Q1 Constraining Forces

Let $V: \mathbb{R}^3 \to \mathbb{R}$ be given.
Let a 2D surface in $\mathbb{R}^3$ be given via a map $f: \mathbb{R}^2 \to \mathbb{R}^3$.

Example: ① The surface is the 1-2 plane:

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} \in \mathbb{R}^3 .$$

② The surface is locally S^2:

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} \sin(x_2)\cos(x_1) \\ \sin(x_2)\sin(x_1) \\ \cos(x_2) \end{bmatrix} \in S^2 \subseteq \mathbb{R}^3$$

Note f does not depend on time.
We denote by $n: \mathrm{im}(f) \to S^2$ the normal unit vector to the surface at each point on it.
A particle moves in $\mathbb{R}^3$ under the influence of a potential given by V and constrained to move along $\mathrm{im}(f)$.
We denote the particle's trajectory on the surface's parameter space ($\mathbb{R}^2$) by $\gamma: \underset{\text{time}}{\mathbb{R}} \to \underset{\text{parameter space}}{\mathbb{R}^2}$

so that the actual trajectory is given by $\underbrace{(f\circ\gamma)}_{=:\Gamma}: \underset{\text{time}}{\mathbb{R}} \to \mathrm{im}(f) \subseteq \underset{\text{space}}{\mathbb{R}^3}$.

We denote by Z the magnitude of the ideal constraint force in the direction of n acting on the particle as a result of the movement being constrained to $\mathrm{im}(f)$. m is the mass

Cl.: $$Z = (n\circ\Gamma)\cdot\left(m\sum_{\alpha,\beta=1}^{2}\left[(\partial_\alpha\partial_\beta f)\circ\gamma\right](\dot\gamma)_\alpha(\dot\gamma)_\beta + (\nabla V)\circ\Gamma\right)$$

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Pf.: Note $\dot{\Gamma} = (f\circ\gamma)^{\cdot}$

$$= \sum_{\alpha=1}^{2} [(\partial_\alpha f)\circ\gamma](\dot{\gamma})_\alpha$$ Chain rule

$$\ddot{\Gamma} = \sum_{\alpha=1}^{2}\left\{[(\partial_\alpha f)\circ\gamma](\ddot{\gamma})_\alpha + \sum_{\beta=1}^{2}[(\partial_\beta\partial_\alpha f)\circ\gamma](\dot{\gamma})_\beta(\dot{\gamma})_\alpha\right\}$$

Note $\partial_\alpha f$ is tangent to the surface, so that

$n\circ(\partial_\alpha f) = 0$ by definition of n.

We start with Newton's 2^{nd} law:

$$m\ddot{\Gamma} = \underbrace{Z(n\circ\Gamma)}_{} - (\nabla V)\circ\Gamma$$

note $\nexists$ constraining force in direction tangent to surface (by def.)

$$\Rightarrow Z(n\circ\Gamma) = m\ddot{\Gamma} + (\nabla V)\circ\Gamma$$

Take an inner product of this eqn w/ $(n\circ\Gamma)$ to get:

$$Z_1 = m(n\circ\Gamma)\ddot{\Gamma} + (n\circ\Gamma)((\nabla V)\circ\Gamma)$$

$$= m\sum_{\alpha,\beta=1}^{2}[(\partial_\alpha\partial_\beta f)\circ\gamma]\cdot(n\circ\Gamma)(\dot{\gamma})_\alpha(\dot{\gamma})_\beta +$$

$$+(n\cdot\nabla V)\circ\Gamma$$

Q2 Illustration of Hamilton's Principle:

Consider a 1D harmonic oscillator of mass 1 and spring const. ω^2.

We denote the trajectory by $\gamma : \underset{\text{time}}{\mathbb{R}} \to \underset{\text{space}}{\mathbb{R}}$ and hence the E.o.M. is $\boxed{\ddot{\gamma} = -\omega^2\gamma}$.

Its general solution is

$$\boxed{\gamma(t) = A\cos(\omega t) + B\sin(\omega t)} \qquad \forall\, t \in \mathbb{R},$$

for some $(A,B) \in \mathbb{R}^2$ (determined by initial cond.).

(a) A family of orbits (parametrized by $(\lambda_1,\lambda_2) \in \mathbb{R}^2$) is given by

$$\boxed{\alpha_{\lambda_1,\lambda_2}(t) = \lambda_1 \sin(W t) + \lambda_2 \sin(2Wt)}$$

for some (fixed) $W \in \mathbb{R}_{>0}$.

Recall a "mechanical orbit" is a solution to the E.o.M.

Cl.: $\alpha_{\lambda_1,\lambda_2}$ is a mechanical orbit if either one of the following holds:

① $W \neq \omega$, $\lambda_1 = \lambda_2 = 0$

② $W = \omega$, $\lambda_2 = 0$, λ_1 arbitrary

③ $W = \frac{1}{2}\omega$, $\lambda_1 = 0$, λ_2 arbitrary

Pf.: We plug in $\alpha_{\lambda_1,\lambda_2}$ into the E.o.M. to get:

$$\dot{\alpha}_{\lambda_1,\lambda_2}(t) = W\lambda_1\cos(Wt) + 2W\lambda_2\cos(2Wt)$$

$$\ddot{\alpha}_{\lambda_1,\lambda_2}(t) = -W^2\lambda_1\sin(Wt) - 4W^2\lambda_2\sin(2Wt)$$

$$\Rightarrow \quad -W^2\lambda_1\sin(Wt) - 4W^2\lambda_2\sin(2Wt) \overset{!}{=} -\omega^2\lambda_1\sin(Wt) - \omega^2\lambda_2\sin(2Wt)$$

Since $\sin(W\cdot)$ and $\sin(2W\cdot)$ are linearly independent, we find: $W = \omega$ and $W = \frac{1}{2}\omega$ if λ_1,λ_2 are arbitrary. However, when either one is zero we don't have to employ the corresponding equation.

(b) Cl.: For the values of λ_1,λ_2 for which $\alpha_{\lambda_1,\lambda_2}$ is a "mechanical orbit", it is an extremal point of the action S.
In particular, we have: $\omega \leq W$: a minimum

$\frac{1}{2}\omega < W < \omega$: a saddle point

$W \leq \frac{1}{2}\omega$: a maximum

Pf.: Recall $$S[\alpha_{\lambda_1,\lambda_2}] \equiv \int_0^{\frac{2\pi}{W}} L[\alpha_{\lambda_1,\lambda_2}]$$

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for the harmonic oscillator we have

$$L[\gamma] \equiv T - V$$
$$= \tfrac{1}{2}m\dot{\gamma}^2 - m\omega^2\gamma^2 \quad \Big\} \ m \equiv 1$$
$$= \tfrac{1}{2}(\dot{\gamma}^2 - \omega^2\gamma^2)$$

Also note $\alpha_{\lambda_1,\lambda_2}(0) = \alpha_{\lambda_1,\lambda_2}(\frac{2\pi}{W}) = 0$.

Hence

$$S[\alpha_{\lambda_1,\lambda_2}] = \int_0^{\frac{2\pi}{W}} \frac{1}{2}\left(\dot{\alpha}_{\lambda_1,\lambda_2}{}^2 - \omega^2 \alpha_{\lambda_1,\lambda_2}{}^2\right)$$

$$= \int_0^{\frac{2\pi}{W}} \frac{1}{2}\Big[\big(W\lambda_1\cos(Wt) + 2W\lambda_2\cos(2Wt)\big)^2 -$$
$$- \omega^2\big(\lambda_1\sin(Wt) + \lambda_2\sin(2Wt)\big)^2\Big]\, dt$$

$$= \int_0^{\frac{2\pi}{W}} \frac{1}{2}\Big[W^2\lambda_1^2\cos(Wt)^2 + 4W^2\lambda_1\lambda_2\cos(Wt)\cos(2Wt) + 4W^2\lambda_2^2\cos(2Wt)^2 -$$
$$-\omega^2\lambda_1^2\sin(Wt)^2 - \omega^2\lambda_2^2\sin(2Wt)^2 - 2\omega^2\lambda_1\lambda_2\sin(Wt)\sin(2Wt)\Big]\, dt$$

Next note:

$$\int_0^{\frac{2\pi}{W}} \cos(Wt)^2\, dt \overset{\alpha := Wt}{=} \int_0^{2\pi} \cos(\alpha)^2 \frac{d\alpha}{W} = \frac{\pi}{W}$$

$$\int_0^{\frac{2\pi}{W}} \cos(2Wt)^2\, dt \overset{\alpha = 2Wt}{=} \int_0^{4\pi} \cos(\alpha)^2 \frac{d\alpha}{2W} = \frac{2\pi}{2W}$$

$$\int_0^{\frac{2\pi}{W}} \cos(Wt)\cos(2Wt)\, dt = \int_0^{2\pi} \cos(\alpha)\cos(2\alpha)\frac{d\alpha}{W} = 0$$

Exactly the same eqns hold if we replace cos w/ sin.

$$\Rightarrow \quad S[\alpha_{\lambda_1,\lambda_2}] = \frac{1}{2}\Big[W^2\lambda_1^2\frac{\pi}{W} + 4W^2\lambda_2^2\frac{\pi}{W} - \omega^2\lambda_1^2\frac{\pi}{W} - \omega^2\lambda_2^2\frac{\pi}{W}\Big]$$

$$= \frac{\pi}{2W}\Big[\lambda_1^2(W^2 - \omega^2) + \lambda_2^2(4W^2 - \omega^2)\Big]$$

Hence the extremal points are precisely what we found in part (a):

$$DS[\alpha_{\lambda_1,\lambda_2}] = \frac{\pi}{2W}\Big[2\lambda_1(W^2-\omega^2) \quad 2\lambda_2(4W^2-\omega^2)\Big]$$

For an extremum point, $DS[\alpha_{\lambda_1,\lambda_2}]=0$

$$\Leftrightarrow \begin{cases} 2\lambda_1(W^2-\omega^2)=0 \\ 2\lambda_2(4W^2-\omega^2)=0 \end{cases}$$

$\Longleftrightarrow$ same condition as in (a).

$\Longleftrightarrow$ The extremum points are exactly the solutions of the E.o.M. ✓ (Hamilton's principle works ✓)

To find the type of extremum, compute Hessian matrix!
(or by "inspection")

$$HS[\alpha_{\lambda_1,\lambda_2}] = \frac{\pi}{2W}\begin{bmatrix} 2(W^2-\omega^2) & 0 \\ 0 & 2(4W^2-\omega^2) \end{bmatrix}$$

Hessian is: ⊛ pos. def. iff $W \geq \omega$ $\Longleftrightarrow$ minimum

⊛ neg. def. iff $W \leq \frac{1}{2}\omega$ $\Longleftrightarrow$ maximum

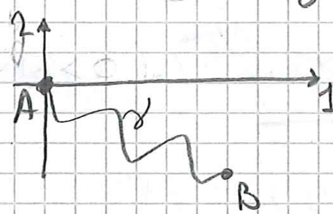
⊛ indef. iff $\frac{1}{2}\omega < W < \omega$ $\Longleftrightarrow$ saddle pt.

Provided (λ_1,λ_2) is at the extremum (as stipulated for (a)).

Q3 The Brachistochrone Curve

Let two points $(A,B)\in\mathbb{R}^2$ be given, s.t. $\begin{cases} (B)_1 > (A)_1 \\ (B)_2 < (A)_2 \end{cases}$.

WLOG $A=0$.



Question: Find a map $\gamma:\mathbb{R}\to\mathbb{R}^2$ s.t.:

① $\gamma(0)=A$, $\gamma(T)=B$ for some $T>0$.

② If a particle of mass m moves along γ under the influence of gravity alone, then T is minimal, where the minimum is over all such γ.

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We start with writing the curve $\underset{\text{time}}{\gamma : \mathbb{R}} \to \underset{\text{space}}{\mathbb{R}^2}$ implicitly instead; (we are only interested in the shape)

Hence define $f : \underset{\text{1-axis}}{\mathbb{R}} \to \underset{\text{2-axis}}{\mathbb{R}}$ s.t. $((\gamma(t))_1, (\gamma(t))_2) \overset{!}{=} (x(t), f(x(t)))$

$\forall\ t \in \mathbb{R}$, for any map $x : \underset{\text{time}}{\mathbb{R}} \to \underset{\text{1-axis}}{\mathbb{R}}$.

We'll find f instead of γ.

The particle's energy is: $E = \frac{1}{2} m \|v\|^2 + mgf$ where v is its instantaneous velocity.

By assumption $\|v\| = 0$ @ A $\Rightarrow E = 0$ @ A.

But we have conservation of energy $\Rightarrow \|v\| = \sqrt{-2gf}$ at all times (note $f \leq 0$ so $\|v\| \in \mathbb{R}$ ✓).

Also note the arclength along the path is $ds = \sqrt{1 + (f')^2}\, dx$.

Hence $T[f] = \int_A^B \frac{ds}{\|v\|} = \int_0^{(B)_1} \frac{\sqrt{1+(f')^2}}{\sqrt{-2gf}} =$

$$= \frac{1}{\sqrt{2g}} \int_0^{(B)_1} \sqrt{-\frac{1+(f')^2}{f}}$$

Since we want to extremize T, we use the Euler-Lagrange equation to find the extremal f:

$$L[f] \equiv \sqrt{-\frac{1+(f')^2}{f}}$$

$$\frac{\partial L}{\partial f'} = \cancel{\frac{1}{2}} \left(-\frac{1+(f')^2}{f}\right)^{-1/2} \left(-\frac{1}{f}\right) \cancel{2} f' = -\frac{f'}{\sqrt{-f\,[1+(f')^2]}}$$

$$\left(\frac{\partial L}{\partial f'}\right)' = -\frac{(f')^2 + (f')^4 - 2ff''}{2f^3 \left(-\frac{1+(f')^2}{f}\right)^{3/2}} \qquad \frac{\partial L}{\partial f} = \sqrt{1+(f')^2}\ \frac{1}{2} (-f)^{-3/2}$$

$$\Rightarrow -\frac{(f')^2 + (f')^4 - 2ff''}{\cancel{2} f^3 \left(-\frac{1+(f')^2}{f}\right)^{3/2}} + \cancel{\frac{1}{2}} \sqrt{1+(f')^2}\ f^{-3/2} = 0$$

$$\Rightarrow +(f')^2 + (f')^4 - 2ff'' - (1+(f')^2)^2 = 0$$

$$\Rightarrow \boxed{2ff'' + (1 + (f')^2) = 0} \Rightarrow \text{Cycloid}$$