Complex Line Integrals

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Recall the expression for the force on an obstacle $\mathcal{B} \subseteq \mathbb{R}^2$:

$$\mathcal{F} = -\oint_{\partial\mathcal{B}} p\, n$$

where $p: \mathbb{R}^2 \to \mathbb{R}$ is the pressure and $n: \partial\mathcal{B} \to S^1$ is the normal to $\partial\mathcal{B}$.

The integral is meant as a line integral, which is defined as follows: Let $\gamma: [0,1] \to \mathbb{R}^2$ be a path tracing out $\partial\mathcal{B}$ ($\mathrm{im}(\gamma) = \partial\mathcal{B}$).

Then $\oint_{\partial\mathcal{B}} f(x) n(x)\, dx = \int_0^1 f(\gamma(t))\, n(\gamma(t)) \|\gamma'(t)\| dt$

for a scalar field $f: \mathbb{R}^2 \to \mathbb{R}$.

What is $n \circ \gamma : [0,1] \to S^1$? It gives the normal to the curve @ each pt.

Recall, $\tau \circ \gamma$ (tangent to the curve) is given by

$$\tau \circ \gamma = \frac{\gamma'}{\|\gamma'\|}$$

Since we are in 2D it's easy to write the normal field to that:

$$n \circ \gamma = \frac{1}{\|\gamma'\|}(+\gamma_2' e_1 - \gamma_1' e_2)$$

②

$$\rightarrow \quad \oint_{\partial B} f n = \int_0^1 (f \circ \gamma)(+\gamma_2' e_1 + \gamma_1' e_2)$$

We now move to complex notation:

Let $S \subseteq \mathbb{C}$, $g : \mathbb{C} \to \mathbb{C}$ and $\tilde{\gamma} : [0,1] \to \mathbb{C}$ be

$\uparrow$ curve

parametrization of S. Then

$$\oint_S g \equiv \underbrace{\int_0^1 \underbrace{(g \circ \tilde{\gamma})(t)\, \tilde{\gamma}'(t)}_{\text{multiplication of two complex numbers}} dt}_{\text{complex number}}$$

A convenient notation for the L.H.S. is:

$\oint_S g(z)\, dz$ (though the meaning is via the formula above)

Let $\eta : \mathbb{R}^2 \xrightarrow{\sim} \mathbb{C}$ be given by $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto x_1 + i x_2$

Cl.: $\oint_{\partial B} f n = \eta^{-1}\left(-i \oint_{\eta(\partial B)} f \circ \eta^{-1}\right)$

Pf.: We have found: $(\text{L.H.S.}) = \int_0^1 (f \circ \gamma)(+\gamma_2' e_1 - \gamma_1' e_2)$

$$= +e_1 \int_0^1 (f \circ \gamma)\gamma_2' - e_2 \int_0^1 (f \circ \gamma)\gamma_1'$$

Applying η on that we find:

$$\eta \oint_{\partial B} f n = +\int_0^1 (f \circ \gamma)\gamma_2' + i \int_0^1 (f \circ \gamma)\gamma_1'$$

$$= \int_0^1 (f \circ \gamma)(-i\gamma_1' + \gamma_2') =$$

$\partial_1 \eta = 1$
$\partial_2 \eta = i$

$$= -i \int_0^1 \underbrace{(f \circ \gamma)}_{f \circ \eta^{-1} \circ \eta \circ \gamma} \underbrace{(\gamma_1' + i \gamma_2')}_{= \eta \circ \gamma' = (\eta \circ \gamma)'} \qquad (\eta \circ \gamma)' = [(\partial_j \eta) \circ \gamma]\, \gamma_j' = \eta \circ \gamma'$$

$$= -i \int_0^1 (f \circ \eta^{-1}) \circ (\eta \circ \gamma)'$$

$$\equiv -i \oint_{\underbrace{\mathrm{im}(\eta \circ \gamma)}_{= \eta(\mathrm{im}(\gamma)) = \eta(\partial \mathcal{B})}} f \circ \eta^{-1}$$

Using the convenient notation and omitting η, we find:

$$\boxed{\oint_{\partial \mathcal{B}} f n = -i \oint_{\partial \mathcal{B}} f(z)\, dz}$$

Next we want also to treat the torque. Recall:

$$M = -\oint_{X \partial \mathcal{B}} \underbrace{(x \wedge p(x) n(x))}_{\text{scalar}} dx$$

In 2D, $a \wedge b \equiv a_1 b_2 - a_2 b_1$ (a scalar).

$$\Rightarrow \quad x \wedge p(x) n(x) = p(x) \left[x_1 n(x)_2 - x_2 n(x)_1 \right]$$

④ We find:

$$\mathcal{M} = -\int_0^1 p(\gamma(t)) \underbrace{\left[\gamma_1(t)\, n(\gamma(t))_2 - \gamma_2(t)\, n(\gamma(t))_1\right]}_{-\frac{\gamma_1(t)\gamma_1'(t)}{\|\gamma'(t)\|} - \frac{\gamma_2(t)\gamma_2'(t)}{\|\gamma'(t)\|}} \|\gamma'(t)\| \, dt$$

$$= +\int_0^1 p(\gamma(t))\, \gamma(t) \cdot \gamma'(t) \; dt$$

Note that if $(a,b) \in (\mathbb{R}^2)^2$, then their scalar product may be expressed as:

$$a \cdot b \equiv a_i b_i = a_1 b_1 + a_2 b_2$$
$$= \mathrm{Re}\{(a_1 + i a_2)(b_1 - i b_2)\}$$
$$= \mathrm{Re}\{\eta(a)\, \overline{\eta(b)}\}$$

$$\Rightarrow \mathcal{M} = +\int_0^1 \underbrace{p(\gamma(t))}_{((p \circ \eta^{-1}) \circ (\eta \circ \gamma))(t)} \mathrm{Re}\{\eta(\gamma(t))\, \underbrace{\overline{\eta(\gamma'(t))}}_{\overline{(\eta \circ \gamma)'(t)}}\}\, dt$$

p is a scalar / integral is linear

$$\stackrel{\downarrow}{=} \mathrm{Re}\left\{\int_0^1 ((p \circ \eta^{-1}) \circ (\eta \circ \gamma))(t)\; \eta(\gamma(t))\, \overline{(\eta \circ \gamma)'(t)}\, dt\right\}$$

$$= \mathrm{Re}\left\{\oint_{z \in \eta(\partial B)} (p \circ \eta^{-1})(z)\, z\, d\bar{z}\right\} \stackrel{\text{omit } \eta}{=} \mathrm{Re}\left\{\oint_{z \in \partial B} p(z)\, z\, d\bar{z}\right\}$$

<u>Remark</u>: In principle one would also have to use the injection $\iota : \mathbb{R} \hookrightarrow \mathbb{C}$: $\oint_{\partial B} f n = \eta^{-1}\left(-i \oint_{\eta(\partial B)} \iota \circ f \circ \eta^{-1}\right)$