JAN 25 2022

WEEK 1

Administrative matters

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(⚠ NOT JNS@P...)

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Office hours: Mondays 4pm-6pm
in my office OR
make appointment by email.

- If you need to quarantine, please let me know ASAP via email.

- Routine very similar to PHY103, but:
  - Quiz in lecture, not precept.
  - Need to hand in one LGT problem.
    (Use LyX: get extra point).

Table of Contents

My conventions if you are unfamiliar

- I would often omit arrows above vectors: I write $v \equiv \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \equiv v_1 e_1 + \dots$ (unit vector)

  instead of $\vec{v}$, $\underline{v}$, $\mathbf{v}$.

  Recall $e_1 \equiv \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 \equiv \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $e_3 \equiv \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

- I use Greek alphabet:

  $\alpha, \beta, \gamma, \delta, \dots$

- $\mathbb{R}^n \equiv$ set of n-vectors w/ real entries

  $\mathbb{R}^2$ – 2D vectors

  $\mathbb{R}^3$ – 3D vectors

- $f: \mathbb{R}^3 \to \mathbb{R}$ is a function from 3D vectors into (real) numbers.

  Sometimes will write

  $$\mathbb{R}^3 \ni x \longmapsto f(x) \in \mathbb{R}$$

  to be explicit about formula, e.g.,

  $$\mathbb{R}^3 \ni x \longmapsto \|x\| \in [0, \infty).$$

- $\|v\|$ norm of vector:

  For $v \in \mathbb{R}^3$, $\|v\|^2 \equiv v_1^{\,2} + v_2^{\,2} + v_3^{\,2}$.

- Better notation for integrals:

$$\int_S f \equiv \int_{x \in S} f(x)\, dx \equiv \int_{x \in S} dx\ f(x) .$$

- $x \mapsto \delta(x) \equiv$ delta-f^n: $\int f\,\delta = f(0)$.

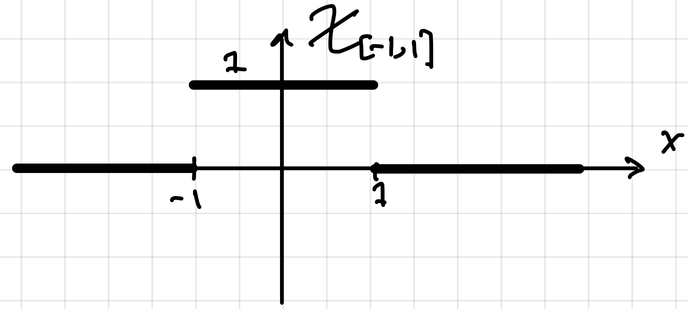
  "symbolic" function useful for densities:

$$\text{"}\ \delta(0) = \infty \ ; \quad \delta(x) = 0 \quad \forall\ x \neq 0\ \text{"}$$

- Characteristic func.:

$$\chi_S(x) \equiv \begin{cases} 1 & x \in S \\ 0 & x \notin S \end{cases}$$

Example:



- 4 units: $k \equiv \dfrac{1}{4\pi\varepsilon_0} \equiv 1$.

  More on that later...

Theory recap

(1) Two point particles w/ charges q_1, q_2 will exert force on each other. The force on q_1 due to q_2 is

$$F_{12} = \frac{q_1 q_2}{\|r_1 - r_2\|^3}(r_1 - r_2)$$

"inverse-square-law"

where r_1, r_2 are the positions of the two particles.

(2) The el. field at r_1 due to the existence of q_2 at r_2 is

$$E_2(r_1) = \frac{q_2}{\|r_1 - r_2\|^3}(r_1 - r_2)$$

(3) The el. field at r_1 due to given charge dist. $\rho : \mathbb{R}^n \to [0, \infty)$ is

$$E(r_1) = \int_{x \in \mathbb{R}^n} dx \, \frac{\rho(x)}{\|r_1 - x\|^3}(r_1 - x)$$

ρ is called "charge density". Defined s.t. $\int \rho \equiv$ total charge in entire volume.

Example ρ's: ① Point charge q at some point $x_0 \in \mathbb{R}^3$:

$$\rho(x) = q\,\delta(x - x_0)$$

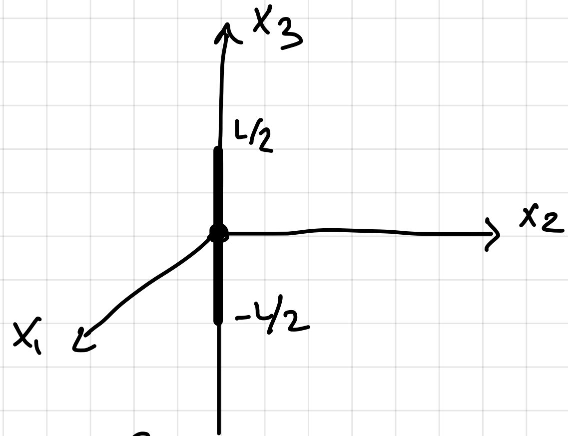
Check: $\int \rho = q \int_{x \in \mathbb{R}^3} \delta(x-x_0)\,dx$

$$= q. \checkmark$$

$$E(r) = \int_{x \in \mathbb{R}^3} \frac{\rho(x)}{\|r-x\|^3}(r-x)\,dx$$

② Linear charge density: wire of length L and total homogeneous charge Q:

$$\rho(x) = \frac{Q}{L}\,\delta(x_1)\,\delta(x_2)\,\chi_{[-\frac{L}{2},\frac{L}{2}]}(x_3)$$



Check: $\int_{x \in \mathbb{R}^3} \rho(x)\,dx = Q. \checkmark$

③ Plate (2D) or cube (3D) also possible.

<u>Example:</u> Infinite thin rod of charge λ per unit length.

$$\Longrightarrow \quad \rho(x) = \delta(x_1)\,\delta(x_2)\,\lambda$$

$$E(r) = \int_{x \in \mathbb{R}^3} dx \, \frac{\rho(x)}{\|r-x\|^3}(r-x)$$

$$= \int_{x_1 \in \mathbb{R}} dx_1 \int_{x_2 \in \mathbb{R}} dx_2 \int_{x_3 \in \mathbb{R}} dx_3 \, \frac{\rho(x_1, x_2, x_3)}{\|r-x\|^3}(r-x)$$

$$= \int_{x_1 \in \mathbb{R}} dx_1 \, \delta(x_1) \int_{x_2 \in \mathbb{R}} dx_2 \, \delta(x_2) \int_{x_3 \in \mathbb{R}} dx_3 \, \frac{\lambda(r-x)}{\|r-x\|^3}$$

$$= \lambda \int_{x_3 \in \mathbb{R}} \frac{r - x_3 e_3}{\|r - x_3 e_3\|^3} \, dx_3$$

$$\int_{x_3\in\mathbb{R}} \frac{r - x_3 e_3}{\|r - x_3 e_3\|^3}\, dx_3 = -\int_{x_3\in\mathbb{R}} \frac{(x_3 - r_3)e_3 - r_1 e_1 - r_2 e_2}{[r_1^2 + r_2^2 + (x_3 - r_3)^2]^{3/2}}\, dx_3$$

$$\overset{\text{transl. invar.}}{=} -\int_{x_3\in\mathbb{R}} \frac{x_3 e_3 - r_1 e_1 - r_2 e_2}{[r_1^2 + r_2^2 + x_3^2]^{3/2}}\, dx_3$$

$R := r_1 e_1 + r_2 e_2$

$$\overset{\downarrow}{=} -\frac{1}{\|R\|^3} \int_{x_3\in\mathbb{R}} \frac{x_3 e_3 - R}{[1 + (x_3/\|R\|)^2]^{3/2}}\, dx_3$$

$z := x_3/\|R\|$

$$\overset{\downarrow}{=} -\frac{1}{\|R\|^2} \int_{z\in\mathbb{R}} \frac{\|R\| z e_3 - R}{(1 + z^2)^{3/2}}\, dz$$

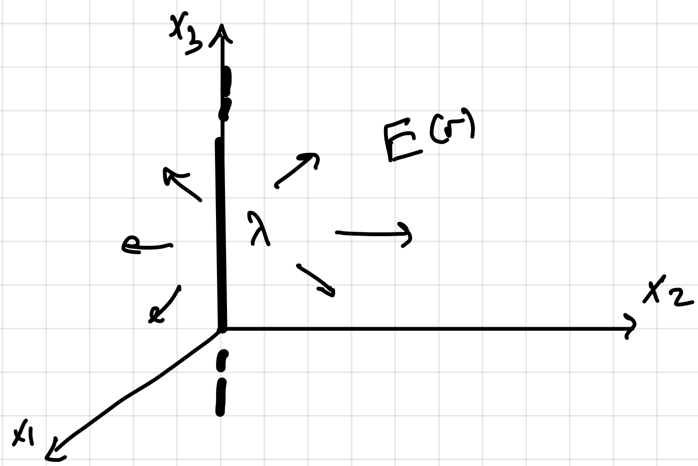
$$= -\frac{1}{\|R\|^2}\left(-R \int_{z\in\mathbb{R}} \frac{dz}{(1+z^2)^{3/2}} + \|R\| e_3 \underbrace{\int_{z\in\mathbb{R}} \frac{z\, dz}{(1+z^2)^{3/2}}}_{\substack{=0 \\ \text{by symmetry}}}\right)$$

$$\left(\frac{z}{\sqrt{1+z^2}}\right)' = \frac{1}{(1+z^2)^{3/2}}$$

$$\Rightarrow \int_{z\in\mathbb{R}} \frac{dz}{(1+z^2)^{3/2}} = \frac{z}{\sqrt{1+z^2}}\Bigg|_{z=-\infty}^{\infty} = 1 - (-1) = 2.$$

$$\Rightarrow E(r) = \lambda\, 2 \frac{R}{\|R\|^2} = 2\lambda \frac{r_1 e_1 + r_2 e_2}{\|r_1 e_1 + r_2 e_2\|^2}$$

$$\boxed{E(r) = 2\lambda \frac{r_1 e_1 + r_2 e_2}{\|r_1 e_1 + r_2 e_2\|^2}} \quad .$$



<u>Note</u>:

- Does NOT dep. on r_3. ✓
- If $R \equiv r_1 e_1 + r_2 e_2 = 0$, undef. ✓
- Mag. propto to $\frac{2\lambda}{\|R\|}$.

Example: Thin rod of charge λ per unit length, of length $2L$.

$$\Longrightarrow \quad \rho(x) = \lambda\,\delta(x_1)\,\delta(x_2)\,\chi_{[-L,L]}(x_3)$$

where $$\chi_S(a) \equiv \begin{cases} 1 & a \in S \\ 0 & a \notin S. \end{cases}$$

$$E(r) = \int_{x \in \mathbb{R}^3} dx \, \frac{\rho(x)}{\|r - x\|^3}(r - x)$$

$$= \int_{x_1 \in \mathbb{R}} dx_1 \int_{x_2 \in \mathbb{R}} dx_2 \int_{x_3 \in \mathbb{R}} dx_3 \, \frac{\rho(x_1, x_2, x_3)}{\|r - x\|^3}(r - x)$$

$$= \int_{x_1 \in \mathbb{R}} dx_1\, \delta(x_1) \int_{x_2 \in \mathbb{R}} dx_2\, \delta(x_2) \int_{x_3 \in \mathbb{R}} dx_3 \, \frac{\lambda\, \chi_{[-L,L]}(x_3)\,(r-x)}{\|r - x\|^3} dx_3$$

$$= \lambda \int_{x_3 = -L}^{L} \frac{r - x_3 e_3}{\| r - x_3 e_3 \|^3} \, dx_3$$

Case 1: $R := r_1 e_1 + r_2 e_2 \neq 0$

$$E(r) = \lambda \int_{x_3 = -L}^{L} \frac{R + (r_3 - x_3) e_3}{(\|R\|^2 + (r_3 - x_3)^2)^{3/2}} \, dx_3$$

$$= \lambda R \int_{x_3=-L}^{L} \frac{dx_3}{\left(\|R\|^2 + (r_3 - x_3)^2\right)^{3/2}} \, dx_3 \; +$$

$$+ \lambda e_3 \int_{x_3=-L}^{L} \frac{(r_3 - x_3)}{\left(\|R\|^2 + (r_3 - x_3)^2\right)^{3/2}} \, dx_3$$

$$z := \frac{x_3 - r_3}{\|R\|}$$

$$\int_{x_3=-L}^{L} \frac{1}{\left(\|R\|^2 + (r_3 - x_3)^2\right)^{3/2}} \, dx_3 =$$

$$= \int_{z=\frac{-L-r_3}{\|R\|}}^{\frac{+L-r_3}{\|R\|}} \frac{1}{\left(\|R\|^2 + \|R\|^2 z^2\right)^{3/2}} \, \|R\| dz$$

$$= \frac{1}{\|R\|^2} \int_{z=\frac{-L-r_3}{\|R\|}}^{\frac{+L-r_3}{\|R\|}} \frac{1}{(1+z^2)^{3/2}} \, dz$$

$$\left(\frac{z}{\sqrt{1+z^2}} \right)' = \frac{1}{(1+z^2)^{3/2}}$$

$$\Longrightarrow \int_{z=\frac{-L-r_3}{\|R\|}}^{\frac{+L-r_3}{\|R\|}} \frac{1}{(1+z^2)^{3/2}}\, dz =$$

$$= \left. \frac{z}{\sqrt{1+z^2}} \right|_{z=\frac{-L-r_3}{\|R\|}}^{z=\frac{+L-r_3}{\|R\|}}$$

$$= \frac{(L-r_3)/\|R\|}{\sqrt{1+(L-r_3)^2/\|R\|^2}} - \frac{(-L-r_3)/\|R\|}{\sqrt{1+(-L-r_3)^2/\|R\|^2}}$$

$$= \frac{L-r_3}{\sqrt{\|R\|^2+(L-r_3)^2}} - \frac{-L-r_3}{\sqrt{\|R\|^2+(L+r_3)^2}}$$

$$= \frac{L-r_3}{\|r-Le_3\|} + \frac{L+r_3}{\|r+Le_3\|}$$

$$\int_{x_3=-L}^{L} \frac{(r_3-x_3)}{(\|R\|^2+(r_3-x_3)^2)^{3/2}}\, dx_3 = \qquad z := \frac{x_3-r_3}{\|R\|}$$

$$= -\frac{1}{\|R\|^2}\int_{z=\frac{-L-r_3}{\|R\|}}^{\frac{+L-r_3}{\|R\|}} \frac{z}{(1+z^2)^{3/2}}\,dz$$

$$\left(\frac{-1}{\sqrt{1+z^2}}\right)' = \frac{z}{(1+z^2)^{3/2}}$$

$$\Rightarrow \int_{z=\frac{-L-r_3}{\|R\|}}^{\frac{+L-r_3}{\|R\|}} \frac{z}{(1+z^2)^{3/2}}\,dz =$$

$$= -\frac{1}{\sqrt{1+z^2}}\Bigg|_{z=\frac{-L-r_3}{\|R\|}}^{z=\frac{+L-r_3}{\|R\|}}$$

$$= \frac{1}{\sqrt{1+(L+r_3)^2/\|R\|^2}} - \frac{1}{\sqrt{1+(L-r_3)^2/\|R\|^2}}$$

$$= \frac{\|R\|}{\|r+Le_3\|} - \frac{\|R\|}{\|r-Le_3\|}$$

All together we find:

$$E(r) = \lambda R \int_{x_3=-L}^{L} \frac{dx_3}{(\|R\|^2+(r_3-x_3)^2)^{3/2}}\, dx_3 \; +$$

$$+ \lambda \int_{x_3=-L}^{L} \frac{(r_3-x_3)}{(\|R\|^2+(r_3-x_3)^2)^{3/2}}\, dx_3$$

$$= \frac{\lambda R}{\|R\|^2}\left[\frac{L-r_3}{\|r-Le_3\|} + \frac{L+r_3}{\|r+Le_3\|}\right] +$$

$$+ \lambda e_3 \frac{1}{\|R\|}\left[\frac{1}{\|r-Le_3\|} - \frac{1}{\|r+Le_3\|}\right].$$

This reduces back to the infinite case if we take the limit $L \to \infty$:

$$\lim_{L\to\infty} E(r) = \frac{2\lambda R}{\|R\|^2}. \quad \checkmark$$

Another limit: $r_3 = 0$

$$\lim_{r_3\to 0} E(r) = \frac{2\lambda R}{\|R\|^2}.$$

Another limit: $\frac{L}{|r_j|} \ll 1$ for $j=1,2,3$.

Then
$$\frac{L \mp r_3}{\| Le_3 \mp r\|} = \frac{r_3\left(\frac{L}{r_3} \mp 1\right)}{\|r\| \, \left\| \frac{L}{\|r\|} \mp \frac{r}{\|r\|} \right\|}$$

$$\| \hat{r} \mp L/\|r\| \|^2 = 1 + \frac{L^2}{\|r\|^2} \mp \langle \hat{r}, \frac{L}{\|r\|} \rangle$$

$$\frac{1}{\|Le_3 \mp r\|} \approx \frac{1}{\|r\|}\left(1 \pm \frac{1}{2}\langle \hat{r}, \frac{L}{\|r\|}\rangle + \dots\right)$$

$$\frac{L \mp r_3}{\|Le_3 \mp r\|} \approx \frac{r_3}{\|r\|}\left(\frac{L}{r_3} \mp 1\right)\left(1 \pm \langle \hat{r}, \frac{L}{\|r\|}\rangle\right) + \dots$$

$$= \frac{r_3}{\|r\|}\left(\frac{L}{r_3} \mp 1 - \langle \hat{r}, \frac{L}{\|r\|}\rangle\right)$$

$$\frac{L - r_3}{\|Le_3 - r\|} - \frac{L + r_3}{\|Le_3 + r\|} \approx 2\frac{r_3}{\|r\|}$$

Similarly,
$$\frac{1}{\|Le_3 - r\|} - \frac{1}{\|Le_3 + r\|} \approx \frac{1}{\|r\|}\left(1 + \frac{1}{2}\langle \hat{r}, \frac{L}{\|r\|}\rangle\right) - \frac{1}{\|r\|}\left(1 - \frac{1}{2}\langle \hat{r}, \frac{L}{\|r\|}\rangle\right)$$
$$= \frac{\langle \hat{r}, L/\|r\| \rangle}{\|r\|}$$

$$\Longrightarrow \quad E(r) \overset{\frac{L}{\|r_0\|} \ll 1}{\approx} \frac{\lambda R}{\|R\|^2}\, 2\, \frac{r_3}{\|r\|} + \lambda e_3 \frac{1}{\|R\|} \langle r$$

Case 2: $\quad r = r_3 e_3, \quad |r_3| > L.$

$$E(r_3 e_3) = \lambda \int_{x_3=-L}^{L} dx_3 \; \frac{(r_3 - x_3) e_3 \, dx_3}{|r_3 - x_3|^3}$$

$$z := x_3 - r_3 \quad \overset{?}{=} -\lambda e_3 \int_{z=-L-r_3}^{L-r_3} \frac{z}{|z|^3} \, dz$$

$r_3 > L$ or $r_3 < -L \iff$

$L - r_3 < 0$ or $-L - r_3 > 0 \implies$

$z < 0$ $\qquad$ $z > 0$

Case 2.1: $z < 0$

$$E(r_3 e_3) = \lambda e_3 \int_{z=-L-r_3}^{L-r_3} \frac{1}{z^2} \, dz$$

$$= \lambda e_3 \; -\frac{1}{z} \Big|_{z=-L-r_3}^{L-r_3}$$

$$= \lambda e_3 \left(\frac{1}{-L-r_3} - \frac{1}{L-r_3} \right)$$

$$= \lambda e_3 \left(\frac{1}{r_3 - L} - \frac{1}{r_3 + L} \right)$$

$$= \lambda e_3 \frac{2L}{r_3^2 - L^2}$$

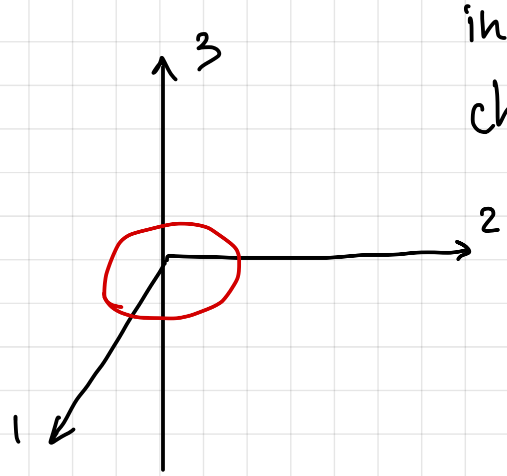
$$= \frac{2L\lambda e_3}{r_3^{\;2}} \frac{1}{1-(L/r_3)^2} \underset{\frac{L}{r_3} \ll 1.}{\approx} \frac{2L\lambda}{r_3^{\;2}} e_3 .$$

Makes sense that very far away from wire it will behave like Coulomb law!

$2\lambda L \equiv$ total charge of wire

$\frac{1}{r_3^{\;2}} \equiv$ inverse square law

Example: Hoop of radius $R>0$ in the 1-2 plane of charge $Q \in \mathbb{R}$.



$\Longrightarrow$ Density is

$$\rho(x) = \delta(x_3)\, \delta(\|x_1 e_1 + x_2 e_2\| - R) \frac{Q}{2\pi R}$$

Change to cylindrical coordinates:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} r\cos(\varphi) \\ r\sin(\varphi) \\ z \end{bmatrix}$$

$$\rho(r,\varphi,z) = \delta(z)\, \delta(r-R) \frac{Q}{2\pi R}$$

Electric field:

$$E(r',\varphi',z') = \int_{r=0}^{\infty} \int_{\varphi=0}^{2\pi} \int_{z=-\infty}^{\infty} r\,dr\,dz\,d\varphi\, \rho(r,\varphi,z) \frac{\begin{bmatrix} r'\cos(\varphi') \\ r'\sin(\varphi') \\ z' \end{bmatrix} - \begin{bmatrix} r\cos(\varphi) \\ r\sin(\varphi) \\ z \end{bmatrix}}{\left\| \begin{bmatrix} r'\cos(\varphi') \\ r'\sin(\varphi') \\ z' \end{bmatrix} - \begin{bmatrix} r\cos(\varphi) \\ r\sin(\varphi) \\ z \end{bmatrix} \right\|^3}$$

$$= \frac{Q}{2\pi R} R \int_{\varphi=0}^{2\pi} d\varphi \frac{\begin{bmatrix} r'\cos(\varphi') \\ r'\sin(\varphi') \\ z' \end{bmatrix} - \begin{bmatrix} R\cos(\varphi) \\ R\sin(\varphi) \\ 0 \end{bmatrix}}{\left\| \begin{bmatrix} r'\cos(\varphi') \\ r'\sin(\varphi') \\ z' \end{bmatrix} - \begin{bmatrix} R\cos(\varphi) \\ R\sin(\varphi) \\ 0 \end{bmatrix} \right\|^3}$$

How to calculate distances in cylindrical coordinates?

$$\left\| \begin{bmatrix} r'\cos(\varphi') \\ r'\sin(\varphi') \\ z' \end{bmatrix} - \begin{bmatrix} R\cos(\varphi) \\ R\sin(\varphi) \\ 0 \end{bmatrix} \right\|^2 = (z')^2 + R^2 + r'^2 - 2r'R\cos(\varphi - \varphi')$$

$$\begin{bmatrix} r'\cos(\varphi') \\ r'\sin(\varphi') \\ z' \end{bmatrix} - \begin{bmatrix} R\cos(\varphi) \\ R\sin(\varphi) \\ 0 \end{bmatrix} = z' e_3 + \left[r'\cos(\varphi') - R\cos(\varphi) \right] e_1 + $$
$$+ \left[r'\sin(\varphi') - R\sin(\varphi) \right] e_2$$

$$\Rightarrow E(r', \varphi', z') = \frac{Q}{2\pi} z e_3 \int_{\varphi=0}^{2\pi} \frac{d\varphi}{\left[(z')^2 + R^2 + r'^2 - 2r'R\cos(\varphi - \varphi') \right]^{3/2}} +$$

$$+ \frac{Q}{2\pi} (r'\cos(\varphi') e_1 + r'\sin(\varphi') e_2) \int_{\varphi=0}^{2\pi} \frac{d\varphi}{\left[(z')^2 + R^2 + r'^2 - 2r'R\cos(\varphi - \varphi') \right]^{3/2}}$$

$$- \frac{Q}{2\pi} R \int_{\varphi=0}^{2\pi} \frac{\left[\cos(\varphi) e_1 + \sin(\varphi) e_2 \right] d\varphi}{\left[(z')^2 + R^2 + r'^2 - 2r'R\cos(\varphi - \varphi') \right]^{3/2}}$$

$$=: \frac{Q}{2\pi} \left(z e_3 + r'\cos(\varphi') e_1 + r'\sin(\varphi') e_2 \right) I_1 - \frac{QR}{2\pi} I_2$$

$$I_1 \equiv \int_{\varphi=0}^{2\pi} \frac{d\varphi}{(\alpha^2 - \beta^2 \cos(\varphi - \varphi'))^{3/2}}$$

$\alpha^2 = (z')^2 + R^2 + r'^2$ 20

$\beta^2 = 2 r' R$

transl. invar.

$$\stackrel{?}{=} \int_{\varphi=0}^{2\pi} \frac{d\varphi}{(\alpha^2 - \beta^2 \cos(\varphi))^{3/2}}$$

$=$ Some terrible elliptic functions

$$I_3 \equiv \int_{\varphi=0}^{2\pi} \frac{\cos(\varphi)}{(\alpha^2 - \beta^2 \cos(\varphi - \varphi'))^{3/2}} d\varphi$$

$=$ not any better ...

Special case: $r' = 0$

$$\Longrightarrow \beta = 0, \quad \alpha^2 = (z')^2 + R^2$$

$$I_1 = \frac{2\pi}{\alpha^3} = \frac{2\pi}{((z')^2 + R^2)^{3/2}}$$

$I_2 = 0$ due to symmetry.

$$\Longrightarrow \boxed{E(r'=0, z') = \frac{Q z e_3}{((z')^2 + R^2)^{3/2}}} .$$

<u>Example</u>: Scatterred finite # of point charges:

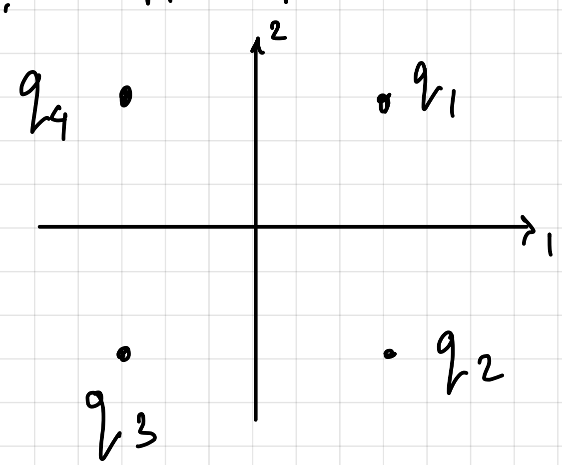
$$\rho(x) = \sum_{j=1}^{N} q_j \, \delta(x - r_j)$$

$$\Rightarrow E(r) = \int_{x \in \mathbb{R}^3} \frac{\rho(x)}{\|r - x\|^3} (r - x) \, dx$$

$$= \sum_{j=1}^{N} q_j \frac{r - r_j}{\|r - r_j\|^3}$$

as expected.

Special case: $N = 4$



$$q_1 = -q, \; q_2 = -2q, \; q_3 = 2q, \; q_4 = q$$

$$r_1 = \begin{bmatrix} a \\ a \end{bmatrix}, \; r_2 = \begin{bmatrix} a \\ -a \end{bmatrix}, \; r_3 = \begin{bmatrix} -a \\ -a \end{bmatrix}, \; r_4 = \begin{bmatrix} -a \\ a \end{bmatrix}$$

yields:

$$E(r) = -q\,\frac{r-\begin{bmatrix}a\\a\end{bmatrix}}{\left\|r-\begin{bmatrix}a\\a\end{bmatrix}\right\|^3} - 2q\,\frac{r-\begin{bmatrix}a\\-a\end{bmatrix}}{\left\|r-\begin{bmatrix}a\\-a\end{bmatrix}\right\|^3}$$

$$+2q\,\frac{r-\begin{bmatrix}-a\\-a\end{bmatrix}}{\left\|r-\begin{bmatrix}-a\\-a\end{bmatrix}\right\|^3} + q\,\frac{r-\begin{bmatrix}-a\\a\end{bmatrix}}{\left\|r-\begin{bmatrix}-a\\a\end{bmatrix}\right\|^3}$$

$$\left\|\begin{bmatrix}\pm a\\\pm a\end{bmatrix}\right\|^2 = a^2 + a^2 = 2a^2$$

$$\rightsquigarrow E(0) = \frac{qa}{(\sqrt{2}a)^3}\underbrace{\left(+\begin{bmatrix}1\\1\end{bmatrix} + 2\begin{bmatrix}1\\-1\end{bmatrix} - 2\begin{bmatrix}-1\\-1\end{bmatrix} - \begin{bmatrix}-1\\1\end{bmatrix}\right)}_{\begin{bmatrix}1+2+2+1\\1-2+2-1\end{bmatrix} = \begin{bmatrix}6\\0\end{bmatrix}}$$

$$\boxed{E(0) = \frac{6q}{2^{3/2}a^2}}$$