Q1 Properties of Deformations

Define a map $f_\gamma : \mathbb{R}^3 \to \mathbb{R}^3$ via $x \longmapsto \begin{bmatrix} x_1 + \gamma x_2 \\ x_2 \\ x_3 \end{bmatrix}$

for some fixed $\gamma \in \mathbb{R}$.

The Frechet derivative of f_γ is given by

$$[(Df_\gamma)(x)]_{i,j} = (\partial_j f_i)(x)$$

$$[(Df_\gamma)(x)] = \begin{bmatrix} (\partial_1 f_1)(x) & (\partial_2 f_1)(x) & (\partial_3 f_1)(x) \\ (\partial_1 f_2)(x) & (\partial_2 f_2)(x) & (\partial_3 f_2)(x) \\ (\partial_1 f_3)(x) & (\partial_2 f_3)(x) & (\partial_3 f_3)(x) \end{bmatrix}$$

[redacted]

$\Rightarrow [(Df_\gamma)(x)]$ does not depend on x.

$\Longleftrightarrow f_\gamma$ is a homogeneous deformation. ✓ (i)

$\det((Df_\gamma)(x)) =$ [redacted] $= 1$

$\Longleftrightarrow f_\gamma$ preserves volume. ✓ (iii)

Let $(x,y) \in (\mathbb{R}^3)^2$.

$\|f_\gamma(x) - f_\gamma(y)\|^2 =$ [redacted]

$=$ [redacted] $\neq \|x-y\|^2$

$\iff$ f_γ is not an isometry. ✓ (ii)

Note $\left[(\mathbb{D}f_\gamma)(x)\right]^T \left[(\mathbb{D}f_\gamma)(x)\right] =$

$=$ [redacted] $\neq \mathbb{1}$

$\Leftrightarrow$ $(\mathbb{D}f_\gamma)(x) \notin O(3)$ (unless $\gamma=0$)

$\Leftrightarrow$ f_γ is not an infinitesimal isometry. ✓ (iv)

Q2 About the Polar Decomposition

Let $A \in Mat_{3\times 3}(\mathbb{R})$ s.t. $\det(A) > 0$.

The left polar decomp. of A is def. as:

$$A = S_A^\ell R_A$$

For some S_A^ℓ symmetric and pos. def., $R_A \in SO(3)$.

$|\cdot|$, "abs. val." of a matrix

In fact, $S_A^\ell \equiv \sqrt{AA^T} =: |A^T|$ as seen in the lecture;

$R_A = |A^T|^{-1} A$ then.

Def. a map $g: SO(3) \longrightarrow \mathbb{R}$ $R \longmapsto \|R-A\|_F^2$

where $\|\cdot\|_F^2 := tr(|\cdot|^2)$ (the Frobenius norm)

Cl.: R_A is equal to the minimum of the map g, and the value of g at R_A, $dist^2(A, SO(3)) = g(R_A) = tr((\mathbb{1}-|A^T|)^2)$.

Pf.: First note that

$g(R_A) \equiv$ [redacted]

$=$ [redacted]

$|A^T|$ symm.

$=$ [redacted] $tr\left((\mathbb{1}-|A^T|)^2\right)$

So the 2^{nd} part of the claim follows if the 1^{st} part holds.

<u>Cl.</u>: If $B \in O(3)$ then $|B_{ij}| \leqslant 1 \quad \forall\ (i,j) \in \{1,\dots,3\}^2$

<u>Pf.</u>: [redacted]

Def. $h(R) := tr(A^T R)$

<u>Cl.</u>: g is minimal $\Longleftrightarrow$ h is maximal.

<u>Pf.</u>: [redacted]

<u>Cl.</u>: If $A = U_A D_A V_A^T$ is an S.V.D. of A then h is max. at $U_A V_A^T$ and the max. is unique if A is inv.

<u>Pf.</u>: [redacted]

Q3 Material & Spatial Coordinates

Let $f : \mathbb{R}^3 \circlearrowleft$ be a deformation (so f is an orientation preserving diffeomorphism).

Recall (Rudin's PMA Theorem $\boxed{10.9}$) that for any cont. $g : \mathbb{R}^3 \circlearrowleft$ w/ cpt. supp. in $f(\mathbb{R}^3)$ then

$$\boxed{\int_{\mathbb{R}^3} g = \int_{x \in \mathbb{R}^3} (g \circ f)(x) \, | \det((\mathbb{D} f)(x)) | \, dx}$$

This is the 1st part of the question.

For the 3rd part, let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be a curve.

Let $v : \mathbb{R}^3 \to \mathbb{R}^3$ be any vector field to be integrated along γ. Then the line integral of v along γ is:

$$\int_{\lambda \in \mathbb{R}} \langle (v \circ \gamma)(\lambda), \gamma'(\lambda) \rangle \, d\lambda$$

Where $\langle \circ, \circ \rangle$ is the inner prod on $\mathbb{R}^3$.

Then if we have a deformation $f : \mathbb{R}^3$ the integral may be computed as:

$$\int_{\lambda\in\mathbb{R}} \langle (v\circ \underbrace{f\circ\gamma}_{\text{new path}})(\lambda), (f\circ\gamma)'(\lambda)\rangle \, d\lambda$$

Chain rule $=$ [redacted]

Finally we deal with the surface integral:

Let $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$ be a 2D surface.

Let $v : \mathbb{R}^3 \to \mathbb{R}^3$ be any vector field to be integrated along σ. Then the surface integral of v along σ is:

$$\int_{(\lambda_1,\lambda_2)\in\mathbb{R}^2} \langle (v\circ\sigma)(\lambda_1,\lambda_2), (\partial_1\sigma)(\lambda_1,\lambda_2) \underset{\uparrow \\ \text{cross prod.}}{\times} (\partial_2\sigma)(\lambda_1,\lambda_2)\rangle \, d\lambda_1 d\lambda_2$$

Then if we perform a deformation $f : \mathbb{R}^3 \circlearrowleft$ we have the new surface $f\circ\sigma$:

$(\partial_i(f\circ\sigma))(\lambda_1,\lambda_2) =$ [redacted]

$\Rightarrow (\partial_1(f\circ\sigma))(\lambda) \times (\partial_2(f\circ\sigma))(\lambda) =$

$=$ [redacted]

$\boxed{\text{A.M. HW\#1}} =$ [redacted]