Q2

The Shape of the Earth

Poisson's equation says:

$$\Delta \varphi = -4\pi G \rho$$

This is solved by $\varphi(x) = -\frac{G}{4\pi}\int_{\mathbb{R}^3} \frac{\rho(x')}{\|x-x'\|}dx'$

Indeed, the Green's function for Δ is $G(x,x') = \frac{-1}{4\pi\|x-x'\|}$:

$$\Delta G \equiv \delta(x-x')$$

$\Longrightarrow$ If $\varphi(x) = \int G(x,x')(-4\pi G\rho(x'))dx'$, we get

$$\Delta\varphi = \Delta\int G(x,x')(-4\pi G\rho(x'))dx'$$

$$\overset{\text{linearity of }\Delta}{=} \int \Delta G(x,x')(-4\pi G\rho(x'))dx'$$

$$\overset{\Delta\text{ only acts on }x}{=} \int \delta(x-x')(-4\pi G\rho(x'))dx'$$

$$= -4\pi G\rho(x) \quad ✓$$

$$\Longrightarrow \varphi(x) = \int \frac{\cancel{-1}}{\cancel{4\pi}\|x-x'\|}(\cancel{-4\pi}G\rho(x'))dx' = G\int_{\mathbb{R}^3}\frac{\rho(x')}{\|x-x'\|}dx'$$

Let $(a_1,a_2,a_3)\in\mathbb{R}^3_{>0}$. Let $\rho\in\mathbb{R}_{>0}$.

Define $B := \left\{x\in\mathbb{R}^3 \;\middle|\; \sum_{i=1}^{3}\frac{x_i^2}{a_i^2} \leq 1\right\}$, the interior of an ellipsoid.

We assume homogeneous mass density ρ within B.

Define $D:\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ via $D(u) := \sqrt{\prod_{i=1}^{3}(a_i^2+u)}$ $\forall u\in\mathbb{R}_{\geq 0}$

$I:\mathbb{R}\to\mathbb{R}$ via $I(\lambda) := a_1a_2a_3\int_\lambda^\infty D(u)^{-1}du$ $\forall\lambda$

$A_i:\mathbb{R}\to\mathbb{R}$ via $A_i(\lambda) := a_1a_2a_3\int_\lambda^\infty [(a_i^2+u)D(u)]^{-1}du$ $\forall\lambda$

[2] Define $\lambda : \mathbb{R}^3 \to \mathbb{R}_{>0}$ via the implicit soln to the eq-n $\sum_{i=1}^{3} \frac{x_i^2}{a_i^2 + \lambda(x)} = 1 \quad \forall x \in \mathbb{R}^3$

<u>Cl:</u>
$$\varphi(x) = -\pi G \rho \begin{cases} I(\lambda(x)) - \sum_{i=1}^{3} A_i(\lambda(x)) x_i^2 & x \notin B \\ I(0) - \sum_{i=1}^{3} A_i(0) x_i^2 & x \in B \end{cases}$$

<u>Pf:</u>

Cl.: $D(u)^{-1} D'(u) = \frac{1}{2}\sum_{i=1}^{3} (a_i^2+u)^{-1} \quad \forall\ u$

Pf.:

Cl.: $\sum_{i=1}^{3} A_i(\lambda) = 2a_1a_2a_3 D(\lambda)^{-1} \quad \forall\ \lambda$

Pf.:

Cl.: $\nabla\left(I(\lambda) - \sum_{i=1}^{3} A_i(\lambda)\pi_i^2\right) = -\sum_{i=1}^{3} A_i(\lambda)\nabla\pi_i^2$ where $\pi_i : \mathbb{R}^3 \to \mathbb{R}$ $x \mapsto x_i$

Pf.:

[4]

Cl.: $\Delta\left(I \circ \lambda - {\sum_{i=1}^{3}}' A_i(\lambda)\, \pi_i^{2}\right) = 0$

Pf.:

Cl: $\Delta\left(I(0)-\sum_{i=1}^{3} A_i(0)\,\pi_i^2\right)=-4$

Cl.: The B.C. are fulfilled: φ cont. on ∂B and $\nabla\varphi$ cont. on ∂B.

Pf.:

Lane-Emden Eq-n

$P: \mathbb{R} \to \mathbb{R}$ defined by $\nabla(P \circ p) = \rho^{-1} \nabla p$.

$$\nabla(P \circ p) = e_i \partial_i (P \circ p) = e_i (P' \circ p) \partial_i p = (P' \circ p)(\nabla p)$$

$\Rightarrow \nabla(P \circ p) = \rho^{-1} \nabla p$ is equivalent to $\boxed{P' \circ p = \rho^{-1}}$.

$$\Rightarrow \boxed{P(p) = \int^{p} \rho^{-1}(p')\, dp' + C_1}$$

We know via the state eq-n that $\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}$

$\exists\ \gamma \in \mathbb{R} \setminus \{1\}$. $\Rightarrow \rho(p) = \rho_0 \left(\frac{p}{p_0}\right)^{\gamma^{-1}} \Rightarrow \rho^{-1}(p') = \rho_0^{-1} \left(\frac{p'}{p_0}\right)^{-\gamma^{-1}}$

We find $P(p) = \int^{p} \rho_0^{-1} \left(\frac{p'}{p_0}\right)^{-\gamma^{-1}} dp' + C_1$

$$= \frac{\rho_0^{-1}}{p_0^{-\gamma^{-1}}} (-\gamma^{-1}+1)^{-1} p^{-\gamma^{-1}+1} + C_1$$

We use the eq-n of state again to get P as a func. of ρ:

$$P(\rho) = \frac{\rho_0^{-1}}{p_0^{-\gamma^{-1}}} (-\gamma^{-1}+1)^{-1} \left(p_0 \left(\frac{\rho}{\rho_0}\right)^{\gamma}\right)^{-\gamma^{-1}+1} + C_1$$

$$= \rho_0^{-1} (-\gamma^{-1}+1)^{-1} p_0 \left(\frac{\rho}{\rho_0}\right)^{-1+\gamma} + C_1$$

Define $C_1 := 0$, $n := (\gamma-1)^{-1} \Rightarrow 1-\gamma^{-1} = (n+1)^{-1}$, $P_0 := \rho_0^{-1}(-\gamma^{-1}+1)^{-1} p_0$

$-1+\gamma = n^{-1}$

to get $\boxed{\frac{P(\rho)}{P_0} = \left(\frac{\rho}{\rho_0}\right)^{1/n}}$

We know that $\nabla p = F_{grav.}$ (see eq-n 1,22 in the script).

We also know the grav. force stems from a pot. $\varphi: \mathbb{R}^3 \to \mathbb{R}$ s.t.

$F_{grav.} = -\rho \Delta \varphi$

$$\Rightarrow \quad \rho^{-1}\nabla p = \rho^{-1} F_{grav.} = -\rho^{-1}\rho\nabla\varphi = -\nabla\varphi$$

$$\Rightarrow \quad \nabla(P \circ p) = -\nabla\varphi$$

$$\Rightarrow \quad \boxed{P(p) = \varphi + \text{const}}$$

We know the Poisson eqn is fulfilled, so that

$$\boxed{\Delta\varphi = 4\pi G\rho}$$

$$\Downarrow$$

$$\boxed{\Delta(P \circ p) = -4\pi G\rho}$$

We found before that $\frac{P}{P_0} = \left(\frac{\rho}{\rho_0}\right)^{1/n} \Rightarrow \rho = \rho_0\left(\frac{P}{P_0}\right)^n$

$$\Rightarrow \quad \boxed{\Delta(P \circ p) = -\frac{4\pi G\rho_0}{P_0^n}(P \circ p)^n} \quad (\text{Poisson})$$

Assume spherical symmetry $\Longrightarrow \exists\ \tilde{p} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ s.t.

$$p(x) = \tilde{p}(\|x\|) \quad \forall\ x \in \mathbb{R}^3$$

Define $\boxed{\tilde{\Theta} := P_0^{-1}\, P \circ \tilde{p}}$

Then the Poisson equation becomes, due to spherical symm.:

$$r^{-2}\left(r^{+2}\,(P_0\tilde{\Theta})'\right)'(r) = -4\pi G\rho_0\left(\tilde{\Theta}(r)\right)^n \quad \forall\ r \in \mathbb{R}_{\geq 0}$$

$$\tilde{\Theta}''(r) + 2r^{-1}\tilde{\Theta}'(r) = -\frac{4\pi G\rho_0}{P_0}\left(\tilde{\Theta}(r)\right)^n$$

Now rescale space: Let $\xi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ by $r \mapsto \alpha^{-1} r$ $\exists\ \alpha > 0$.

Define $\boxed{\theta := \tilde{\Theta} \circ \xi^{-1}}$. $\Longrightarrow \tilde{\Theta} = \theta \circ \xi$

Then $\tilde{\Theta}''(r) = (\theta \circ \xi)''(r) = ((\theta' \circ \xi)\underbrace{\xi'}_{\alpha^{-1}})'(r) = \alpha^{-1}(\theta' \circ \xi)'(r) =$

$$= \alpha^{-1}(\theta'' \circ \xi)(r)\,\xi' = \alpha^{-2}\theta''(\xi(r))$$

$$2r^{-1}\tilde{\Theta}'(r) = 2r^{-1}\alpha^{-1}(\theta' \circ \xi)(r) = 2(\alpha\,\xi(r))^{-1}\alpha^{-1}\theta'(\xi(r))$$

We find $\quad \theta'' \circ \xi + 2\xi^{-1}\theta' \circ \xi = -\frac{4\pi G\rho_0}{P_0}\alpha^2(\theta \circ \xi)^n$

We hence pick $\boxed{\alpha := \sqrt{\frac{P_0}{4\pi G \rho_0}}}$ and find

$$\boxed{r^{-2}(r^2\Theta')'(r) + \Theta^n(r) = 0 \qquad \forall\, r \in \mathbb{R}_{\geq 0}} \quad (*)$$

Lane–Emden eq-n

Our initial conditions are:

$$\Theta(0) = (\rho_0^{-1}\,\rho \circ \tilde{\rho} \circ \xi^{-1})(0) = \rho_0^{-1}\,\underbrace{\rho(\underbrace{p(0)}_{=P_0})}_{=\rho_0} = 1.$$

We determine the other initial condition:

Write $\Theta(r) = \Theta(0) + \Theta'(0)r + O(r^2) = 1 + \Theta'(0)r + O(r^2)$

Plug it into $(*)$ to get:

$$\underbrace{r^{-2}\underbrace{(r^2\,\Theta'(0))'(r)}_{2r\,\Theta'(0)}}_{2r^{-1}\,\Theta'(0)} + \underbrace{\underbrace{(1+\Theta'(0)\,r)^n}_{1+\Theta'(0)r}}_{\text{finite as } r\to 0} = 0$$

The only way to make sure the 1^{st} term is finite as $r \to 0$ is to set $\Theta'(0) = 0$.

Note that then the eq-n is not fulfilled at $r=0$ (that is ok, it's a singular point.)

$$\Longrightarrow \boxed{\begin{aligned} \Theta'(0) &= 0 \\ \Theta(0) &= 1 \end{aligned}}$$

Note that when Θ is zero, ρ is zero $\Rightarrow \varphi$ is zero

$\Rightarrow$ No grav. pot. $\Rightarrow$ Outside of star.

Hence the first zero of Θ is the radius of the star.

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Define $\boxed{K := P_0 \rho_0^{-n^{-1}}}$. (So now eq-n of state is $P = K\rho^{n^{-1}}$)

Express α w/ K and ρ_0 instead of P_0 and ρ_0:

$$\alpha \equiv \sqrt{\frac{P_0}{4\pi G \rho_0}} = \sqrt{\frac{K \rho_0^{n^{-1}}}{4\pi G \rho_0}} = \sqrt{\frac{K}{4\pi G}}\, \rho_0^{-\frac{n-1}{2n}}$$

Next express ρ in terms of Θ:

$$\rho = \rho_0 \left(\frac{P}{P_0}\right)^n = \rho_0\, \tilde{\Theta}^n = \rho_0\, \Theta(\xi)^n$$

Hence the mass of the star is given by:

$$M = 4\pi \int_0^R r^2 \rho(r)\, dr \qquad R \equiv \text{radius of star (still unknown)}$$

$=$

change of var $\xi := \xi(r) = \alpha^{-1} r$

$=$

$=$

Eq-n ⊛

$=$

fund. thm. of calc.

$=$

Cl.: $\Theta'(\alpha^{-1}R) < 0$

Pf.: Recall $\Theta(0) = 0$ and R is def. s.t. $\alpha^{-1}R$ is the first zero of Θ. $\Rightarrow$ Θ must decrease $\Rightarrow$ Its derivative is negative.

$$\Rightarrow \boxed{M = 4\pi \alpha^3 \rho_0 (\alpha^{-1}R)^2 \left[-\Theta'(\alpha^{-1}R)\right]}$$

Let $\beta \in \mathbb{R}$. Then

$MR^\beta =$ [redacted]

If we want MR^β to be indep. of ρ_0, then β must be:

$$1 - \frac{(3+\beta)(n-1)}{2n} \overset{!}{=} 0$$

$$(3+\beta)(n-1) \overset{!}{=} 2n$$

$$\boxed{\beta = \frac{2n}{n-1} - 3 = \frac{2n-3n+3}{n-1} = \frac{3-n}{n-1}}$$

Then define $\boxed{C := 4\pi \left(\frac{K}{4\pi G}\right)^{\frac{n}{n-1}} \xi(R)^{\frac{n+1}{n-1}} (-1)\, \Theta'(\xi(R))}$

$\Rightarrow \boxed{MR^\beta = C}$ If $\beta > 0$, R is smaller as M is larger.

If $n = 3/2$, $\beta = \frac{3 - 3/2}{3/2 - 1} = \frac{3/2}{1/2} = 3 \Rightarrow$ non-rel. case

$n = 3$, $\beta = 0 \Rightarrow$ rel. case

$\Downarrow$

$M = C$

If $M < C$, we're in the non-rel. case anyway.

If $M > C$, we're in the ultra-rel. case & $\nexists$ sol. $\Rightarrow$ instability.