Q1 Stability of Dipping & Swimming Body

$\mathcal{C} \equiv$ geometric middle of object.

The torque corresponding to a pt. O (origin) inside the object is:

$$\tau = c \wedge A$$

where c is the position of the center $\mathcal{C}$ w.r.t. $O = 0$:

$$c = \frac{1}{|V|}\int_V x\, d^3x$$

Indeed, by Stokes':

$$\tau \equiv \int_{\partial V} x \wedge (-p\, do) = \int_{\partial V} do \wedge (px) = \int_V \mathrm{curl}(px)\, d^3x$$

$$= \int_V ((\nabla p) \wedge x)\, d^3x \underset{\nabla p = \text{const}}{=} \nabla p \wedge \int_V x\, d^3x = \nabla p \wedge (|V| c)$$

$$= +c \wedge (-|V| \nabla p) \underset{\text{Archimedes}}{=} c \wedge \underbrace{(\rho g |V| e_3)}_{\text{Buoyancy force, } A} \equiv c \wedge A$$

Place O at S (C.o.M.) (should not matter when calculating the total torque, as the grav. force and buoyancy are equal), so:

$$\tau_{total} = c_0 \wedge A + C.o.M._0 \wedge F_{grav}$$

$$= (c_0 - C.o.M._0) \wedge A = (c_0 + v - C.o.M._0 - v) \wedge A \quad \forall v \in \mathbb{R}^3$$

So pick $O := S$, in which case $\nexists\ \tau_{grav.}$, $\tau_{tot} = c \wedge A$.

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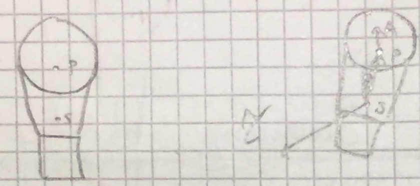
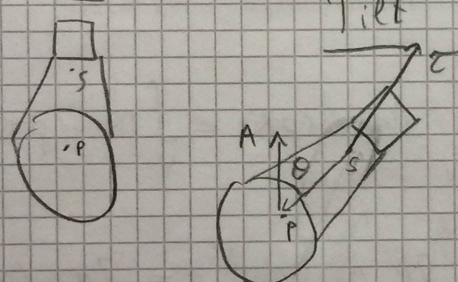
(a) Neutral   Tilt



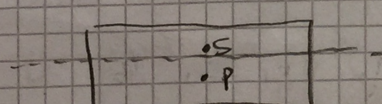
If we had S above P:

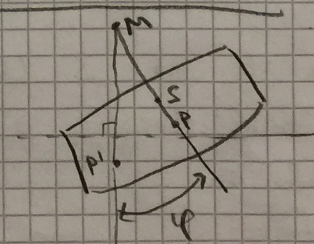
Neutral   Tilt



(b) Metacentric height of ship: M

Original pos.   Tilted Position





Principal axis: $\int_{W} x_1 x_2 \, dx_1 \, dx_2 = 0$

Cl.: The distance of M rel. to the ship is indep. of φ if φ is small, and is given by:

$$\overline{PM} = \frac{\Theta}{|V|}$$

where $\Theta := \int_W x_1^2 \, dx_1 dx_2$ is the moment of inertia of the surface W corresponding to the tilt axis.

Pf.:

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Then as before, $\exists$ stability by considering the direction torque:

$$\tau' =$$

$$=$$

$$=$$

$$=$$

$$\Rightarrow$$

The Lane-Emden Eq-n, Part 1

Recall the def. of the "pressure force potential":

Find $\mathcal{P} : \mathbb{R} \to \mathbb{R}$ s.t. $\nabla(\mathcal{P} \circ p) = \rho^{-1} \nabla p$

Then we have:

$$\nabla p = F_{grav.}$$

If $F_{grav.}$ comes from a potential (and it does) then

$$F_{grav.} = -\rho \nabla \varphi$$

$$\Rightarrow \quad \rho^{-1} \nabla p = -\nabla \varphi \iff \nabla(\mathcal{P} \circ p) = -\nabla \varphi$$

$$\iff \boxed{\mathcal{P} \circ p + \varphi = \text{const}}$$

If the fluid is self-grav. then the Poisson eq-n must be fulfilled: $\boxed{\Delta \varphi = 4\pi G \rho}$

Assume polytrope eq-n of state: $\boxed{\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}}$ where $\gamma := 1 + n^{-1}$.

Then $\frac{\mathcal{P}}{\mathcal{P}_0} = \left(\frac{\rho}{\rho_0}\right)^{n^{-1}}$ (?)

Assume spherical symmetry (star).

Define $\xi := \alpha^{-1} \|x\|$ (α to be chosen later)

$\Theta(\xi) := \frac{(\mathcal{P} \circ p)(\alpha \xi)}{\mathcal{P}_0}$ (spherical symm., so only one arg.)

Cl.: The following eq-n holds

clear by def.

$$\boxed{\xi^{-2}(\xi^2 \Theta')' + \Theta^n = 0}$$

w/ initial cond. $\Theta(0) \overset{\Delta}{=} 1$, $\Theta'(0) = 0$.

Pf.: We know $\Delta \varphi = 4\pi G \rho \Rightarrow \boxed{\Delta \mathcal{P} = -4\pi G \rho}$

$$\nabla P = \rho^{-1} \nabla p$$

P is a function of p, so we really have:

1^{st} order diff. eq-n:

$$dP = \rho^{-1}(p)\, dp \implies P(p) = \int_{p_0}^{p} \rho^{-1}(p')\, dp'$$

Now plug in the eq-n of state $\rho(p) = \rho_0 \left(\frac{p}{p_0}\right)^{1/\gamma}$:

$$P(p) = \int_{p_0}^{p} \rho_0^{-1} \left(\frac{p_0}{p'}\right)^{1/\gamma} dp' \underset{\gamma \neq 1}{=} \rho_0^{-1} p_0^{1/\gamma} (-\gamma^{-1}+1)^{-1} \, p'^{-\gamma^{-1}+1} \Big|_{p_0}^{p} =$$

$$= \rho_0^{-1} p_0^{\gamma^{-1}} (-\gamma^{-1}+1)^{-1} \left(p^{-\gamma^{-1}+1} - p_0^{-\gamma^{-1}+1}\right)$$

$$\Rightarrow \quad P(\rho) = \frac{1}{(-\gamma^{-1}+1)\,\rho_0\, p_0^{\gamma^{-1}}}\left(p^{-\gamma^{-1}+1} - p_0^{-\gamma^{-1}+1}\right)$$

Now use the eq-n of state again to find P as a function of ρ:

$$P(\rho) = \frac{1}{(-\gamma^{-1}+1)\,\rho_0\, p_0^{\gamma^{-1}}}\left(\left(p_0\left(\frac{\rho}{\rho_0}\right)^{\gamma}\right)^{-\gamma^{-1}+1} - p_0^{-\gamma^{-1}+1}\right) =$$

$$= \frac{1}{(-\gamma^{-1}+1)\,\rho_0\, p_0^{\gamma^{-1}}}\left(\frac{p_0^{-\gamma^{-1}+1}}{\rho_0^{-1+\gamma}}\,\rho^{-1+\gamma} - p_0^{-\gamma^{-1}+1}\right)$$

$$n := (\gamma-1)^{-1} \Rightarrow \; = \frac{n+1}{\rho_0\, p_0^{\frac{n}{n+1}}}\, p_0^{\frac{1}{n+1}}\left(\frac{\rho}{\rho_0}\right)^{1/n} - \frac{n+1}{\rho_0\, p_0^{\frac{n}{n+1}}}\, p_0^{\frac{1}{n+1}}$$

$$\Rightarrow \gamma = 1 + \frac{1}{n}$$

$$\Rightarrow \gamma^{-1} = \frac{n}{n+1} \qquad 1-\gamma^{-1} = 1 - \frac{n}{n+1}$$

$$\Rightarrow \; = \frac{n+1-n}{n+1} = \frac{1}{n+1}$$

$$= \underbrace{\frac{n+1}{\rho_0\, p_0^{\frac{n}{n+1}}}\, p_0^{\frac{1}{n+1}}}_{=:\, P_0}\left[\left(\frac{\rho}{\rho_0}\right)^{1/n} - 1\right]$$

$$\Rightarrow \quad \boxed{\frac{P(\rho)}{P_0} = \left(\frac{\rho}{\rho_0}\right)^{1/n} - 1}$$