Mechanics of Continua–Spring 2017–Recitation Session of Week 2

Jacob Shapiro

March 13, 2017

1 Curvilinear Coordinates

1.1 Directional Derivatives

Claim 1. If $u : \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable vector field and $v \in \mathbb{R}^n$ then

$$(Du)\, v = \langle v,\, \nabla \rangle\, u = \partial_v u \equiv \left[\partial_\varepsilon u\left(\cdot + \varepsilon v\right)\right]\big|_{\varepsilon=0}$$

Proof. Since u is continuously differentiable, Du is a matrix given by components

$$(Du)_{i,\,j} \quad = \quad \partial_j u_i$$

Then if e_i is the ith standard basis vector of $\mathbb{R}^n$ and we are using repeating-index-sum convention, then

$$\begin{aligned}(Du)\, v &= e_i\, (Du)_{ij}\, v_j \\ &= e_i\, (\partial_j u_i)\, v_j \\ &= v_j\, (\partial_j u_i)\, e_i \\ &\equiv \langle v,\, \nabla \rangle\, u\end{aligned}$$

This takes care of the first equal sign in the claim. The last equal sign is a definition (not part of the claim). For the middle equal sign: Let $x \in \mathbb{R}^3$ be given. Then

$$\begin{aligned}\left[\partial_\varepsilon u\left(x + \varepsilon v\right)\right]\big|_{\varepsilon=0} &\equiv \lim_{\varepsilon \to 0} \frac{u\left(x + \varepsilon v\right) - u\left(x\right)}{\varepsilon} \\ &\quad (u \text{ is differentiable and so may be linearly approximated}) \\ &= \lim_{\varepsilon \to 0} \frac{u\left(x\right) + \varepsilon\left(\left(Du\right)\left(x\right)\right)\left(v\right) + O\left(\varepsilon^2\right) - u\left(x\right)}{\varepsilon} \\ &= \left(\left(Du\right)\left(x\right)\right)\left(v\right)\end{aligned}$$

Since x was arbitrary the claim follows. $\square$

1.2 Differential Geometry

Notation 2. Let $\mathcal{M}$ be a smooth manifold over $\mathbb{R}^n$. That means that for any $p \in \mathcal{M}$ there is some $U \in Nbhd_{\mathcal{M}}(p)$ such that we have a chart, that is, a homeomorphism $\varphi : U \to \varphi(U) \in Open(\mathbb{R}^n)$ and that transition maps between different charts are smooth.

We call $\mathcal{F}(\mathcal{M})$ the algebra of smooth maps $\mathcal{M} \to \mathbb{R}$.

At any point $p \in \mathcal{M}$ we have a vector space $T_p\mathcal{M}$, called the tangent space, built from maps $\mathcal{F}(\mathcal{M}) \to \mathbb{R}$ which are linear and Leibniz at p: If $X \in T_p\mathcal{M}$, f and g are in $\mathcal{F}(\mathcal{M})$ and $\alpha \in \mathbb{R}$ then

1. Leibniz: $X(fg) = f(p)X(g) + X(f)g(p)$.
2. Linear: $X(\alpha f + g) = aX(f) + X(g)$.

Given a chart φ, $T_p\mathcal{M}$ has a basis induced by φ which we label as $\{ X_i^\varphi \}_{i=1}^n$ and is given by

$$X_i^\varphi \quad := \quad \left[\partial_i \left(\cdot \circ \varphi^{-1}\right)\right]\big|_{\varphi(p)}$$

A metric $g \in \Gamma\left((TM \otimes TM)^*\right)$ at any point $p \in \mathcal{M}$ takes two tangent vectors and gives a number

$$g_p(X_p, Y_p) \quad \in \quad \mathbb{R}$$

and then one can verify g_p is an inner product at any p.

In the basis $\{ X_i^\varphi \}_{i=1}^n$, the components of the metric g are given by $\left\{ g_p\left(X_i^\varphi, X_j^\varphi\right) \right\}_{i,j=1}^n$.

Definition 3. (Musical Isomorphism) We define $\flat : T_p\mathcal{M} \to T_p^*\mathcal{M}$ by

$$v^\flat \quad := \quad g(v, \cdot) \qquad \forall v \in T_p\mathcal{M}$$

Furthermore, by the Riesz representation theorem as g is an inner-product, any $\omega \in T_p^*\mathcal{M}$ is of the form $\omega = g(v_\omega, \cdot)$ for some $v_\omega \in T_p\mathcal{M}$ so that we can define $\sharp : T_p^*\mathcal{M} \to T_p\mathcal{M}$ as

$$(g(v, \cdot))^\sharp \quad := \quad v$$

Thus $\flat^{-1} = \sharp$. Then

$$g\left(\omega^\sharp, v\right) \quad := \quad \omega(v) \qquad \forall v \in T_p\mathcal{M}$$

Definition 4. (The covariant derivative induced by a given metric) A metric g defines a covariant derivative via the Christoffel symbols: Let $\{ v_i \}_{i=1}^n$ be a basis of $T_p\mathcal{M}$, and $g_{ij} := g(v_i, v_j)$. Then we denote by g^{ij} the ij components of the inverse of the matrix $\{ g_{ij} \}_{i,j=1}^n$. Then the Christoffel symbols are defined as

$$\Gamma^i{}_{kl} \quad := \quad \frac{1}{2} g^{im}\left(\partial_l g_{mk} + \partial_k g_{ml} - \partial_m g_{kl}\right)$$

The covariant derivative ∇ induced by g is then defined via Γ as follows: For a scalar f, we have

$$\nabla f = v_i(f)\, v_i^*$$

For a vector X we have

$$\nabla X = \left(\Gamma^k{}_{ij} v_j(X) + v_i(v_k(X))\right) v_k \otimes v_i^*$$

and then there are recursive formulas for how ∇ acts on general tensors (but we won't need those formulas here).

Remark 5. We assume everything stated so far is well known to the reader. Now starts the part about the differential operators and their definitions in differential geometry.

Definition 6. The gradient *grad* is a map $grad : \mathcal{F}(\mathcal{M}) \to TM$ defined as

$$grad(f) := (\nabla f)^\sharp \quad \forall f \in \mathcal{F}(\mathcal{M})$$

Thus if $f \in \mathcal{F}(\mathcal{M})$ and $X \in T\mathcal{M}$ then

$$\begin{aligned} g(grad(f), X) &\equiv g\left((\nabla f)^\sharp, X\right) \\ &\equiv (\nabla f)(X) \\ &\equiv \nabla_X f \\ &\equiv X(f) \end{aligned}$$

Definition 7. The divergence *div* is a map $div : T\mathcal{M} \to \mathcal{F}(\mathcal{M})$ given by

$$div(X) := tr(\nabla X)$$

(Note the trace is defined via contraction: If T is a tensor of type $(1, 1)$ then its trace is the following expression

$$tr(T) \equiv \sum_{i=1}^{n} T(v_i^*, v_i)$$

)

Definition 8. The Laplacian Δ is a map $\Delta : \mathcal{T}(k, l) \to \mathcal{T}(k, l)$ given by

$$\Delta(T) := div(\nabla T)$$

Question: What about $div \circ grad = div\left((\nabla f)^\sharp\right)$?

Example 9. Consider the manifold $\mathbb{R}^2$. Define on it an open subset $U \in \mathit{Open}\left(\mathbb{R}^2\right)$ given by deleting from $\mathbb{R}^2$ the positive horizontal axis:

$$U \quad := \quad \mathbb{R}^2 \backslash \left\{\, x \in \mathbb{R}^2 \,\middle|\, x_1 \geq 0 \wedge x_2 = 0 \,\right\}$$

Define a chart $\psi : U \to \psi(U)$ by the following formula: $\psi(x) := \begin{bmatrix} \|x\| \\ \arctan\left(\frac{x_2}{x_1}\right) \end{bmatrix}$. Then this defines a homeomorphism (verify...) and $\psi(U) = (0,\, \infty) \times (0,\, 2\pi) \in \mathit{Open}\left(\mathbb{R}^2\right)$. Also check that $\psi^{-1} = \begin{bmatrix} r \\ \varphi \end{bmatrix} \mapsto \begin{bmatrix} r\cos(\varphi) \\ r\sin(\varphi) \end{bmatrix}$. Then the basis of $T\mathbb{R}^2$ corresponding to ψ is given by

$$X_r^{\psi} \quad = \quad \cos(\varphi)\,\partial_1 + \sin(\varphi)\,\partial_2$$

$$X_{\varphi}^{\psi} \quad = \quad -r\sin(\varphi)\,\partial_1 + r\cos(\varphi)\,\partial_2$$

Then we take the Euclidean metric, which is given by

$$g(\partial_i,\, \partial_j) \quad = \quad \delta_{i,\,j}$$

in the standard basis $\{\, \partial_i \,\}_{i=1}^2$ and write it in our new basis to get:

$$\begin{aligned} g\left(X_{\alpha}^{\psi},\, X_{\beta}^{\psi}\right) \quad &= \quad \left(X_{\alpha}^{\psi}\right)_i \left(X_{\beta}^{\psi}\right)_j \underbrace{g(\partial_i,\, \partial_j)}_{\delta_{i,\,j}} \\ &= \quad \left(X_{\alpha}^{\psi}\right)_i \left(X_{\beta}^{\psi}\right)_i \end{aligned}$$

Thus

$$\begin{aligned} \left[g\left(X_{\alpha}^{\psi},\, X_{\beta}^{\psi}\right)\right]_{(\alpha,\,\beta)\in\{\, r,\,\varphi \,\}^2} \quad &= \quad \begin{bmatrix} \cos(\varphi)^2 + \sin(\varphi)^2 & \cos(\varphi)(-r\sin(\varphi)) + \sin(\varphi)\, r\cos(\varphi) \\ -r\sin(\varphi)\cos(\varphi) + r\cos(\varphi)\sin(\varphi) & r^2\sin(\varphi)^2 + r^2\cos(\varphi)^2 \end{bmatrix} \\ &= \quad \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix} \end{aligned}$$

We find

$$\boxed{[g] = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}} \tag{1}$$

which implies

$$\left[g^{-1}\right] \quad = \quad \begin{bmatrix} 1 & 0 \\ 0 & r^{-2} \end{bmatrix}$$

Now we write the gradient of a scalar $f \in \mathcal{F}(\mathcal{M})$ in this basis:

$$\begin{aligned} \mathit{grad}(f) \quad &\equiv \quad (\nabla f)^{\sharp} \\ &= \quad \left[g\left(X_{\alpha}^{\psi},\, X_{\beta}^{\psi}\right)^{-1}\right] X_{\alpha}^{\psi}(f)\, X_{\beta}^{\psi} \\ &= \quad X_r^{\psi}(f)\, X_r^{\psi} + r^{-2} X_{\varphi}^{\psi}(f)\, X_{\varphi}^{\psi} \end{aligned}$$

This is still not the formula we know. This is because the basis $\left\{ X^{\psi}_{\alpha} \right\}_{\alpha}$ is not normalized. In fact

$$g\left(X^{\psi}_{\varphi},\, X^{\psi}_{\varphi}\right) \quad = \quad r^2$$

so that with a normalized vector $\hat{X}^{\psi}_{\varphi} \equiv \frac{1}{r} X^{\psi}_{\varphi}$ we obtain

$$\boxed{grad\left(f\right) = X^{\psi}_{r}\left(f\right) X^{\psi}_{r} + r^{-1} X^{\psi}_{\varphi}\left(f\right) \hat{X^{\psi}_{\varphi}}} \tag{2}$$

which is the formula we know from young age (perhaps with the notation $X^{\psi}_{r} = \partial_r$, $X^{\psi}_{\varphi} = \partial_{\varphi}$ and we are used to write the (orthonormal) basis of the tangent space as e_r and e_{φ}).

We go on to tackle the divergence. Recall that if $u \in TM$ then

$$\nabla u \quad = \quad \left(X^{\psi}_{\beta}\left(X^{\psi}_{\alpha}\left(u\right)\right) + \Gamma^{\alpha}{}_{\beta\gamma} X^{\psi}_{\gamma}\left(u\right)\right) X^{\psi}_{\alpha} \otimes X^{\psi *}_{\beta}$$

So if compute the Christoffel symbols for (1) we'll know how to compute the covariant derivative of a vector field.

$$\partial_r \left[g\left(X^{\psi}_{\alpha},\, X^{\psi}_{\beta}\right)\right]_{(\alpha,\,\beta)\in\{\, r,\,\varphi\,\}^2} \quad = \quad \begin{bmatrix} 0 & 0 \\ 0 & 2r \end{bmatrix}$$

$$\partial_{\varphi} \left[g\left(X^{\psi}_{\alpha},\, X^{\psi}_{\beta}\right)\right]_{(\alpha,\,\beta)\in\{\, r,\,\varphi\,\}^2} \quad = \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence

$$\partial_r \left[g\left(X^{\psi}_{\varphi},\, X^{\psi}_{\varphi}\right)\right] \quad = \quad 2r$$

and all other derivatives are zero. There are three indices, each with two possible values, thus eight entries to compute:

$$\begin{aligned}
\Gamma^{\alpha}{}_{\beta\gamma} \quad &\equiv \quad \frac{1}{2}\left[g\left(X^{\psi}_{\alpha},\, X^{\psi}_{\delta}\right)^{-1}\right]\left(\partial_{\gamma}\left[g\left(X^{\psi}_{\beta},\, X^{\psi}_{\delta}\right)\right] + \partial_{\beta}\left[g\left(X^{\psi}_{\delta},\, X^{\psi}_{\gamma}\right)\right] - \partial_{\delta}\left[g\left(X^{\psi}_{\gamma},\, X^{\psi}_{\beta}\right)\right]\right) \\
&= \quad \frac{1}{2}\left[g\left(X^{\psi}_{\alpha},\, X^{\psi}_{r}\right)^{-1}\right]\left(\partial_{\gamma}\left[g\left(X^{\psi}_{\beta},\, X^{\psi}_{r}\right)\right] + \partial_{\beta}\left[g\left(X^{\psi}_{r},\, X^{\psi}_{\gamma}\right)\right] - \partial_{r}\left[g\left(X^{\psi}_{\gamma},\, X^{\psi}_{\beta}\right)\right]\right) + \\
&\quad\quad + \frac{1}{2}\left[g\left(X^{\psi}_{\alpha},\, X^{\psi}_{\varphi}\right)^{-1}\right]\left(\partial_{\gamma}\left[g\left(X^{\psi}_{\beta},\, X^{\psi}_{\varphi}\right)\right] + \partial_{\beta}\left[g\left(X^{\psi}_{\varphi},\, X^{\psi}_{\gamma}\right)\right] - \partial_{\varphi}\left[g\left(X^{\psi}_{\gamma},\, X^{\psi}_{\beta}\right)\right]\right) \\
&= \quad \frac{1}{2}\delta_{\alpha,\,r}\left(-\delta_{\gamma,\,\varphi}\delta_{\beta,\,\varphi}2r\right) + \frac{1}{2}\delta_{\alpha,\,\varphi}r^{-2}\left(\delta_{\beta,\,\varphi}\delta_{\gamma,\,r}2r + \delta_{\beta,\,r}\delta_{\gamma,\,\varphi}2r\right) \\
&= \quad -r\delta_{\alpha,\,r}\delta_{\gamma,\,\varphi}\delta_{\beta,\,\varphi} + r^{-1}\delta_{\alpha,\,\varphi}\left(\delta_{\beta,\,\varphi}\delta_{\gamma,\,r} + \delta_{\beta,\,r}\delta_{\gamma,\,\varphi}\right)
\end{aligned}$$

$$\Gamma^{r}{}_{\cdot\cdot} \quad = \quad \begin{bmatrix} 0 & 0 \\ 0 & -r \end{bmatrix}$$

$$\Gamma^{\varphi}{}_{\cdot\cdot} = \begin{bmatrix} 0 & r^{-1} \\ r^{-1} & 0 \end{bmatrix}$$

Hence the covariant derivative is

$$\begin{aligned}
\nabla u &= \left(X^{\psi}_{\beta}\left(X^{\psi}_{\alpha}(u)\right) + \Gamma^{\alpha}{}_{\beta\gamma} X^{\psi}_{\gamma}(u)\right) X^{\psi}_{\alpha} \otimes X^{\psi *}_{\beta} \\
&= \left(X^{\psi}_{r}\left(X^{\psi}_{r}(u)\right) + \underbrace{\Gamma^{r}{}_{r\gamma}}_{=0} X^{\psi}_{\gamma}(u)\right) X^{\psi}_{r} \otimes X^{\psi *}_{r} + \\
&\quad \left(X^{\psi}_{\varphi}\left(X^{\psi}_{r}(u)\right) + \underbrace{\Gamma^{r}{}_{\varphi\gamma}}_{\delta_{\gamma,\,\varphi}(-r)} X^{\psi}_{\gamma}(u)\right) X^{\psi}_{r} \otimes X^{\psi *}_{\varphi} + \\
&\quad \left(X^{\psi}_{r}\left(X^{\psi}_{\varphi}(u)\right) + \underbrace{\Gamma^{\varphi}{}_{r\gamma}}_{\delta_{\gamma,\,\varphi} r^{-1}} X^{\psi}_{\gamma}(u)\right) X^{\psi}_{\varphi} \otimes X^{\psi *}_{r} + \\
&\quad \left(X^{\psi}_{\varphi}\left(X^{\psi}_{\varphi}(u)\right) + \underbrace{\Gamma^{\varphi}{}_{\varphi\gamma}}_{\delta_{\gamma,\,r} r^{-1}} X^{\psi}_{\gamma}(u)\right) X^{\psi}_{\varphi} \otimes X^{\psi *}_{\varphi} \\
&= X^{\psi}_{r}\left(X^{\psi}_{r}(u)\right) X^{\psi}_{r} \otimes X^{\psi *}_{r} + \\
&\quad \left(X^{\psi}_{\varphi}\left(X^{\psi}_{r}(u)\right) - r X^{\psi}_{\varphi}(u)\right) X^{\psi}_{r} \otimes X^{\psi *}_{\varphi} + \\
&\quad \left(X^{\psi}_{r}\left(X^{\psi}_{\varphi}(u)\right) + r^{-1} X^{\psi}_{\varphi}(u)\right) X^{\psi}_{\varphi} \otimes X^{\psi *}_{r} + \\
&\quad \left(X^{\psi}_{\varphi}\left(X^{\psi}_{\varphi}(u)\right) + r^{-1} X^{\psi}_{r}(u)\right) X^{\psi}_{\varphi} \otimes X^{\psi *}_{\varphi}
\end{aligned}$$

Now that we have the covariant derivative, we can compute the divergence:

$$\begin{aligned}
div(u) &\equiv tr(\nabla u) \\
&= X^{\psi}_{r}\left(X^{\psi}_{r}(u)\right) + X^{\psi}_{\varphi}\left(X^{\psi}_{\varphi}(u)\right) + r^{-1} X^{\psi}_{r}(u)
\end{aligned}$$

Next note that if we expand u as

$$u = u_r X^{\psi}_{r} + u_{\varphi} \hat{X}^{\psi}_{\varphi}$$

(again, we want to use the orthonormal basis $\left\{ X^{\psi}_{r},\, \hat{X}^{\psi}_{\varphi} \right\}$ instead of the merely orthogonal basis $\{ X^{\psi}_{r},\, X^{\psi}_{\varphi} \}$) then

$$u = u_r X^{\psi}_{r} + u_{\varphi} \frac{1}{r} X^{\psi}_{\varphi}$$

so that we find

$$div(u) = X^{\psi}_{r}(u_r) + \frac{1}{r} u_r + X^{\psi}_{\varphi}\left(\frac{1}{r} u_{\varphi}\right)$$

and finally since X^ψ_φ and r commute we get

$$\boxed{div\,(u) = X^\psi_r\,(u_r) + \frac{1}{r}u_r + \frac{1}{r}X^\psi_\varphi\,(u_\varphi)} \tag{3}$$

which is the result we know.

To compute the Laplacian of a scalar, we use the definition

$$\Delta f \quad \equiv \quad div\,(grad\,(f))$$

and we have already computed the gradient of a scalar f (see (2)) from which we find find

$$\begin{aligned} div\,(grad\,(f)) &= X^\psi_r\left(X^\psi_r\,(f)\right) + \frac{1}{r}X^\psi_r\,(f) + \frac{1}{r}X^\psi_\varphi\left(r^{-1}X^\psi_\varphi\,(f)\right) \\ &= X^\psi_r\left(X^\psi_r\,(f)\right) + \frac{1}{r}X^\psi_r\,(f) + \frac{1}{r^2}X^\psi_\varphi\left(X^\psi_\varphi\,(f)\right) \end{aligned}$$

Again the result we are familiar with (with the notation $X^\psi_r \to \partial_r$, $X^\psi_\varphi \to \partial_\varphi$).

The Laplacian of a vector field is left as an exercise to the reader.

2 Liouville's Theorem

Claim 10. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be smooth. If Df is constant, then f is affine.

Proof. We can always write f as a Taylor expansion:

$$f\,(x_0 + x) \quad = \quad f\,(x_0) + (Df)\,(x_0)\,x + \dots$$

where the dots denote higher order derivatives, which are written using, for instance, multi-index notation: $\alpha \in (\mathbb{N}_{\geq 0})^n$ is an n-index. Then

$$D^\alpha f \quad := \quad \partial_1^{\alpha_1}\dots\partial_n^{\alpha_n} f$$

Then it's clear that since Df which is expressed via $\partial_j f_i$ for instance, is constant, then any higher order derivatives will vanish and we'll be left with an affine map. □

3 Some Remarks about HW1

3.1 Question 3

- In the change of variables formula there *is* an absolute value on the determinant.

3.2 Question 2

Claim 11. Let $A \in Mat_{n\times n}(\mathbb{R})$. If $\mathrm{Tr}(AB) = 0$ for all $B \in Mat_{n\times n}(\mathbb{R})$ such that $B = -B^T$ then $A = A^T$.

Proof. Write $A = \underbrace{\frac{1}{2}\left(A + A^T\right)}_{=:A_S} + \underbrace{\frac{1}{2}\left(A - A^T\right)}_{=:A_{AS}}$ and observe

Claim 12. $\mathrm{Tr}(UV) = 0$ for all U symmetric and V anti-symmetric.

Proof. We have by the fact that $\mathrm{Tr}(X) = \mathrm{Tr}\left(X^T\right)$,

$$\begin{aligned}
\mathrm{Tr}(UV) &= \mathrm{Tr}\left((UV)^T\right) \\
&= \mathrm{Tr}\left(V^T U^T\right) \\
&= \mathrm{Tr}(-VU) \\
&= -\mathrm{Tr}(UV)
\end{aligned}$$

□

then by the above,

$$\begin{aligned}
\mathrm{Tr}(AB) &= \mathrm{Tr}\left(\left(A_S + A_{AS}\right)B\right) \\
&= \mathrm{Tr}\left(A_{AS}B\right)
\end{aligned}$$

Since this is true *for all* B, we can pick $B = {A_{AS}}^T$ to get

$$\begin{aligned}
\mathrm{Tr}\left(A_{AS}{A_{AS}}^T\right) &= 0 \\
\|A_{AS}\|_F^2 &= 0 \\
A_{AS} &= 0
\end{aligned}$$

Using the fact that $\|\cdot\|_F$ is a norm. Hence $A = A_S$, that is, $A = A^T$. □

Now we are given the matrix $A \in Mat_{3\times 3}(\mathbb{R})$ such that $\det(A) > 0$, and write

$$A = \left|A^T\right| R_A$$

We define the (square) distance function $g : SO(3) \to \mathbb{R}$:

$$g(R) = \|R - A\|_F^2$$

We want the derivative of g at an arbitrary $R \in SO(3)$, $(Dg)(R)$. This will be a linear map on the tangent space of $SO(3)$ at R. We know that:

$$((Dg)(R))\left(\tilde{R}\right) = \partial_\varepsilon|_{\varepsilon=0}\, g\left(R + \varepsilon\tilde{R}\right)$$

And $\left(R+\varepsilon\tilde{R}\right)\overset{!}{\in}SO\left(3\right)$ so

$$\begin{aligned}\left(R+\varepsilon\tilde{R}\right)\left(R+\varepsilon\tilde{R}\right)^{T} & \overset{!}{=} & \mathbb{1}\\ \tilde{R}R^{T}+R\tilde{R}^{T} & \overset{!}{=} & 0\end{aligned}$$

similarly,

$$\begin{aligned}\left(R+\varepsilon\tilde{R}\right)^{T}\left(R+\varepsilon\tilde{R}\right) & \overset{!}{=} & \mathbb{1}\\ R^{T}\tilde{R}+\tilde{R}^{T}R & \overset{!}{=} & 0\end{aligned}$$

Now the computation of $\left(Dg\right)\left(R\right)$:

$$\begin{aligned}\partial_{\varepsilon}|_{\varepsilon=0}\,g\left(R+\varepsilon\tilde{R}\right) & = & \partial_{\varepsilon}|_{\varepsilon=0}\,\mathrm{Tr}\left(\left(R+\varepsilon\tilde{R}-A\right)^{T}\left(R+\varepsilon\tilde{R}-A\right)\right)\\ & = & \partial_{\varepsilon}|_{\varepsilon=0}\,\mathrm{Tr}\left(\left(R-A\right)^{T}\left(R-A\right)+\varepsilon\tilde{R}^{T}\left(R-A\right)+\varepsilon\left(R-A\right)^{T}\tilde{R}\right)\\ & = & -\mathrm{Tr}\left(\tilde{R}^{T}A+A^{T}\tilde{R}\right)\\ & = & -2\mathrm{Tr}\left(A^{T}\tilde{R}\right)\\ & = & -2\mathrm{Tr}\left(A^{T}\tilde{R}R^{T}R\right)\\ & = & -2\mathrm{Tr}\left(RA^{T}\tilde{R}R^{T}\right)\\ & \overset{!}{=} & 0\end{aligned}$$

This constraint should hold for all $\tilde{R}$ such that $\tilde{R}R^{T}$ and $\tilde{R}^{T}R$ are anti-symmetric. Since given any $W\in Mat_{3\times3}\left(\mathbb{R}\right)$ with $W=-W^{T}$, there is some $\tilde{R}_{R,\,W}$ such that $W=\tilde{R}_{R,\,W}R^{T}$ (in particular, take $\tilde{R}_{R,\,W}:=RW$), we find using the above claim that R is a critical point of g iff

$$\boxed{RA^{T}\text{ is symmetric}}$$

The constraint on R is thus that

$$RR_{A}{}^{T}\left|A^{T}\right| = \left|A^{T}\right|\left(RR_{A}{}^{T}\right)^{T}$$

Hence

$$\begin{aligned}\left|A^{T}\right|^{2} & = & \left|A^{T}\right|\left|A^{T}\right|\\ & = & \left|A^{T}\right|\left(RR_{A}{}^{T}\right)^{T}\left(RR_{A}{}^{T}\right)\left|A^{T}\right|\\ & = & \left(RR_{A}{}^{T}\right)^{T}\left|A^{T}\right|\left|A^{T}\right|\left(RR_{A}{}^{T}\right)^{T}\\ & = & \left(RR_{A}{}^{T}\right)^{T}\left|A^{T}\right|^{2}\left(RR_{A}{}^{T}\right)^{T}\end{aligned}$$

so

$$\boxed{\left[\left(R{R_A}^T\right)^T,\ \left|A^T\right|^2\right]=0}$$

Hence by the spectral theorem, $\left[\left(R{R_A}^T\right)^T,\ f\left(\left|A^T\right|^2\right)\right]=0$ for any function f, in particular, $f=\sqrt{\cdot}$ as $\left|A^T\right|>0$, so that $\left[\left(R{R_A}^T\right)^T,\ \left|A^T\right|\right]=0$ and thus these two matrices may be simultaneously diagonalized by some orthogonal $M\in O(3)$:

$$\left(R{R_A}^T\right)^T = M^T D_{\left(R{R_A}^T\right)^T} M$$

and

$$\left|A^T\right| = M^T D_{\left|A^T\right|} M$$

where the D's are diagonal matrices containing the eigenvalues of the respective matrices. We find that if R is a critical point of g then:

$$\begin{aligned}
g(R) &= \|R-A\|_F^2 \\
&= \left\|(R-A)^T\right\|_F^2 \\
&= \left\|R^T-A^T\right\|_F^2 \\
&= \left\|{R_A}^T R_A R^T - {R_A}^T\left|A^T\right|\right\|_F^2 \\
&= \left\|{R_A}^T\left(R_A R^T-\left|A^T\right|\right)\right\|_F^2 \\
&\quad (\|\cdot\|_F \text{ invariant under } O(3)) \\
&= \left\|R_A R^T-\left|A^T\right|\right\|_F^2 \\
&= \left\|\left(R{R_A}^T\right)^T-\left|A^T\right|\right\|_F^2 \\
&= \left\|M^T D_{\left(R{R_A}^T\right)^T} M - M^T D_{\left|A^T\right|} M\right\|_F^2 \\
&\quad (\|\cdot\|_F \text{ invariant under } O(3)) \\
&= \left\|D_{\left(R{R_A}^T\right)^T} - D_{\left|A^T\right|}\right\|_F^2 \\
&\equiv \sum_{i=1}^{3}\left|\left(D_{\left(R{R_A}^T\right)^T}\right)_{ii}-\left(D_{\left|A^T\right|}\right)_{ii}\right|^2
\end{aligned}$$

We know that $\det(A)>0$ so that $\left(D_{\left|A^T\right|}\right)_{ii}$, which is the ith singular value of A, is necessarily (strictly) positive. Furthermore, $\left(R{R_A}^T\right)^T$ is orthogonal, so that $\left|\left(D_{\left(R{R_A}^T\right)^T}\right)_{ii}\right|=1$, and $\left(D_{\left(R{R_A}^T\right)^T}\right)_{11}\left(D_{\left(R{R_A}^T\right)^T}\right)_{22}\left(D_{\left(R{R_A}^T\right)^T}\right)_{33}=+1$ as $\det\left(\left(R{R_A}^T\right)^T\right)=1$ as $\left(R{R_A}^T\right)^T\in SO(3)$. Hence to minimize g we must pick R such that $\left(D_{\left(R{R_A}^T\right)^T}\right)_{ii}=1$, which implies $\left(R{R_A}^T\right)^T=\mathbb{1}$, or $R=R_A$ as desired.