Mechanics of Continua–Spring 2017–Recitation Session of Week 1

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1 Haunted by Analysis 2

1.1 The Frechet Derivative

We give a reminder of what a Frechet derivative is.

Definition 1. Let $f : V \to W$ be a mapping between two Banach spaces V and W. The Frechet derivative of f at some $x \in V$, denoted by $(Df)(x)$, is a bounded linear operator $\mathcal{B}(V, W)$ which is an approximation of f near x in the following sense:

$$\lim_{h \to 0} \frac{\|f(x+h) - f(x) - ((Df)(x))(h)\|_W}{\|h\|_V} = 0$$

f is called Frechet-differentiable iff $(Df)(x)$ exists for all $x \in V$.

Claim 2. Assume that V and W are finite dimensional. Then every linear operator is bounded. Furthermore, If all partial derivatives of f exist and are continuous, then f is Frechet differentiable and $(Df)(x)$ is identified with the matrix given with entries $(\partial_j f_i)(x)$ (the Jacobian matrix). The converse is false as seen in some pathological examples.

Claim 3. If f is linear itself then $(Df)(x)$ is independent of x and is equal to f.

Proof. $(Df)(x)$ is unique if it exists (...). Then assuming f is linear, we have

$$\frac{\|f(x+h) - f(x) - ((Df)(x))(h)\|_W}{\|h\|_V} = \frac{\|f(h) - ((Df)(x))(h)\|_W}{\|h\|_V}$$

so that $(Df)(x) := f$ does the job. $\square$

Remark 4. Note that $(Df)(x)$ can also be seen as a map $V \ni x \mapsto (Df)(x) \in \mathcal{B}(V, W)$. In this sense, this map is *not* generically linear. Indeed, here's an example: $f:\mathbb{R}^3\to\mathbb{R}^3$ given by $f\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right):=\begin{bmatrix}(x_1)^3\\(x_2)^3\\(x_3)^3\end{bmatrix}$. The Frechet derivative of this map is given by the matrix

$$\begin{aligned}\left[(Df)(x)\right]_{i,\,j=1}^3 &\equiv \begin{bmatrix}(\partial_1 f_1)(x) & (\partial_2 f_1)(x) & (\partial_3 f_1)(x)\\(\partial_1 f_2)(x) & (\partial_2 f_2)(x) & (\partial_3 f_2)(x)\\(\partial_1 f_3)(x) & (\partial_2 f_3)(x) & (\partial_3 f_3)(x)\end{bmatrix}\\ &= \begin{bmatrix}3(x_1)^2 & 0 & 0\\0 & 3(x_2)^2 & 0\\0 & 0 & 3(x_3)^2\end{bmatrix}\end{aligned}$$

and this matrix, as a function of x, is clearly not linear.

2 Linear Algebra

2.1 Orientation

Let V be a finite dimensional vector space. We know that there is an isomorphism $V\cong\mathbb{R}^n$ for some $n\in\mathbb{N}_{>0}$.

Definition 5. A choice of such an isomorhism $f:V\to\mathbb{R}^n$ is an orientation on V.

Definition 6. Two orientations $f_1:V\to\mathbb{R}^n$ and $f_2:V\to\mathbb{R}^n$ are called "equivalent" iff the linear map $f_1\circ {f_2}^{-1}:\mathbb{R}^n\to\mathbb{R}^n$, which is an $n\times n$ matrix, has positive determinant.

Claim 7. There are exactly two equivalence classes for orientations.

Definition 8. A map $f:V\to V$ is orientation preserving iff $\det((Df)(x))>0$ for all $x\in V$.

Example 9. Consider the reflection on $\mathbb{R}^3$, given by $-\mathbb{1}_{3\times 3}$. Its determinant is $(-1)^3=-1$ so it is not orientation preserving.

Remark 10. Deformations of rigid bodies should preserve orientation.

2.2 Symmetric positive definite matrices

Definition 11. (Cholesky decomposition) A matrix $P\in Mat_{n\times n}(\mathbb{R})$ is called positive iff there is some $A_P\in Mat_{n\times n}(\mathbb{R})$ such that $(A_P)^T A_P=P$.

Claim 12. The following are equivalent:

1. P is positive.

2. P is symmetric and has eigenvalues in $[0, \infty)$.

3. P is symmetric and $\langle x, Px \rangle \geq 0$ for all $x \in \mathbb{R}^n$.

Proof. 1. implies 2.: Assume that P is positive. Then $P = A^T A$ for some A. Then $P^T = \left(A^T A\right)^T = A^T \left(A^T\right)^T = A^T A = P$ so that P is symmetric. Let $\lambda \in \sigma(P)$. Then there is some $v \in \mathbb{R}^n \setminus \{ 0 \}$ such that $Pv = \lambda v$. If $\lambda = 0$ we are finished. Otherwise, $A^T A v = \lambda v$ implies

$$\begin{aligned} 1 &= \frac{\|v\|^2}{\|v\|^2} \\ &= \frac{\langle v, v \rangle}{\|v\|^2} \\ &= \frac{\frac{1}{\lambda} \langle v, \lambda v \rangle}{\|v\|^2} \\ &= \frac{\frac{1}{\lambda} \left\langle v, A^T A v \right\rangle}{\|v\|^2} \\ &= \frac{\frac{1}{\lambda} \langle Av, Av \rangle}{\|v\|^2} \\ &= \frac{\frac{\|Av\|^2}{\lambda}}{\|v\|^2} \end{aligned}$$

which implies that $\lambda = \frac{\|Av\|^2}{\|v\|^2} > 0$. Since λ was an arbitrary eigenvalue of P, we find $\sigma(P) \subseteq \mathbb{R}_{\geq 0}$.

2. implies 3.: Any symmetric matrix may be orthogonally diagonalized: $P = O^T D O$ where $O \in O(n)$ and D is a diagonal matrix whose entries are the eigenvalues of P. Since we assume $\sigma(P) \in \mathbb{R}_{\geq 0}$, the entries of D are in $\mathbb{R}_{\geq 0}$. Then if $x \in \mathbb{R}^n$ is given,

$$\begin{aligned} \langle x, Px \rangle &= \left\langle x, O^T D O x \right\rangle \\ &= \langle Ox, DOx \rangle \\ &= \sum_{i=1}^n \sum_{j=1}^n (Ox)_i D_{ij} (Ox)_j \\ & \quad (D \text{ is diagonal}) \\ &= \sum_{i=1}^n (Ox)_i D_{ii} (Ox)_i \\ &= \sum_{i=1}^n \left[(Ox)_i\right]^2 D_{ii} \\ &\geq 0 \end{aligned}$$

Each term in the last sum is non-negative numbers due to it being the product of two non-negative numbers.

3. implies 1.: P is symmetric, so we diagonalize it as $P = O^T DO$ as above. Then note that by the above calculation, $D_{ii} \geq 0$ for all i (otherwise we reach a contradiction). As a result, $\sqrt{D}$ is defined and is a diagonal matrix whose entries are $\sqrt{D_{ij}}$. Define $A := \sqrt{D}O$. Then

$$\begin{aligned} A^T A &= \left(\sqrt{D}O\right)^T \sqrt{D}O \\ &= O^T \sqrt{D}\sqrt{D}O \\ &= O^T DO \\ &= P \end{aligned}$$

□

2.3 Polar Decomposition

Let $A \in Mat_{n\times n}(\mathbb{R})$ be given. As we've seen in the lecture, there are unique left and right polar decompositions given by

$$\begin{aligned} A &= O\,|A| \\ &= \left|A^T\right| O \end{aligned}$$

where $|A| \equiv \sqrt{A^T A}$ and $O := A\,|A|^{-1} = \left|A^T\right|^{-1} A$.

Example 13. (Thanks to Hansueli) Note that in general $|A| \neq \left|A^T\right|$. Indeed, Let $A = \frac{1}{\sqrt{2}}\begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix}$. We have

$$\begin{aligned} A^T A &= \left(\frac{1}{\sqrt{2}}\begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix}\right)^T \frac{1}{\sqrt{2}}\begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 2 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 5 & -3 \\ -3 & 5 \end{bmatrix} \end{aligned}$$

whereas

$$\begin{aligned} AA^T &= \frac{1}{\sqrt{2}}\begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix}\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix}\right)^T \\ &= \frac{1}{2}\begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -2 & 1 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

This example corresponds to $\left|A^T\right| \equiv \sqrt{AA^T}$ being stretch along the e_1 axis and then $A\,|A|^{-1} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$ being rotation by 45 degrees.