n \}_n$ converges and is hence Cauchy. Hence $\lim_n \eta_n = \xi$ for some $\xi \in \mathcal{H}$ (because regardless of the status of $\operatorname{im}(A)$, $\mathcal{H}$ certainly *is* complete and hence Cauchy sequence converge). Since A is bounded it is continuous, and so we find that

$$\begin{aligned} A\xi &= A\lim_n \eta_n \\ &= \lim_n A\eta_n \\ &= \lim_n \varphi_n \\ &= \psi . \end{aligned}$$

We obtain then that $\psi \in \operatorname{im}(A)$ as desired.

((2)<=>(3)): We have $\ker\left(|A|^2\right)^\perp = \ker(A)^\perp$ using Lemma B.5. Now, since $|A|^2 \geq 0$, (2) is equivalent to $|A|^2 \geq \varepsilon\mathbb{1}$ on $\ker(A)^\perp$. Hence

$$\begin{aligned} |A|^2 - \varepsilon\mathbb{1} \geq 0 &\iff \left\langle\psi, \left(|A|^2 - \varepsilon\mathbb{1}\right)\psi\right\rangle \forall\psi \in \ker(A)^\perp \\ &\iff \|A\psi\|^2 \geq \varepsilon^2\|\psi\|^2 \forall\psi \in \ker(A)^\perp \end{aligned}$$

□

Thus, we could alternatively define A to be Fredholm when $\ker A, \ker A^*$ are finite dimensional *and* $\operatorname{im}A$ is closed, and we could just as well write for its index

$$\operatorname{index}(A) = \dim\ker A - \dim\ker A^* \in \mathbb{Z} .$$

From this formula it is clear that if a self-adjoint operator is Fredholm then its index must be zero.

Example A.5. Here are a few trivial examples for the notion of a Fredholm operator:

1. The identity operator $\mathbb{1}$ is Fredholm and its index is zero.
2. The zero operator $\mathcal{H} \ni v \mapsto 0$ is *not* Fredholm.
3. Recall the position operator X from (1.4), which is not even bounded. Its inverse $A := X^{-1}$ is however bounded: $\|A\| \leq 1$. It is however *not* Fredholm, even though it is self-adjoint and has an empty kernel. To see that $\operatorname{im}A \notin \operatorname{Closed}\left(\ell^2(\mathbb{N})\right)$, use the second characterization of Lemma A.4 and note that while zero is not in $\sigma\left(|A|^2\right)$, it is an accumulation point and hence not isolated in $\sigma\left(|A|^2\right)$. What is the cokernel of A? Is it finite dimensional?
4. The right-shift operator R from Example 1.9 on $\ell^2(\mathbb{N})$ is Fredholm. Indeed, one checks that

   $$|R|^2 = R^*R = \mathbb{1}$$

   and hence by the second characterization of Lemma A.4 it has closed image. Its kernel is empty and the kernel of its adjoint, the left shift operator, is spanned by δ_1 and is hence one dimensional.

   $$\operatorname{index}(R) = -1 .$$

   Note that, considered on $\ell^2(\mathbb{Z})$, R is also Fredholm, but it is now invertible and hence has zero index. The right shift operator is the most important example of a Fredholm operator, and in a sense, all other non-zero index operators may be connected to a power of the right shift, as we shall see.

Claim A.6. If $A \in \mathcal{B}(\mathcal{H}_1 \to \mathcal{H}_2)$ with $\mathcal{H}_1, \mathcal{H}_2$ finite dimensional, then A is Fredholm and its index equals

$$\text{index}(A) = \dim(\mathcal{H}_1) - \dim(\mathcal{H}_2) .$$

Proof. The rank-nullity theorem [KK07] states that

$$\dim(\mathcal{H}_1) = \dim(\ker(A)) + \dim(\text{im}(A)) .$$

Furthermore, since $\text{coker}(A) \equiv \mathcal{H}_2 / \text{im}(A)$, we have

$$\dim(\text{coker}(A)) = \dim(\mathcal{H}_2) - \dim(\text{im}(A)) .$$

Thus, we have

$$\begin{aligned} \text{index}(A) &\equiv \dim(\ker(A)) - \dim(\text{coker}(A)) \\ &= \dim(\mathcal{H}_1) - \dim(\text{im}(A)) - \dim(\mathcal{H}_2) + \dim(\text{im}(A)) \\ &= \dim(\mathcal{H}_1) - \dim(\mathcal{H}_2) . \end{aligned}$$

as desired. ☐

In particular, any square matrix is Fredholm with index zero: finite dimensions are not very interesting for Fredholm theory. Be that as it may some mechanical models in physics have been studied of finite *non*-square matrices, which have a non-zero index.

Definition A.7. An operator $F \in \mathcal{B}(\mathcal{H})$ is called *finite rank* iff $\dim(\text{im}(F)) < \infty$ iff it may be written in the form

$$F = \sum_{i=1}^{N} f_i \varphi_i \otimes \psi_i^*$$

where $\{ f_i \}_{i=1}^{N} \geq 0$ are the singular values of F and $\{ \varphi_i \}_i$, $\{ \psi_i \}$ are orthonormal bases of $\mathcal{H}$.

For the next definition, we use the open ball definition

$$B_\varepsilon(v) \equiv \{ u \in \mathcal{H} \mid \|u - v\| < \varepsilon \} .$$

Definition A.8. An operator $K \in \mathcal{B}(\mathcal{H})$ is called *compact iff* $\overline{K\left(\overline{B_1(0)}\right)}$ *is compact in* $\mathcal{H}$ iff it is the operator-norm limit of finite rank operators iff it may be written as

$$K = \lim_{N\to\infty} \sum_{i=1}^{N} f_i \varphi_i \otimes \psi_i^*$$

where the limit is meant in the operator-norm and f_i may accumulate only at zero. The set of all compact operators is denoted as $\mathcal{K}(\mathcal{H})$. It is a two-sided ideal within $\mathcal{H}$.

Lemma A.9. *(Riesz)* $\mathbb{1} - K \in \mathcal{F}(\mathcal{H})$ *for all* $K \in \mathcal{K}(\mathcal{H})$ *and* $\text{index}(\mathbb{1} - K) = 0$.

Proof. Write $K = \lim_n F_n$ (in operator norm) where F_n is finite rank. Hence $\mathbb{1} - K + F_n$ is invertible for n sufficiently large since $\|K - F_n\|$ may be made arbitrarily small. Then,

$$\mathbb{1} - K = (\mathbb{1} - K + F_n)\left(\mathbb{1} - (\mathbb{1} - K + F_n)^{-1} F_n\right)$$

so that

$$\mathbb{1} - K = G(\mathbb{1} - F)$$

with G invertible and F finite rank. Hence $\ker(\mathbb{1} - K) = \ker(\mathbb{1} - F)$. Now, $v \in \ker(\mathbb{1} - F)$ iff $v = Fv$ which implies that v is an eigenvector of F with eigenvalue 1. But this if F is finite rank its eigenspaces are finite dimensional. Same for $\mathbb{1}-F^*$. The two kernels are of the same dimension since F is of finite rank. ☐

Theorem A.10. *(Atkinson) $A\in\mathcal{F}(\mathcal{H})$ iff A is invertible up to compacts, i.e., iff there is some operator $B\in\mathcal{B}(\mathcal{H})$, called* the parametrix of A, *such that,*

$$\mathbb{1}-AB,\mathbb{1}-BA\in\mathcal{K}(\mathcal{H})\ .$$

We note that we may have $\mathbb{1}-AB\neq\mathbb{1}-BA$ indeed. Furthermore, $\operatorname{index}(B)=-\operatorname{index}(A)$.

Proof. If $\mathbb{1}-AB,\mathbb{1}-BA\in\mathcal{K}(\mathcal{H})$ then $BA=\mathbb{1}-K$ for some compact K and using Lemma A.9 we have that BA is Fredholm of index zero. Hence $\ker(BA)$ is finite dimensional. But $\ker(A)\subseteq\ker(BA)$ so that $\ker(A)$ is finite dimensional. Also, $\mathcal{K}(\mathcal{H})$ is closed under adjoint, so that $\ker(A^*)$ is also finite. Finally, let $v\in\ker(A)^\perp$. Then $v\in\ker(BA)^\perp$. Since BA is Fredholm, it has closed image, so that by Lemma A.4 we have

$$\begin{aligned}\varepsilon\|\varphi\| &\leq \|BA\varphi\|\\ &\leq \|B\|\|A\varphi\|\,.\end{aligned}$$

Consequently, A has closed range. Thus A is Fredholm. Now, using the logarithmic law Theorem A.18 further below, since $AB=\mathbb{1}-K$, $0=\operatorname{index}(AB)=\operatorname{index}(A)+\operatorname{index}(B)$.

Also

$$\begin{aligned}\ker(B) &\subseteq \ker(AB)\\ &= \ker(B^*A^*)\end{aligned}$$

Conversely, assume $A\in\mathcal{F}(\mathcal{H})$. Want to construct two partial inverses: let P,Q be the orthogonal projections onto $\ker(A)$ and $\ker(A^*)$ resp. We claim that $|A|^2+P$ and $|A^*|^2+Q$ are bijections. Indeed, $\ker(A)=\ker\left(|A|^2\right)$ so if $\mathcal{H}\cong\ker\left(|A|^2\right)^\perp\oplus\ker\left(|A|^2\right)$, $|A|^2+P\cong\left.|A|^2\right|_{\operatorname{im}(P)^\perp}\oplus\mathbb{1}$ and similarly for the other operator. Hence $B:=|A|^2+P$ is invertible, and

$$\mathbb{1} = B^{-1}A^*A+B^{-1}P\,.$$

But now, $B^{-1}P$ is of finite rank and $C:=B^{-1}A^*$ is the sought-after parametrix. ☐

Definition A.11. The *essential spectrum* $\sigma_{\text{ess}}(A)$ of an operator $A\in\mathcal{B}(\mathcal{H})$ is the set of all points $z\in\mathbb{C}$ such that $A-z\mathbb{1}$ is *not* Fredholm.

Claim A.12. If $A\in\mathcal{F}(\mathcal{H})$ and $\operatorname{index}(A)=0$ then $A=G+K$ for some G invertible and K compact.

Proof. Since $\operatorname{index}(A)=0$, $\dim\ker A=\dim\ker A^*$. Thus,

$$\mathcal{H} \cong \ker(A)\oplus\ker(A)^\perp\cong\ker(A^*)\oplus\ker(A^*)^\perp\ .$$

But we know that since $\dim\ker A=\dim\ker A^*$, there is a natural linear isomorphism $\eta:\ker(A)\to\ker(A^*)$. We also know that $\operatorname{im}(A)\cong\ker(A^*)^\perp$. Hence, $A|_{\ker(A)^\perp}$ is just an isomorphism

$$\ker(A)^\perp \to \operatorname{im}(A)\ .$$

Hence, the map

$$G:=\eta\oplus A|_{\ker(A)^\perp}:\mathcal{H} \to \mathcal{H}$$

is an isomorphism and $\mathcal{H}\ni(v_1,v_2)\overset{K}{\mapsto}(\eta(v_1),0)\in\mathcal{H}$ is compact and hence the result. ☐

Theorem A.13. *We have the inclusion*

$$\mathcal{F}(\mathcal{H}) + \mathcal{K}(\mathcal{H}) \quad \subseteq \quad \mathcal{F}(\mathcal{H})$$

and the Fredholm index is stable under compact perturbations.

Proof. If $A \in \mathcal{F}(\mathcal{H})$ and $K \in \mathcal{K}(\mathcal{H})$, then by Atkinson Theorem A.10, there is some parametrix B such that $AB - \mathbb{1}, BA - \mathbb{1}$ is compact. But B will be a parametrix of $A + K$ too:

$$(A + K) B - \mathbb{1} \quad = \quad AB + KB - \mathbb{1}$$

which is compact since KB is compact (as the compacts form an ideal). Hence $A + K$ is Fredholm. We postpone the proof that the index remains stable until the next theorem. ☐

We see from Atkinson's theorem that the essential spectrum is stable under compact perturbations.

Theorem A.14. *(Dieudonne)* index : $\mathcal{F}(\mathcal{H}) \to \mathbb{Z}$ *is operator-norm-continuous and if* $B \in \mathcal{F}(\mathcal{H})$ *is any parametrix of* $A \in \mathcal{F}(\mathcal{H})$ *then*

$$B_{\|B\|^{-1}}(A) \quad \subseteq \quad \mathcal{F}(\mathcal{H}) \, .$$

In particular, $\mathcal{F}(\mathcal{H}) \in \mathrm{Open}(\mathcal{B}(\mathcal{H}))$.

Proof. Let $A \in \mathcal{F}(\mathcal{H})$ and B be any parametrix of it. Take any $\tilde{A} \in B_{\|B\|^{-1}}(A)$. We have

$$\left\|B\left(A - \tilde{A}\right)\right\| \leq \|B\|\left\|A - \tilde{A}\right\| < 1$$

by assumption, so that $\mathbb{1} - B\left(A - \tilde{A}\right)$ is invertible. We claim that $\left(\mathbb{1} - B\left(A - \tilde{A}\right)\right)^{-1} B$ is a parametrix for $\tilde{A}$. Now,

$$\begin{aligned} 0 &= \mathrm{index}(\mathbb{1}) \\ &= \mathrm{index}(\mathbb{1} - K) \\ &= \mathrm{index}\left(\left(\mathbb{1} - B\left(A - \tilde{A}\right)\right)^{-1} B\tilde{A}\right) \\ &= \mathrm{index}\left(B\tilde{A}\right) \end{aligned}$$

Now, via Theorem A.18 further below we have $\mathrm{index}(B) + \mathrm{index}\left(\tilde{A}\right)$ and since $\mathrm{index}(B) = -\mathrm{index}(A)$ we obtain the result. ☐

Finally, we finish the proof of Theorem A.13: if $A \in \mathcal{F}(\mathcal{H})$ and $K \in \mathcal{K}(\mathcal{H})$, the homotopy $[0,1] \ni t \mapsto A + tK \in \mathcal{F}(\mathcal{H})$ interpolates in a norm continuous way between A and $A+K$ and thus the index is constant along this path.

Claim A.15. A is invertible up to compacts iff it is invertible up to finite ranks.

Proof. Since finite rank operators are compact one direction is trivial. Now, assume that there is some $B \in \mathcal{F}(\mathcal{H})$ with which $\mathbb{1} - AB, \mathbb{1} - BA \in \mathcal{K}(\mathcal{H})$. Let $\{F_n\}_n$ be a sequence of finite rank operators which converges to $K := \mathbb{1} - BA$ in operator norm. Then $\|F_n - K\|$ can be made arbitrarily small and hence $W_n := \mathbb{1} - K + F_n$ is invertible for n sufficiently large. Then,

$$\begin{aligned} BA &= \mathbb{1} - K \\ &= W_n\left(\mathbb{1} - W^{-1} F_n\right) \end{aligned}$$

and hence

$$\mathbb{1}-W_n^{-1}BA = W^{-1}F_n\,.$$

Since finite rank operators form an ideal, $W_n^{-1}F_n$ is finite rank too. This same logic shows that

$$\mathbb{1}-AB\tilde{W}_n^{-1} = \tilde{F}_n\tilde{W}_n^{-1}$$

where now $\tilde{F}_n\tilde{W}_n^{-1}$ is finite rank. So Now, $W_n^{-1}B$ is a partial left inverse and $B\tilde{W}_n^{-1}$is a partial right inverse. Then

$$\begin{aligned}
W_n^{-1}BA &= \mathbb{1}-W^{-1}F_n \\
W_n^{-1}BAB\tilde{W}_n^{-1} &= B\tilde{W}_n^{-1}-W^{-1}F_nB\tilde{W}_n^{-1} \\
W_n^{-1}B\left(\mathbb{1}-\tilde{F}_n\tilde{W}_n^{-1}\right) &= B\tilde{W}_n^{-1}-W^{-1}F_nB\tilde{W}_n^{-1} \\
W_n^{-1}B-B\tilde{W}_n^{-1} &= W_n^{-1}B\tilde{F}_n\tilde{W}_n^{-1}-W^{-1}F_nB\tilde{W}_n^{-1}
\end{aligned}$$

and since the finite rank operators form an ideal within $\mathcal{B}(\mathcal{H})$, we find that $W_n^{-1}B$ equals $B\tilde{W}_n^{-1}$ up to finite rank operators and hence there is just one parametrix, say, $W_n^{-1}B$.

The following is taken from [Mur94]: □

Theorem A.16. *(Fedosov) If $A\in\mathcal{F}(\mathcal{H})$ and B is* any *parametrix of A such that $\mathbb{1}-AB,\mathbb{1}-BA$ is finite rank, then*

$$\operatorname{index}(A) = \operatorname{tr}([A,B])\,.$$

Proof. First we note that if B,B' are two such parametrii, then $B'=B+F$ for some F finite rank. Then $[A,B']=[A,B]+[A,F]$ and since F is finite rank, $\operatorname{tr}([A,F])=0$. So we are free to choose *any* such finite rank parametrix.

We claim there is a finite-rank parametrix B such that $A=ABA$. If we can find such a parametrix, then $\mathbb{1}-AB$ and $\mathbb{1}-BA$ are idempotents, and their traces equal the dimension of the cokernel and kernel of A respectively and hence the result.

To find this special B, since A induces an isomorphism $\varphi:\ker(A)^\perp\to\operatorname{im}(A)$, by the bounded inverse theorem, φ^{-1} is bounded. Let B be any extension of φ^{-1} to $\mathcal{H}$. Then it fulfills $A=ABA$. □

Another useful trace formula is the following:

Claim A.17. If there is some $n\in\mathbb{N}$ such that $\mathbb{1}-|A|^2$ and $\mathbb{1}-|A^*|^2$ are n-Schatten class, then

$$\operatorname{index}(A) = \operatorname{tr}((\mathbb{1}-|A|)^n)-\operatorname{tr}((\mathbb{1}-|A^*|)^n)\,.$$

The proof is left as an exercise to the reader.

Theorem A.18. *(Logarithmic law) If $A,B\in\mathcal{F}(\mathcal{H})$ then*

$$\begin{aligned}
\operatorname{index}(AB) &= \operatorname{index}(A)+\operatorname{index}(B) \\
\operatorname{index}(A\oplus B) &= \operatorname{index}(A)+\operatorname{index}(B)\,.
\end{aligned}$$

Proof. The easiest proof is via Fedosov Theorem A.16: If $\tilde{A}$ is a parametrix for A and $\tilde{B}$ is a parametrix for B then $\tilde{B}\tilde{A}$ is a parametrix for AB. If we let $F:=\mathbb{1}-B\tilde{B}$ be finite rank, then some algebra implies

$$\operatorname{index}(AB) = \operatorname{index}(A)+\operatorname{tr}\left(\tilde{A}AB\tilde{B}-\tilde{B}\tilde{A}AB\right)$$

the last trace is seen to equal $\operatorname{index}(B)$ since $\tilde{A}A-\mathbb{1}$ is finite rank.

The statement about the direct sum is trivial. □

Theorem A.19. *(Atiyah-Jähnich) We have* $\pi_0(\mathcal{F}(\mathcal{H})) \cong \mathbb{Z}$*.*

Proof. We already know that index : $\mathcal{F}(\mathcal{H}) \to \mathbb{Z}$ is continuous and is constant on the path-connected components of $\mathcal{F}(\mathcal{H})$. Thus index lifts to a well-defined map on $\pi_0(\mathcal{F}(\mathcal{H}))$. To see that it's surjective it suffices to consider powers of the right-shift operator. So we only need to show it is injective.

First we claim that if $\text{index}(A) = 0$ then there is a path from A to $\mathbb{1}$: Via Claim A.12 we have $A = G + K$ for some invertible G and compact K. By Theorem B.6, there is a path γ from $(A-K)^{-1}$ to $\mathbb{1}$, and γA is a path from $\mathbb{1} - \tilde{K}$ to A. From there we can define a further homotopy to reduce $\tilde{K}$ to zero.

Next, we need that if $\text{index}(A) = \text{index}(B)$ then there is a path between them. To that end, let $\tilde{B}$ be the parametrix of B. Then $\text{index}\left(A\tilde{B}\right) = 0$, whence by the above there is a path $\gamma : A\tilde{B} \mapsto \mathbb{1}$. The path $\tilde{\gamma} := \gamma B$ interpolates between $A - K$ and B. Again, a further homotopy brings us to A. □

B Some linear algebra

Claim B.1. $\ker(A^*) = (\text{im}A)^\perp$ for any operator $A : \mathcal{H} \to \mathcal{H}$.

Proof. We have the following sequence of equivalent statements for any $v \in \mathcal{H}$:

1. $v \in \ker(A^*)$.
2. $A^* v = 0$.
3. $\langle u, A^* v\rangle = 0$ for all $u \in \mathcal{H}$.
4. $\langle Au, v\rangle = 0$ for all $u \in \mathcal{H}$.
5. $v \in (\text{im}A)^\perp$.

□

Claim B.2. $W^\perp \in \text{Closed}(\mathcal{H})$ for any subspace $W \subseteq \mathcal{H}$.

Proof. Write

$$\begin{aligned} W^\perp &= \bigcap_{v \in W} \{\, u \in \mathcal{H} \mid \langle v, u\rangle = 0 \,\} \\ &= \bigcap_{v \in W} \langle v, \cdot\rangle^{-1}(\{\, 0 \,\}) \,. \end{aligned} \tag{B.1}$$

But $\{\, 0 \,\} \in \text{Closed}(\mathbb{C})$ and $\langle v, \cdot\rangle^{-1} : \mathcal{H} \to \mathbb{C}$ is continuous, so that $\langle v, \cdot\rangle^{-1}(\{\, 0 \,\}) \in \text{Closed}(\mathcal{H})$. But now, arbitrary intersections of closed subsets are again closed (cf. **??**). □

Claim B.3. For any subspace $W \subseteq \mathcal{H}$, $\left(\overline{W}\right)^\perp = W^\perp$.

Proof. Since $W \subseteq \overline{W}$, $\left(\overline{W}\right)^\perp \subseteq W^\perp$ via (B.1). Conversely, let $v \in W^\perp$. WTS $v \in \left(\overline{W}\right)^\perp$, i.e., that for all $w \in \overline{W}$,

$\langle v, w \rangle = 0$. Let $\{ w_n \}_n \subseteq W$ such that $\lim_n w_n = w$. Then

$$\begin{aligned} \langle v, w \rangle &= \left\langle v, \lim_n w_n \right\rangle \\ &= \lim_n \langle v, w_n \rangle && (\langle v, \cdot \rangle \text{ is continuous}) \\ &= \lim_n 0 = 0\,. \end{aligned}$$

□

Claim B.4. For any subspace $W \subseteq \mathcal{H}$, $\left(W^\perp\right)^\perp = \overline{W}$.

Proof. Let $w \in \overline{W}$. Then $\langle w, v \rangle = 0$ for all $v \in \left(\overline{W}\right)^\perp$, which, via Claim B.3, implies that $\langle w, v \rangle = 0$ for all $v \in W^\perp$, which is equivalent to saying that $w \in \left(W^\perp\right)^\perp$.

Conversely, by [Rud91, 12.4], for any closed subspace,

$$\begin{aligned} \mathcal{H} &= \overline{W} \oplus \left(\overline{W}\right)^\perp \\ &= \overline{W} \oplus W^\perp\,. && (\text{Via Claim B.3}) \end{aligned}$$

Now since $W^\perp \in \mathrm{Closed}\left(\mathcal{H}\right)$ via Claim B.2, we may also write

$$\mathcal{H} = W^\perp \oplus \left(W^\perp\right)^\perp\,.$$

Hence we learn that

$$W^\perp \oplus \left(W^\perp\right)^\perp \quad = \quad W^\perp \oplus \overline{W}\,.$$

Now if $\overline{W}$ were a proper subspace of $\left(W^\perp\right)^\perp$, this line would lead to a contradiction. □

Lemma B.5. *We have* $\ker\left(A\right) = \ker\left(\left|A\right|^2\right)$ *with* $\left|A\right|^2 \equiv A^* A$.

Proof. We have the chain of implications

$$\begin{aligned} \varphi \in \ker\left(A\right) \quad &\iff \quad A\varphi = 0 \\ &\iff \quad A^* A\varphi = \left|A\right|^2 \varphi = 0 \\ &\implies \quad v \in \ker\left(\left|A\right|^2\right)\,. \end{aligned}$$

Conversely,

$$\begin{aligned} \varphi \in \ker\left(\left|A\right|^2\right) \quad &\iff \quad \left|A\right|^2 \varphi = 0 \\ &\iff \quad \left\langle \left|A\right|^2 \varphi, \psi \right\rangle = 0 \forall \psi \in \mathcal{H} \\ &\iff \quad \langle A\varphi, A\psi \rangle = 0 \forall \psi \in \mathcal{H}\,. \end{aligned}$$

In particular, choose $\psi = \varphi$ to get $\|A\varphi\| = 0$ which implies $A\varphi = 0$ and so $\varphi \in \ker\left(A\right)$ as desired. □

Theorem B.6. *(Kuiper's theorem) The invertible operators are contractible within* $\mathcal{B}\left(\mathcal{H}\right)$.

Proof. Let $A \in \mathcal{B}(\mathcal{H})$. We wish to find a continuous path to $\mathbb{1}$. Using the polar decomposition, write

$$A = |A| U$$

where $|A| \equiv \sqrt{A^* A}$ and $U = A |A|^{-1}$ is unitary. By the Hille-Yosida theorem [RS80], there is some self-adjoint operator H such that $U = e^{iH}$. Define the map

$$\gamma : [0,1] \to \mathrm{GL}(\mathcal{B}(\mathcal{H}))$$

via

$$\gamma(t) = ((1-t)|A| + t\mathbb{1}) e^{i(1-t)H}.$$

Then $\gamma(0) = A$ and $\gamma(1) = \mathbb{1}$, γ is norm continuous and $\gamma(t)$ is invertible because $|A| > 0$. $\square$

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