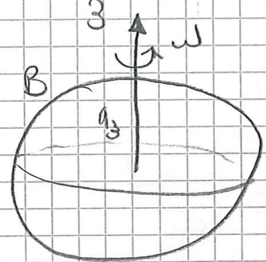
$\boxed{Q2}$ The Shape of the Earth – Part 2

B. The Hydrostatic Equilibrium



Assume the ellipsoid from $\boxed{HW4Q2}$ is rotating about the 3-axis w/ angular velocity ωe_3.

Goal: Get $\{a_i\}_{i=1}^{3}$. ← Only two var. $\left\{\frac{a_3}{a_1}, \frac{a_2}{a_1}\right\}$.

Recall that in a rotating system the centrifugal force is $m\omega^2 R$, hence the acceleration is $\omega^2 R$.

$\Rightarrow$ Acceleration is $\frac{\omega^2}{2}\nabla(\pi_1^2+\pi_2^2)$ (recall $\nabla\pi_i^2 = 2e_i\pi_i$)

$\Rightarrow$ The "centrifugal potential" is $-\frac{\omega^2}{2}(\pi_1^2+\pi_2^2)$.

We know $\nabla p = -\rho\nabla(\text{potential})$

On ∂B $\nexists$ pressure so that ∂B is a surface on which $\nabla(\text{potential})=0$ $\Leftrightarrow$ equipotential surface.

$$\zeta := \left[\varphi - \frac{\omega^2}{2}(\pi_1^2+\pi_2^2)\right]\Big|_{\partial B}$$

$$= -\pi G\rho\left(I(0) - \sum_{i=1}^{3} A_i(0)\pi_i^2\right) - \frac{\omega^2}{2}(\pi_1^2+\pi_2^2)$$

$$\Rightarrow \frac{\zeta}{\pi G\rho} + I(0) = A_3(0)\pi_3^2 + \sum_{i=1}^{2}\Big(A_i(0) - \underbrace{\frac{\omega^2}{2\pi G\rho}}_{=:\Omega}\Big)\pi_i^2$$

This equation must obey $\sum_{i=1}^{3}\frac{\pi_i^2}{a_i^2} = 1$ because it holds on ∂B:

$$\frac{\zeta}{\pi G\rho} + I(0) = A_3(0)a_3^2\frac{\pi_3^2}{a_3^2} + \sum_{i=1}^{2}(A_i(0)-\Omega)a_i^2\frac{\pi_i^2}{a_i^2}$$

Hence $\boxed{q = \text{[redacted]}}$ and

$$\begin{cases} \text{[redacted]} = (A_i(0) - \Omega)\, a_i^2 & \forall\ i \in \{1,2\} \\ \text{[redacted]} = (A_2(0) - \Omega)\, a_2^2 \end{cases}$$

equations giving rel. among $\{a_i\}_i$. Could be solved ...

The 1^{st} eq-n w/ $i=1$ reads:

[redacted]

$$\Leftrightarrow \quad \boxed{\Omega = A_1(0) - A_3(0)\,\frac{a_3^2}{a_1^2}} \quad ⊛$$

and same w/ $1 \leftrightarrow 2$.

(eq $i=2$) − (eq $i=1$) $\Rightarrow$ [redacted]

$\Leftrightarrow$

Case 1: $\boxed{a_1 = a_2} \Longrightarrow$ [redacted]

Case 2: $\boxed{a_1 \neq a_2} \Longrightarrow$ [redacted]

Recall

$$A_i(0) = a_1 a_2 a_3 \int_0^\infty \left[(a_i^2 + u)\,\Delta(u)\right]^{-1} du$$

$$= \cancel{a_1^3}\,\frac{a_2}{a_1}\,\frac{a_3}{a_1} \int_0^\infty \left[\left(\frac{a_i^2}{a_1^2} + \frac{u}{a_1^2}\right) a_1^2 \sqrt{\cancel{a_1^6} \prod_{j=1}^{3} \left(\frac{a_j^2}{a_1^2} + \frac{u}{a_1^2}\right)}\,\right]^{-1} du$$

$$v := \frac{u}{a_1^2} \quad \Rightarrow \quad = \frac{a_2}{a_1}\,\frac{a_3}{a_1} \int_0^\infty \left[\left(\frac{a_i^2}{a_1^2} + v\right) \sqrt{\prod_{j=1}^{3} \left(\frac{a_j^2}{a_1^2} + v\right)}\,\right]^{-1} dv$$

$\Rightarrow$ Indeed, $A_i(0)$ depends only on the ratios $\frac{a_3}{a_1}$, $\frac{a_2}{a_1}$ and not on a_1, a_2, a_3 independently.

$$A_2(0) - A_1(0) = a_1 a_2 a_3 \int_0^\infty \mathcal{D}(u)^{-1} \underbrace{\left(\frac{1}{a_2^2+u} - \frac{1}{a_1^2+u}\right)}_{\frac{a_1^2-a_2^2}{(a_1^2+u)(a_2^2+u)}} du$$

✓ bcs. only depends on ratios. really means $\frac{a_2}{a_1} = 1$.

Assume $a_1 = a_2 \overset{!}{=} 1$. Searching for a_3.

Then $\mathcal{D}(u) = (1+u)(a_3^2+u)^{1/2}$

$$\underbrace{\int_0^\infty \left[(1+u)^3 (a_3^2+u)^{1/2}\right]^{-1} du}_{=: f(a_3)} = \underbrace{a_3 \int_0^\infty \left[(1+u)(a_3^2+u)^{3/2}\right]^{-1} du}_{=: g(a_3)}$$

$f(0) = \int_0^\infty \left[\sqrt{u}\,(1+u)^3\right]^{-1} du \qquad g(0) = 0 \Rightarrow f(0) > g(0)$

$f(1) = \int_0^\infty (1+u)^{-7/2} du \qquad g(1) = \int_0^\infty (1+u)^{-5/2} du \Rightarrow f(1) < g(1)$

<u>Cl.</u>: f, g are cont. <u>Pf.</u>: ...

<u>Corollary</u>: $\exists\, a_3 \geq 0 : f(a_3) = g(a_3)$ <u>Pf.</u>: Middle value thm.

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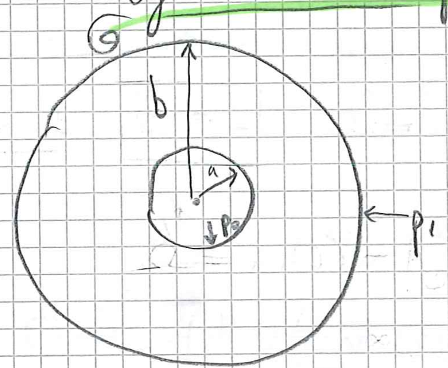
C. Graphical Rep.

$\boxed{Q1}$

Cylindrical Pipe



An elastic cylindrical pipe w/
inner pressure p_0
outer pressure p_1

No volume forces!

By cylindrical symmetry, $\exists$ only radial component to u and it only depends on the radius: $u = u_r e_r$ for some $u_r : [0,\infty) \to \mathbb{R}$.

Then by (1.14) we find

$$\begin{array}{c|ccc} & r & \varphi & z \\ \hline \varepsilon = \begin{matrix} r \\ \varphi \\ z \end{matrix} & \multicolumn{3}{c}{\begin{bmatrix} u_r' & 0 & 0 \\ 0 & \frac{1}{r}u_r & 0 \\ 0 & 0 & 0 \end{bmatrix}} \end{array}$$

HW2Q2

Then $tr(\varepsilon) = tr(\frac{1}{2}(Du + (Du)^T)) = tr(Du) = div(u) = u_r' + \frac{1}{r}u_r$.

Note $\frac{1}{r}(rf)' = \frac{1}{r}rf' + \frac{1}{r}f = f' + \frac{1}{r}f$

$\Rightarrow \quad tr(\varepsilon) = \frac{1}{r}(ru_r)'$

Recall from (2.5) (Hooke's law) $\sigma = 2\mu\varepsilon + \lambda(tr(\varepsilon))\mathbb{1}_{3\times 3}$

where σ was the Cauchy stress tensor, giving the surface forces, μ, λ were elastic constants.

$\Rightarrow \quad \sigma =$

We then employ the field equations (2.19) to find u. Due to no volume forces, $F=0$ in that eq-n.

Use again $\boxed{HW2Q2}$ to get:

$$\Rightarrow \boxed{u_r(r) = \frac{1}{2} C_1 r + \frac{C_2}{r}}$$

To find C_1, C_2 we need to employ B.C.
Unfortunately the B.C. are specified in terms of p_0, p_1, not $u(a), u(b)$. So we plug in this sol-n of u into σ and then use $\boxed{\sigma n = -p n}$ where n is the normal vector to the surface.

[redacted]

$$\Rightarrow \quad \boxed{c_2 = \frac{1}{2\mu}\,\frac{p_0 - p_1}{b^2 - a^2}\,a^2 b^2} \qquad \boxed{c_1 = \frac{a^2 p_0 - b^2 p_1}{(\mu+\lambda)(b^2 - a^2)}}$$

Assume $p_0 > p_1$. $\Rightarrow$ [redacted]

$\Rightarrow$

It's value is max when r is min, i.e., $r = a$!

$\sigma_{max} =$ [redacted]

(ii) when $b \to \infty$, $c_2 \longrightarrow$ [redacted]

$c_1 \longrightarrow$

when $a \to 0$, $c_2 \longrightarrow$

$c_1 \longrightarrow$

[redacted]