Analytical Mechanics — HW#11 — 6/12/2016

Q1 Legendre Transformations

Let $f: I \to \mathbb{R}$ be a continuously differentiable strictly convex (or strictly concave) function where $I \subseteq \mathbb{R}$ is an interval.
Its Legendre transform $\mathcal{L}f$ is yet another map defined as

$$y \xmapsto{\mathcal{L}f} y\,(f')^{-1}(y) - f((f')^{-1}(y))$$

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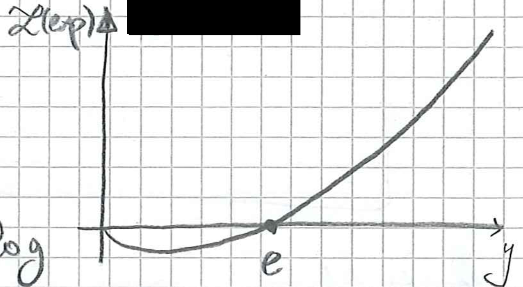

(a) **Claim:** $(\mathcal{L}(\exp))(y) = y \log\left(\frac{y}{e}\right)$

Proof: $\exp' = \exp$ and $\exp^{-1} = \log$

Hence we find

$$\begin{aligned}(\mathcal{L}(\exp))(y) &= y\log(y) - \exp(\log(y)) \\ &= y\log(y) - y \\ &= y\,[\log(y) - 1] \\ &= y\,[\log(y) - \log(e)] \\ &= y \log\left(\frac{y}{e}\right)\end{aligned}$$

Note $\exp$ is indeed strictly convex (see notes).

Claim: $f(x) := -\frac{1}{2}(x-a)^2 + \frac{a^2}{4}$ $\forall x \in I$ has

$$\mathcal{L}f = f$$

Proof: First note that f is smooth as a polynomial.
We verify it is strictly concave:
For f is twice differentiable, it is strictly concave iff its second derivative is (strictly) negative:
$f'(x) = -(x-a)$
$f''(x) = -1 < 0$

2 $(f')^{-1}$ is defined s.t.

$$(f')^{-1}(f'(x)) = x \quad \wedge \quad \underbrace{f'((f')^{-1}(y))} = y$$

$$-((f')^{-1}(y)-a) = y$$

verify

$$-(-(x-\cancel{a})-\cancel{a}) = x \quad ✓$$

$$\Rightarrow \quad (f')^{-1}(y) = -(y-a)$$

Hence $(\mathcal{L}f)(y) = y[-(y-a)] - \left\{-\frac{1}{2}\underbrace{(-(y-\cancel{a})-\cancel{a})^2}_{y^2} + \frac{a^2}{4}\right\}$

$$= -y^2 + ay + \frac{1}{2}y^2 - \frac{a^2}{4}$$

$$= -\frac{1}{2}(y-a)^2 + \frac{a^2}{4}$$

$$= f(y)$$

(b)(i) Claim: $\mathcal{L}f$ is also continuously differentiable.

Proof: $(\mathcal{L}f)'(y) = (f')^{-1}(y) + \cancel{y((f')^{-1})'(y)} - \cancel{\underbrace{f'((f')^{-1}(y))}_{y}((f')^{-1})'(y)}$

$$= (f')^{-1}(y)$$

Assuming $(f')^{-1}$ is differentiable.

It is indeed differentiable: f' is strictly monotone increasing

$\Rightarrow (f')^{-1}$ is also strictly monotone increasing, and hence differentiable.

Claim: $\mathcal{L}f$ is also strictly convex.

Proof: Want $(\mathcal{L}f)'' > 0$.

$(\mathcal{L}f)'' = ((f')^{-1})'$

But we know $(f')^{-1}$ is strictly monotone increasing.

(ii) Claim: $\mathcal{L}^2 = \mathbb{1}$ (involution on space of functions)

Proof: $(\mathcal{L}^2(f))(x) = x(((\mathcal{L}f)')^{-1})(x) - (\mathcal{L}f)((((\mathcal{L}f)')^{-1})(x))$

We have found $(\mathcal{L}f)' = (f')^{-1} \Rightarrow \boxed{((\mathcal{L}f)')^{-1} = f'}$

$$= x f'(x) - (\mathcal{L}f)(f'(x))$$

$$= x f'(x) - f'(x)((f')^{-1})(f'(x)) + f((f')^{-1}(f'(x)))$$

$$= \cancel{xf'(x)} - \cancel{f'(x)x} + f(x)$$
$$= f(x)$$

Q2 Magnetic Symplectic Structure

A charged particle of charge $q \in \mathbb{R}$ and mass $m \in \mathbb{R}_{>0}$ moves in $\mathbb{R}^3$ under the influence of an electric field $E : \mathbb{R}^3 \to \mathbb{R}^3$ and a magnetic field $B : \mathbb{R}^3 \to \mathbb{R}^3$.

We know the Lorentz force is given by

$$q\left[E \circ \gamma + \frac{1}{c}\dot{\gamma} \times (B \circ \gamma)\right]$$

where $\gamma : \mathbb{R} \to \mathbb{R}^3$ is the trajectory of the particle.

Hence by Newton's 2nd law

$$m\ddot{\gamma} = q\left[E \circ \gamma + \frac{1}{c}\dot{\gamma} \times (B \circ \gamma)\right] \qquad (1)$$

We want to define a Hamiltonian whose corresponding canonical EoM will be equivalent to (1).

Consider $H : \mathbb{R}^6 \to \mathbb{R}$ given by (minimal substitution)

$$H(x,p) := \frac{1}{2m}\left(p - \frac{q}{c}A(x)\right)^2 + q\varphi(x) \qquad \forall\, (x,p) \in (\mathbb{R}^3)^2$$

where A is the magnetic vector potential defined with the rel.

$$B \equiv \nabla \times A$$

φ is the electric potential, defined with the rel.

$$E \equiv -\nabla\varphi$$

Cl.: The canonical eqns of motion corresponding to H are equivalent to (1).

Pf.: The Lagrangian corresponding to H is its Legendre transf. w.r.t. the 2nd argument $p \in \mathbb{R}^3$:

$$(\dot\gamma)_i \underset{\substack{\uparrow \\ \text{eq-n (7.4)}}}{=} (\partial_{i+3} H)(\gamma, \pi)$$

where $\pi : \mathbb{R} \to \mathbb{R}^3$ is the trajectory in the space of canonical momentum.

$$(\partial_{i+3} H)(x,p) = \frac{1}{2m} 2 \left(p - \frac{q}{c} A(x) \right)_i = \frac{p_i}{m} - \frac{q}{mc} A_i(x) \qquad \forall\ i \in \{1,2,3\},\ (x,p) \in (\mathbb{R}^3)^2$$

We find the canonical momentum π:

$$\boxed{\pi = m\dot\gamma + \frac{q}{c} A \circ \gamma}$$

The second eq-n of motion in (7.4) gives

$$\boxed{(\dot\pi)_i = -(\partial_i H)(\gamma, \pi)} \qquad \forall\ i \in \{1,2,3\}$$

$$(\partial_i H)(x,p) = \frac{1}{m}\left(p - \frac{q}{c} A(x)\right)_j \left(-\frac{q}{c}\right)(\partial_i A_j)(x) + q(\partial_i \varphi)(x)$$

$$= -\frac{q}{mc} p_j (\partial_i A_j)(x) + \frac{q^2}{mc^2} A_j(x) (\partial_i A_j)(x) + q(\partial_i \varphi)(x)$$

$$\left.\begin{aligned} (\dot\pi)_i &= m(\ddot\gamma)_i + \frac{q}{c}(\dot{A \circ \gamma})_i \\ &= m(\ddot\gamma)_i + \frac{q}{c}\left((\partial_j A_i) \circ \gamma\right) \dot\gamma_j \end{aligned}\right\} \text{chain rule}$$

Hence

$$m(\ddot\gamma)_i = -\frac{q}{c}\left((\partial_j A_i)\circ\gamma\right)(\dot\gamma)_j - (\partial_i H)\left(\gamma, m\dot\gamma + \frac{q}{c} A\circ\gamma\right)$$

$$\overset{E = -\nabla\varphi}{=} -\frac{q}{c}\left((\partial_j A_i)\circ\gamma\right)(\dot\gamma)_j + \frac{q}{mc}\left(m(\dot\gamma)_j + \cancel{\frac{q}{c} A_j \circ \gamma}\right)\left((\partial_i A_j)\circ\gamma\right)$$

$$- \cancel{\frac{q^2}{mc^2}(A_j\circ\gamma)\left((\partial_i A_j)\circ\gamma\right)} + q E_i \circ \gamma$$

$$= q\Big[\underbrace{-\frac{1}{c}\left((\partial_j A_i)\circ\gamma\right)(\dot\gamma)_j + \frac{1}{c}(\dot\gamma)_j\left((\partial_i A_j)\circ\gamma\right)}_{\frac{1}{c}(\dot\gamma)_j (\partial_i A_j - \partial_j A_i)\circ\gamma} + E_i\circ\gamma\Big]$$

Note that

$$(\dot{\gamma} \times B)_i \equiv \varepsilon_{ijk} (\dot{\gamma})_j B_k$$

$$\equiv \varepsilon_{ijk} (\dot{\gamma})_j \varepsilon_{klm} \partial_l A_m$$

$$= \underbrace{\varepsilon_{ijk} \varepsilon_{klm}}_{\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}} (\dot{\gamma})_j \partial_l A_m$$

$$= (\dot{\gamma})_j (\partial_i A_j - \partial_j A_i)$$

We thus recover the EoM (1).

Hence we find that if we define the trajectory in phase space $\Gamma : \underbrace{\mathbb{R}}_{\text{time}} \longrightarrow \underbrace{\mathbb{R}^6}_{\text{phase space}}$ $\Gamma := (\gamma, \pi)$

then (7, 4) may be written as:

$$\begin{cases} (\dot{\Gamma})_i = (\partial_{i+3} H) \circ \Gamma \\ (\dot{\Gamma})_{i+3} = -(\partial_i H) \circ \Gamma \end{cases}$$

If we define a matrix $\Omega := \begin{bmatrix} 0_{3\times3} & -\mathbb{1}_{3\times3} \\ +\mathbb{1}_{3\times3} & 0_{3\times3} \end{bmatrix}$ then these two equations may also be written together as:

$$\boxed{\Omega \dot{\Gamma} = \nabla H \circ \Gamma} \qquad (2)$$

Ω is called the standard symplectic form.

Now we define yet another Hamiltonian by

$$\tilde{H}(x,p) := \frac{1}{2m} p^2 + q\,\varphi(x) \qquad \forall\, (x,p) \in (\mathbb{R}^3)^2.$$

What is the canonical momentum now?

6

$$(\mathring{\gamma})_i = (\partial_{i+3}\tilde{H})(\gamma, \tilde{\pi}) = \frac{1}{m}(\tilde{\pi})_i$$

$$\Rightarrow \boxed{\tilde{\pi} = m\mathring{\gamma}} \quad \text{as usual.}$$

We also define a (phase-dependent) symplectic form

$\tilde{\Omega} : \mathbb{R}^6 \longrightarrow \mathrm{Mat}_{6\times 6}(\mathbb{R})$ (the magnetic symplectic form)

via $\tilde{\Omega} := \Omega + \frac{q}{c}\mathcal{M}_b$

constant matrix from above

new matrix-valued function on $\mathbb{R}^6$

where $\mathcal{M}_b$ is defined as

$$\mathcal{M}_b := \begin{bmatrix} \mathcal{B} & 0_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} \end{bmatrix}$$

and $\mathcal{B}$ a new matrix-valued (3×3 matrix) function on $\mathbb{R}^3$ defined by:

① $\mathcal{B}^T := -\mathcal{B}$

② $\mathcal{B}_{i+1,i+2} := B_i \quad \forall i \in \{1,2,3\}$ (all indices mod 3)

Hence $\mathcal{B}_{2,3} = B_1$, $\mathcal{B}_{3,1} = B_2$, $\mathcal{B}_{1,2} = B_3$ so:

$$\mathcal{B} = \begin{bmatrix} 0 & B_3 & -B_2 \\ -B_3 & 0 & B_1 \\ B_2 & -B_1 & 0 \end{bmatrix} \quad \boxed{\text{or}} \quad \mathcal{B}v = v \times B \quad \forall v \in \mathbb{R}^3$$

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We define next the phase space trajectory corresponding to $\tilde{\pi}$: $\tilde{\Gamma}: \mathbb{R} \to \mathbb{R}^6$ $\tilde{\Gamma} := (\gamma, \tilde{\pi})$

and we have then the new canonical EoM corresponding to $\tilde{H}, \tilde{\Gamma}$ and $\tilde{\Omega}$:

$$\boxed{(\tilde{\Omega} \circ \gamma)\dot{\tilde{\Gamma}} = (\nabla \tilde{H}) \circ \tilde{\Gamma}} \qquad (3)$$

Claim: (3) $\Longleftrightarrow$ (1) as well.

Proof: We unpack (3) to find its equivalence to (1):

$$\tilde{\Omega}\,\dot{\tilde{\Gamma}} = \Omega\,\dot{\tilde{\Gamma}} + \frac{q}{c}\begin{bmatrix} \mathcal{B} & 0 \\ 0 & 0 \end{bmatrix}\dot{\tilde{\Gamma}}$$

$$= \begin{bmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix}\begin{bmatrix} \dot{\gamma} \\ \dot{\tilde{\pi}} \end{bmatrix} + \frac{q}{c}\begin{bmatrix} \mathcal{B} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{\gamma} \\ \dot{\tilde{\pi}} \end{bmatrix}$$

$$= \begin{bmatrix} -\dot{\tilde{\pi}} \\ \dot{\gamma} \end{bmatrix} + \frac{q}{c}\begin{bmatrix} \mathcal{B}\dot{\gamma} \\ 0 \end{bmatrix} = \begin{bmatrix} -\dot{\tilde{\pi}} + \frac{q}{c}\dot{\gamma} \times B \\ \dot{\gamma} \end{bmatrix}$$

$$(\nabla \tilde{H}) \circ \tilde{\Gamma} = \begin{bmatrix} \nabla_{1,2,3}\tilde{H} \\ \nabla_{4,5,6}\tilde{H} \end{bmatrix} \circ \begin{bmatrix} \gamma \\ \tilde{\pi} \end{bmatrix} = \begin{bmatrix} q(\nabla\varphi)\circ\gamma \\ \frac{1}{m}\tilde{\pi} \end{bmatrix}$$

We thus find the two equations:

8

$$\begin{cases} -\dot{\tilde{\pi}} + \frac{q}{c}\,\dot{\gamma} \times (B\circ\gamma) & = q\,(\nabla\varphi)\circ\gamma \\ \dot{\gamma} & = \frac{1}{m}\,\tilde{\pi} \end{cases}$$

combining the two we find:

$$m\ddot{\gamma} = q\left[\frac{1}{c}\,\dot{\gamma} \times (B\circ\gamma) + E\circ\gamma\right]$$

which is simply eq-n (1). ✓

Claim: $\forall\,(i,j,k) \in \{1,\dots,6\}$ we have

$$\boxed{\partial_i \tilde{\Omega}_{jk} + \partial_j \tilde{\Omega}_{ki} + \partial_k \tilde{\Omega}_{ij} = 0}$$

Proof: Recall $\tilde{\Omega} \equiv \Omega + \frac{q}{c}\mathcal{M}$

$$\Rightarrow \partial_i \tilde{\Omega} = \frac{q}{c}\partial_i \mathcal{M} = \frac{q}{c}\begin{bmatrix} \partial_i B & 0 \\ 0 & 0 \end{bmatrix} = \frac{q}{c}\begin{bmatrix} \begin{bmatrix} 0 & \partial_i B_3 & -\partial_i B_2 \\ & 0 & \partial_i B_1 \\ & & 0 \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix}$$

Recall Maxwell's eq-n $\operatorname{div}(B) = 0$.

$$\Rightarrow \sum_i \partial_i B_i = 0$$

$(j,k) = (1,2)$ if $i = 3$ we get:

$$\frac{q}{c}\left(\partial_3 B_3 + \partial_1 B_1 + \partial_2 B_2\right) = \frac{q}{c}\operatorname{div}(B) = 0 \ ✓$$

In fact any combination of indices which is not zero a priori gives $\operatorname{div}(B)$, and hence zero. (Check!)

$\Rightarrow d\tilde{\Omega} = 0$ and we already saw $\tilde{\Omega}^T = -\tilde{\Omega}$.

As $\tilde{\Omega}$ is non-degenerate, we find that $\tilde{\Omega}$ defines a symplectic structure on $\mathbb{R}^6$.

Note that via the minimal substitution we may get back to the standard symplectic vector space d/. by Ω:

$$\begin{bmatrix} x \\ p \end{bmatrix} \longmapsto \begin{bmatrix} x \\ p + \frac{q}{c}A(x) \end{bmatrix}$$

Time Dependent Canonical Transf.

Let $\Gamma: \mathbb{R} \to (\mathbb{R}^f)^2$ be the trajectory of a system with $f \in \mathbb{N}_{\geq 1}$ degrees of freedom, with

$\Gamma = (\gamma, \pi)$ and $\gamma: \mathbb{R} \to \mathbb{R}^f$, $\pi: \mathbb{R} \to \mathbb{R}^f$ the trajectories in space and momentum respectively.

Then eq-n (7.11) is $\boxed{\Omega \dot{\Gamma} = (\nabla H) \circ \Gamma}$ (4) where $\Omega \equiv \begin{bmatrix} 0 & -\mathbb{1}_{f\times f} \\ \mathbb{1}_{f\times f} & 0 \end{bmatrix}$.

Let $f: (\mathbb{R}^f)^2 \times \underset{\text{time}}{\mathbb{R}} \to (\mathbb{R}^f)^2$ be a time-dependent coordinate transformation, which is assumed to be canonical (pp. 71):

$$\Gamma \equiv f \circ (\overline{\Gamma}, \mathbb{1}_{\mathbb{R}}) \text{ with } \overline{\Gamma}: \mathbb{R} \to (\mathbb{R}^f) \text{ some}$$

other trajectory: $\Gamma(t) = f(\overline{\Gamma}(t), t) \quad \forall\ t \in \mathbb{R}$.

Cl.: $\exists\ K: (\mathbb{R}^f)^2 \times \mathbb{R} \to \mathbb{R}$ (new time-dep. Hamiltonian)

such that (4) may be written as

$$\boxed{\Omega \dot{\overline{\Gamma}} = (\nabla_{1,\dots,2f} K) \circ \overline{\Gamma}}$$

Pf.: We have $\dot{\Gamma} = f(\dot{\overline{\Gamma}}, \cdot) = \sum_{j=1}^{2f} \left((\partial_j f) \circ (\overline{\Gamma}, \mathbb{1}_{\mathbb{R}})\right)(\dot{\overline{\Gamma}})_j +$

$$+ (\partial_t f) \circ (\overline{\Gamma}, \mathbb{1}_{\mathbb{R}})$$

Define $A_{ij} := \partial_j f_i \quad \forall\ (i,j) \in \{1, \dots, 2f\}$

Then $\dot{\Gamma} = \left(A \circ (\overline{\Gamma}, \mathbb{1}_{\mathbb{R}})\right) \dot{\overline{\Gamma}} + (\partial_t f) \circ (\overline{\Gamma}, \mathbb{1}_{\mathbb{R}})$

By assumption that f is canonical we have

$$\boxed{A^T \Omega A = \Omega}$$

We also have $\overline{H}: (\mathbb{R}^f)^2 \to \mathbb{R}$ defined by

$$\overline{H}(x, p, t) := H(f(x, p, t)) \quad \forall\ (x, p, t) \in \mathbb{R}^{2f} \times \mathbb{R}$$

10

$$\Rightarrow \quad H(x,p) = \bar{H}(f^{-1}(x,p))$$

$$(\nabla H)_i = \partial_i H = \sum_{j=1}^{2f} \left((\partial_j \bar{H})\circ f^{-1}\right) \underbrace{\partial_i (f^{-1})_j}_{(A^{-1})_{ji} = ((A^{-1})^T)_{ij}} = (A^{-1})^T (\nabla \bar{H})\circ f^{-1}$$

Finally note

$$A^{-1} A = \mathbb{1} \iff \sum_{k=1}^{2f} \left(\partial_k (f^{-1})_i\right)\left(\partial_j f_k\right) = \delta_{ij}$$

We follow the hint, to start from (4):

$$\Omega \dot{\Gamma} = \Omega \left[\left(A\circ(\bar{\Gamma}, \mathbb{1}_{\mathbb{R}})\right)\dot{\bar{\Gamma}} + (\partial_t f)\circ(\bar{\Gamma}, \mathbb{1}_{\mathbb{R}}) \right]$$

$$\overset{\Omega A = (A^T)^{-1}\Omega}{=} (A^T)^{-1} \Omega \dot{\bar{\Gamma}} + \Omega (\partial_t f) \;\overset{!}{}$$

$$\nabla H = (A^{-1})^T \nabla \bar{H} \circ f^{-1}$$

$$\Longrightarrow \quad (A^T)^{-1} \Omega \dot{\bar{\Gamma}} + \Omega \partial_t f = (A^{-1})^T \nabla \bar{H} \circ f^{-1}$$

$$\Longrightarrow \quad \boxed{\Omega \dot{\bar{\Gamma}} = \nabla \bar{H} \circ f^{-1} - A^T \Omega \partial_t f}$$

Define $g : (\mathbb{R}^f)^2 \times \mathbb{R} \to (\mathbb{R}^f)^2$ by $\boxed{g := -A^T \Omega \partial_t f}$.

For fixed t, consider $g_t : (\mathbb{R}^f)^2 \circlearrowleft$.

Claim: $Dg_t = (Dg_t)^T$

Proof: We have

$$(Dg_t)_{ij} \equiv \partial_j (g_t)_i = -\partial_j (A^T \Omega \partial_t f)_i$$

$$= - \underbrace{\partial_j (A^T)_{il}}_{\partial_j \partial_i f_l = \partial_i \partial_j f_l \;\checkmark} (\Omega \partial_t f)_l - (A^T \Omega)_{il} \partial_j \partial_t f_l$$

Remains to show $(A^T\Omega)_{il} \underbrace{\partial_j \partial_t f_l}_{\partial_t A_{lj}} = (A^T \Omega)_{jl} \underbrace{\partial_i \partial_t f_l}_{\partial_t A_{li}}$.

$$\iff A^T \Omega \partial_t A \overset{?}{=} (A^T \Omega \partial_t A)^T = (\partial_t A^T) \Omega^T A = -(\partial_t A^T) \Omega A$$

$\Leftrightarrow \partial_t (A^T \Omega A) = 0$

$\Leftrightarrow \partial_t \Omega = 0$ ✓ (true)

Claim: $Dg_t = (Dg_t)^T \Rightarrow \exists\ F : (\mathbb{R}^f)^2 \times \mathbb{R} \to (\mathbb{R}^f)^2$

s.t. $g = \nabla F$

Proof: This is the integrability condition given in the hint.
Note that symmetric matrices may be diagonalized and $Dg_t > 0$.

...

$$\Rightarrow \boxed{\Omega \dot{\Gamma} = \nabla(\bar{H} + F)}$$

So define $\boxed{K := \bar{H} + F}$.