# The Neumann Problem

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## 1 Formulation of the Problem

Let $D$ be a bounded open subset in $\mathbb{R}^d$ with $\partial D$ its boundary such that $D$ is sufficiently nice (to be stipulated later as Lipschitz). Let $f \in L^2(D)$ and $g : L^2(\partial D)$ be two given scalar fields and $n : \partial D \to S^{d-1}$ be the normal unit vector to the boundary. Prove that

$$\begin{cases} \Delta\varphi & = f \\ (\nabla\varphi) \cdot n|_{\partial D} & = g \end{cases} \tag{1}$$

has a unique solution up to a constant for the unknown scalar field $\varphi : D \to \mathbb{R}$ in $H^1(D)$ if and only if

$$\int_D f = \int_{\partial D} g \tag{2}$$

(This last condition makes sense because $L^2 \subseteq L^1$)

### 1.1 Sketch of Solution

1. Verify that if a solution of (1) exists, then (2) must be satisfied using the divergence theorem.

2. Formulate (1) as a variational problem: $\varphi$ solves (1) iff

$$\int_D (\nabla\varphi) \cdot (\nabla\psi) = -\int_D f\psi + \int_{\partial D} g\psi \quad \forall\psi \tag{3}$$

   Assuming (2) is satisfied.

3. Write (3) using the bilinear and linear respectively forms

$$\omega(\varphi, \psi) := \int_D (\nabla\varphi) \cdot (\nabla\psi)$$

   and

$$\eta(\psi) := -\int_D f\psi + \int_{\partial D} g\psi$$

4. Use the Lax-Milgram theorem, which says that if $\omega$ is continuous, $\eta$ is continuous, and $\omega$ is elliptic (meaning $\omega(\psi,\,\psi)\geq\alpha\|\psi\|^2$ for all $\psi$ for some $\alpha>0$) then there is a unique solution $\varphi$ to the equation

$$\omega(\varphi,\,\cdot)-\eta \quad = \quad 0$$

In order to show that $\omega$ and $\eta$ are continuous, use the Cauchy-Schwarz inequality; in order to show that $\omega$ is elliptic, use the Poincare inequality

$$\|\psi\| \quad \leq \quad C\|\nabla\psi\|$$

for some $C>0$.

## 1.2 Solution

(We follow notes by Hervé Le Dret found on https://www.ljll.math.upmc.fr/~ledret/M1ApproxPDE.html)

*1 Note.* Regarding (2), we see that if $\varphi$ solves (1), then using the divergence theorem we find

$$\begin{aligned}\int_D\Delta\varphi &\equiv \int_D\nabla\cdot(\nabla\varphi)\\ &\quad\text{(Div. thm.)}\\ &= \int_{\partial D}(\nabla\varphi)\cdot n\end{aligned}$$

so that using $\Delta\varphi=f$ and the boundary condition $\left.(\nabla\varphi)\cdot n\right|_{\partial D}=g$, we arrive at the compatability condition (2). Conversely, if (2) does not hold then as seen, there cannot exist a solution.

First we introduce some notation:

**2 Definition.** $\mathcal{D}(D)$ is the space of all infinitely differentiable scalar-fields on $D$ such that which have compact support. Its dual, $\mathcal{D}'(D)$, is the space of all distributions on $D$. It is the space of all continuous linear forms $\mathcal{D}(D)\to\mathbb{R}$.

**3 Definition.** $Cpt(D)$ is the set of all compact subsets of $D$.

**4 Definition.** We define

$$L^p_{\text{loc}}(D) \quad := \quad \{\,u:D\to\mathbb{R}\mid \left.u\right|_K\in L^p(K)\,\forall K\in Cpt(D)\,\}$$

Note that $L^1_{\text{loc}}(D)$ contains all $L^p(D)$ spaces, and that $L^1_{\text{loc}}(D)$ is continuously and injectively embedded in $\mathcal{D}'(D)$.

*5 Claim.* If $u\in L^1_{\text{loc}}\,(D)$ is such that

$$\int_D u\varphi \quad = \quad 0 \quad \forall\varphi\in\mathcal{D}(D)$$

then $u=0$ almost-everywhere.

*Proof.* Omitted. □

**6 Definition.** Let $m \in \mathbb{N}$ and $p \in [1, \infty]$. The Sobolev space $W^{m,p}(D)$ is defined as

$$W^{m,p}(D) \;:=\; \left\{ u \in L^p(D) \,\middle|\, \partial^\alpha u \in L^p(D) \;\forall \alpha \in \mathbb{N}^d : |\alpha| \leq m \right\}$$

where we are using the multi-index notation for $\alpha$. Note also that as $u \in L^p(D)$, it is not necessarily differentiable in the usual sense, but it is in a distribution $L^p(D) \subseteq L^1_{\text{loc}}(D) \subseteq \mathcal{D}'(D)$; as distributions are infinitely differentiable, $\partial^\alpha u$ makes sense. Then the requirement is that $\partial^\alpha u$ is a distribution that comes from a function in $L^p(D)$. We also define for convenience

$$H^m(D) \;:=\; W^{m,2}(D)$$

and note $W^{0,p}(D) \equiv L^p(D)$. We also have a natural norm:

$$\|u\|_{W^{m,p}(D)} \;:=\; \left( \sum_{\alpha \in \mathbb{N}^d : |\alpha| \leq m} \left( \|\partial^\alpha u\|_{L^p(\Omega)} \right)^p \right)^{\frac{1}{p}}$$

for all $p \in [1, \infty)$ and

$$\|u\|_{W^{m,\infty}(D)} \;:=\; \max_{\alpha \in \mathbb{N}^d : |\alpha| \leq m} \|\partial^\alpha u\|_{L^\infty(D)}$$

Note that for $p = 2$ we can see that $\|\cdot\|_{H^m(D)}$ comes from the inner product

$$\langle u, v \rangle_{H^m(D)} \;:=\; \sum_{\alpha \in \mathbb{N}^d : |\alpha| \leq m} \langle \partial^\alpha u, \partial^\alpha v \rangle_{L^2(\Omega)}$$

as $L^2(\Omega)$ is a Hilbert space.

Note that $W^{m,p}(D)$ are Banach spaces and so $H^m(D)$ is a Hilbert space. For example, the step function $H$ is in $L^1$ but is its (distributional) derivative, the delta function $\delta_0$ is not a function.

**7 Definition.** The closure of $\mathcal{D}(D)$ in $H^m(D)$ is denoted by $H^m_0(D)$. This is a sub-Hilbert-space in $H^m(D)$.

**8 Definition.** A bounded open subset $C \subseteq \mathbb{R}^d$ is called Lipschitz if its boundary is "sufficiently regular" in the sense that it can be thought of as locally being the graph of a Lipschitz continuous function.

*9 Claim.* If $D$ is Lipschitz then $\exists! \gamma_0 : H^1(D) \to L^2(\partial D)$ continuous linear such that for all $u \in C^1\left(\overline{D}\right)$, $\gamma_0\left(u|_D\right) = u|_{\partial D}$. There is also a well defined continuous linear mapping $\gamma_1 : H^2(D) \to L^2(\partial D)$ given by

$$\gamma_1(u) \;:=\; \sum_{i=1}^{d} \gamma_0(\partial_i u)\, n_i$$

and such that for all $u \in C^2\left(\overline{D}\right)$,

$$\gamma_1\left(u|_D\right) = (\nabla u)\cdot n$$

where $n$ is the normal unit vector to $\partial D$.

**10 Definition.** Define $\mathcal{H} := \left\{ \psi \in H^1(D) \,\middle|\, \int_D \psi = 0 \right\}$ which makes sense as $H^1(D) \subseteq L^2(D) \subseteq L^1(D)$.

*11 Claim.* $\mathcal{H}$ is a Hilbert space using the same scalar product as that of $H^1(\Omega)$.

*Proof.* We show that $\mathcal{H} \in \mathit{Closed}\left(H^1(\Omega)\right)$. Let $\{ h_n \}_n$ be a sequence in $\mathcal{H}$ such that $h_n \to h$ in $H^1(D)$ for some $h \in H^1(D)$. Want $h \in \mathcal{H}$. Then of course as $H^1(D) \subseteq L^2(D)$, we have $h_n \to h$ in $L^2(D)$, and then by Cauchy-Schwarz in $L^1(D)$ as well. Thus, since each $h_n \in \mathcal{H}$, its integral is zero and so

$$\begin{aligned} 0 &= \int_D h_n \\ &\to \int_D h \end{aligned}$$

so that $h \in \mathcal{H}$ as well. □

*12 Claim.* (Variational formulation of the Neumann problem) $\varphi \in H^2(D)$ solves the Neumann problem above iff for any $\psi \in \mathcal{H}$,

$$\int_D (\nabla\varphi)\cdot(\nabla\psi) = -\int_D f\psi + \int_{\partial D} g\gamma_0(\psi)$$

Assuming $f$ and $g$ obey the compatability condition above.

*Proof.* Take an arbitrary $\psi \in \mathcal{H}$ and multiply the equation $\Delta\varphi = f$ with it to get

$$(\Delta\varphi)\,\psi = f\psi$$

Note that since $\varphi \in H^2(D)$, $\Delta\varphi \in L^2(D)$; also, $\psi \in H^1(D)$ implies $\psi \in L^2(D)$. Then via Hoelder's inequality that $(\Delta\varphi)\,\psi \in L^1(D)$ so that the left hand side is integrable. Since $f \in L^2(D)$ and $\psi \in L^2(D)$, again by Hoelder $f\psi \in L^1(D)$ so that we can integrate the equation and obtain:

$$\int_D (\Delta\varphi)\,\psi = \int_D f\psi$$

We now use Green's first identity on the left hand side to get

$$\begin{aligned}\int_D (\Delta\varphi)\,\psi &= -\int_D (\nabla\varphi)\cdot(\nabla\psi) + \int_{\partial D} \psi\,(\nabla\varphi)\cdot n \\ &= -\int_D (\nabla\varphi)\cdot(\nabla\psi) + \int_{\partial D} \psi g\end{aligned}$$

Of course this cannot really be written since $\varphi$ is not a map on $\partial D$ but only on $D$ so that we must use 9 and then Green's first identity is written as

$$\int_D (\Delta\varphi)\,\psi = -\int_D (\nabla\varphi)\cdot(\nabla\psi) + \int_{\partial D} \gamma_0(\psi)\,\gamma_1(\varphi)$$

So that we find using the boundary condition that $\varphi$ fulfills:

$$\int_D (\Delta\varphi)\,\psi = -\int_D (\nabla\varphi)\cdot(\nabla\psi) + \int_{\partial D} \gamma_0(\psi)\,g$$

We find

$$\int_D (\nabla\varphi)\cdot(\nabla\psi) = \int_{\partial D} g\gamma_0(\psi) - \int_D f\psi$$

which is what we wanted to show.

Conversely, if we have some $\varphi \in H^2(D)$ such that

$$\int_D (\nabla\varphi)\cdot(\nabla\psi) = \int_{\partial D} g\gamma_0(\psi) - \int_D f\psi \qquad \forall\psi\in\mathcal{H} \tag{4}$$

Now because $\mathcal{D}(D)$ is not actually contained within $\mathcal{H}$, we need a little song and dance about defining, for each $\psi \in \mathcal{D}(D)$,

$$\tilde{\psi} := \psi - \frac{1}{\int_D}\int_D \psi$$

and now $\tilde{\psi} \in \mathcal{H}$. Note $\psi$ and $\tilde{\psi}$ differ by a constant, namely, $\frac{1}{\int_D}\int_D \psi =: k$, so that $\nabla\psi=\nabla\tilde{\psi}$. Hence if $\psi\in\mathcal{D}(D)$,

$$\begin{aligned}\int_D(\nabla\varphi)\cdot(\nabla\psi) &= \int_D(\nabla\varphi)\cdot\left(\nabla\tilde{\psi}\right)\\ &\quad\text{(By hypothesis)}\\ &= \int_{\partial D}g\gamma_0\left(\tilde{\psi}\right)-\int_D f\tilde{\psi}\\ &= \int_{\partial D}g\gamma_0(\psi-k)-\int_D f(\psi-k)\\ &\quad(\psi\in\mathcal{D}(D)\Longrightarrow\gamma_0(\psi)=0\wedge\gamma_0(k)=k)\\ &= -\int_D f\psi-k\left(\int_D f-\int_{\partial D}g\right)\\ &\quad\text{(Using the compatability condition)}\\ &= -\int_D f\psi\end{aligned}$$

Since this holds for all $\psi\in\mathcal{D}(D)$, we can use (5) to conclude $\Delta\varphi=f$ in the distributional sense. But $f\in L^2(D)$, so this holds in $L^2(D)$ as well.

So now we need to establish that $\varphi$ obeys the boundary conditions.

Using Green's formula now on the left-hand side of (4) again we find

$$-\int_D(\Delta\varphi)\psi+\int_{\partial D}\gamma_0(\psi)\gamma_1(\varphi) = \int_{\partial D}g\gamma_0(\psi)-\int_D f\psi \qquad \forall\psi\in\mathcal{H}$$

But now we may use $\Delta\varphi=f$ to find

$$\int_{\partial D}\gamma_0(\psi)(\gamma_1(\varphi)-g) = 0 \qquad \forall\psi\in\mathcal{H}$$

If $g\in H^{\frac{1}{2}}(D)$ then via $\varphi\in H^2(D)$, $\gamma_1(\varphi)\in H^{\frac{1}{2}}(\partial D)$, so that there is some $\psi\in\mathcal{H}$ so that $\gamma_0(\psi)=\gamma_1(\varphi)-g$ and we find

$$\int_{\partial D}(\gamma_1(\varphi)-g)^2 = 0$$

so it must be that $\gamma_1(\varphi)-g=0$ and $\varphi$ obeys the Neumann boundary solution of (1) as needed. □

*13 Note.* Defining the bilinear form $\omega_\Delta:\mathcal{H}\times\mathcal{H}\to\mathbb{R}$ by

$$\omega_\Delta(\varphi,\psi) := \int_D(\nabla\varphi)\cdot(\nabla\psi) \qquad \forall(\varphi,\psi)\in\mathcal{H}\times\mathcal{H}$$

and a linear form $\eta_{f,g}:\mathcal{H}\to\mathbb{R}$ via

$$\eta_{f,g}(\psi) := -\int_D f\psi+\int_{\partial D}g\gamma_0(\psi) \qquad \forall\psi\in\mathcal{H}$$

We see via that $\varphi$ solves (1) iff

$$\omega_\Delta(\varphi,\cdot) = \eta_{f,g}$$

*14 Claim.* (Lax-Milgram) Let $\mathcal{H}$ be a Hilbert space, $\omega$ be a bilinear form and $\eta$ a linear form, such that:

1. $\omega$ is continuous: $\exists M > 0$ such that

$$|\omega(\varphi, \psi)| \leq M\|\varphi\|\|\psi\| \qquad \forall (\varphi, \psi) \in \mathcal{H}^2$$

2. $\omega$ is $\mathcal{H}$-elliptic: $\exists \alpha > 0$ such that

$$\omega(\psi, \psi) \geq \alpha\|\psi\|^2 \qquad \forall v \in \mathcal{H}$$

3. $\eta$ is continuous: $\exists C > 0$ such that

$$|\eta(\psi)| \leq C\|\psi\| \quad \forall v \in \mathcal{H}$$

Then $\exists! \varphi \in \mathcal{H}$ such that

$$\omega(\varphi, \cdot) = \eta \tag{5}$$

*Proof.* We start with uniqueness: Let $\varphi_1$ and $\varphi_2$ both satisfy (5). Using linearity of $\omega$ in its first argument we have

$$\omega(\varphi_1 - \varphi_2, \cdot) = 0$$

In particular,

$$\omega(\varphi_1 - \varphi_2, \varphi_1 - \varphi_2) = 0$$

Now using the fact that $\omega$ is $\mathcal{H}$-elliptic we have actually that

$$\alpha\|\varphi_1 - \varphi_2\|^2 \leq 0$$

and so $\|\varphi_1 - \varphi_2\| = 0$ as $\alpha > 0$. But a norm is zero iff its argument is zero, so that $\varphi_1 = \varphi_2$.

We turn to existence:

Define $\Omega : \mathcal{H} \to \mathcal{H}'$ by $\psi \mapsto \omega(\psi, \cdot)$. Then the variational problem is to find a $\varphi$ such that

$$\Omega(\varphi) = \eta$$

Since $\omega$ is continuous, $\Omega(\varphi)$ is continuous (and also linear by bilinearity) so that $\Omega$ is well defined. $\eta$ is also continuous so that the variational problem is in fact an equation to be solve in $\mathcal{H}'$, the dual of $\mathcal{H}$.

Since $\eta$ is given and $\varphi$ is the unknown, the question is whether $\Omega$ is an epimorphism.

*Claim.* $im(\Omega) \in Closed(\mathcal{H}')$.

*Proof.* Let $\{ \pi_n \}_n$ be a sequence in $im\,(\Omega)$ such that $\pi_n \to \pi$ for some $\pi \in \mathcal{H}'$. If we can show that $\pi \in im\,(\Omega)$ then our result is implied.

Note that since $\pi_n$ converges, it is Cauchy. Since it is in $im\,(\Omega)$, we have a sequence $\{ \psi_n \}_n$ in $\mathcal{H}$ such that $\Omega\,(\psi_n) = \pi_n$ for all $n$. By $\mathcal{H}$-ellipticity, we have

$$\begin{aligned} \|\psi_n - \psi_m\|^2 &\leq \frac{1}{\alpha}\omega\,(\psi_n - \psi_m,\, \psi_n - \psi_m) \\ &= \frac{1}{\alpha}\,\langle \Omega\,(\psi_n) - \Omega\,(\psi_m)\,,\, \psi_n - \psi_m \rangle_{\mathcal{H}',\,\mathcal{H}} \\ &= \frac{1}{\alpha}\,\langle \pi_n - \pi_m,\, \psi_n - \psi_m \rangle_{\mathcal{H}',\,\mathcal{H}} \\ &\phantom{=}\ \text{(Cauchy-Schwarz)} \\ &\leq \frac{1}{\alpha}\|\pi_n - \pi_m\|\|\psi_n - \psi_m\| \end{aligned}$$

Thus if $\|\psi_n - \psi_m\| = 0$ we are finished, as then that means $\{ \psi_n \}_n$ converges. Otherwise, we have

$$\|\psi_n - \psi_m\| \leq \frac{1}{\alpha}\|\pi_n - \pi_m\|$$

so that $\{ \psi_n \}_n$ is Cauchy, and by completeness of $\mathcal{H}$, converges. So that there is some $\psi \in \mathcal{H}$ such that $\psi_n \to \psi$. But $\Omega$ is continuous, so that

$$\begin{aligned} \pi &= \lim_n \pi_n \\ &= \lim_n \Omega\,(\psi_n) \\ &= \Omega\left(\lim_n \psi_n\right) \\ &= \Omega\,(\psi) \end{aligned}$$

and we find $\pi \in im\,(\Omega)$ as desired. □

*Claim.* $\overline{im\,(\Omega)} = \mathcal{H}'$ (density)

*Proof.* We show this by showing that $(im\,(\Omega))^{\perp} = \{\,0\,\}$. Let $\pi \in (im\,(\Omega))^{\perp}$. Then

$$\langle \Omega\,(\psi)\,,\,\pi\rangle_{\mathcal{H}'} \quad = \quad 0$$

for all $\psi \in \mathcal{H}$. If $\delta : \mathcal{H}' \to \mathcal{H}$ is the isomorphism furnished by the Riesz representation theorem, then

$$\begin{aligned} \langle \Omega\,(\psi)\,,\,\pi\rangle_{\mathcal{H}'} &= \langle \omega\,(\psi,\,\cdot)\,,\,\pi\rangle_{\mathcal{H}'} \\ &= \omega\,(\psi,\,\delta\,(\pi)) \end{aligned}$$

so that

$$\omega\,(\psi,\,\delta\,(\pi)) \quad = \quad 0$$

for all $\psi \in \mathcal{H}$. So pick $\psi = \delta\,(\pi)$ to get

$$\begin{aligned} 0 &= \omega\,(\delta\,(\pi)\,,\,\delta\,(\pi)) \\ &\geq \alpha\|\delta\,(\pi)\|^2 \end{aligned}$$

by $\mathcal{H}$-ellipticity. But $\alpha > 0$ so that $\delta\,(\pi) = 0$, hence $\pi = 0$. But $\pi$ was arbitrary, so that $(im\,(\Omega))^{\perp} = \{\,0\,\}$. □

We then have as an immediate result that $\Omega$ is an epimorphism. □

*15 Claim.* (Poincare-Wirtinger inequality) Let $D$ be a Lipschitz open subset of $\mathbb{R}^d$. Then there exists a constant $C$ depending on $D$ such that for all $\psi \in H^1\,(D)$,

$$\left\|\psi - \frac{1}{\int_D}\int_D \psi\right\|_{L^2(D)} \quad \leq \quad C\|\nabla\psi\|_{L^2(D)} \tag{6}$$

*Proof.* Omitted. □

*16 Claim.* (The Lax-Milgram theorem may be used) The conditions of 14 are fulfilled by $\omega_\Delta$ and $\eta_{f,\,g}$.

*Proof.* We first show the elipticity of $\omega_\Delta$:

$$\begin{aligned} \omega_\Delta\,(\psi,\,\psi) &\equiv \int_D (\nabla\psi)\cdot(\nabla\psi) \\ &\equiv \|\nabla\psi\|^2_{L^2(D)} \end{aligned}$$

Because $\psi \in \mathcal{H}$, $\int_D \psi = 0$ so that (6) implies

$$\|\psi\|^2_{L^2(D)} \quad \leq \quad C^2\|\nabla\psi\|^2_{L^2(D)}$$

Hence

$$\begin{aligned}\|\psi\|_{\mathcal{H}}^{2} & \equiv \|\psi\|_{H^{1}(D)}^{2} \\ & \equiv \|\psi\|_{L^{2}(D)}^{2}+\|\nabla\psi\|_{L^{2}(D)}^{2} \\ & \leq \left(1+C^{2}\right)\|\nabla\psi\|_{L^{2}(D)}^{2} \\ & = \left(1+C^{2}\right)\omega_{\Delta}(\psi,\,\psi)\end{aligned}$$

So that $\omega_{\Delta}$ is $\mathcal{H}$-elliptic with constnat $\alpha:=\left(1+C^{2}\right)^{-1}$.

We now show continuity of $\omega_{\Delta}$:

$$\begin{aligned}|\omega_{\Delta}(u,\,v)| & \equiv \left|\int_{D}(\nabla u)\cdot(\nabla v)\right| \\ & \leq \int_{D}|(\nabla u)\cdot(\nabla v)| \\ & \quad \text{(Cauchy-Schwarz)} \\ & \leq \|\nabla u\|_{L^{2}(D)}\|\nabla v\|_{L^{2}(D)} \\ & \leq \|u\|_{H^{1}(D)}\|v\|_{H^{1}(D)}\end{aligned}$$

For $\eta_{f,\,g}$, we have

$$\begin{aligned}|\eta_{f,\,g}(\psi)| & \equiv \left|-\int_{D}f\psi+\int_{\partial D}g\gamma_{0}(\psi)\right| \\ & \quad \text{(Cauchy-Schwarz)} \\ & \leq \|f\|_{L^{2}(D)}\|\psi\|_{L^{2}(D)}+\|g\|_{L^{2}(\partial D)}\|\gamma_{0}(\psi)\|_{L^{2}(\partial D)} \\ & \quad (\gamma_{0}\text{ is continuous, so for some constant }c>0) \\ & \leq \|f\|_{L^{2}(D)}\|\psi\|_{H^{1}(D)}+\|g\|_{L^{2}(\partial D)}c\|\psi\|_{H^{1}(D)}\end{aligned}$$

□

*17 Remark.* $\mathcal{H}$ is a subspace of $H^{1}(D)$ which is $L^{2}(D)$-orthogonal to the constant maps: If $c$ is a constant map, and $\psi\in\mathcal{H}$:

$$\begin{aligned}\langle\psi,\,c\rangle_{L^{2}(D)} & \equiv \int_{D}\psi c \\ & = c\int_{D}\psi \\ & \quad (\psi\in\mathcal{H}) \\ & = 0\end{aligned}$$

and since the constants maps are also solutions of (1), we conclude that the general solution of (1) is taken from $\mathcal{H}\oplus\mathbb{R}$.