Analytical Mechanics — HW12 — 18/12/16

Q1 Virial Theorem

Let $f \in \mathbb{N}_{\geq 1}$ be given (# of D.o.F. of a "Mechanical System"). $\mathbb{R}^{2f}$ is the phase space, comprised of all possible combinations $(q,p) \in \mathbb{R}^{2f}$ of position and (generalized) momentum.

Let $[\cdot,\cdot] : \mathbb{R}^f \times \mathbb{R}^f \longrightarrow \mathbb{R}$ be a pos. def. inner product on $\mathbb{R}^f$.

Def.: A map $V: \mathbb{R}^f \longrightarrow \mathbb{R}$ is called homogeneous of degree n (for some $n \in \mathbb{Z}$) iff $V(\alpha x) = \alpha^n V(x)$ $\forall x \in \mathbb{R}^f, \alpha \in \mathbb{R}$.

Let $V: \mathbb{R}^f \to \mathbb{R}$ be given s.t. V is homogeneous of deg. $-n$, for some given $n \in \mathbb{N}_{\geq 1}$.

Define $H: \mathbb{R}^{2f} \to \mathbb{R}$ by $(q,p) \longmapsto \frac{1}{2}[p,p] + V(q)$

(a) Define $F: \mathbb{R}^{2f} \to \mathbb{R}$ by $(q,p) \longmapsto \langle p, q \rangle_{\mathbb{R}^f}$

Define the flow of dilations $\varphi: \mathbb{R} \to \mathrm{Aut}(\mathbb{R}^{2f})$ by

$$(\varphi(\lambda))(q,p) := (\exp(\lambda) q, \exp(-\lambda) p)$$

Claim: φ is really a flow.

Proof: We must establish that φ is a gp. morphism:

$$(\varphi(\lambda_1+\lambda_2))(q,p) \equiv (e^{\lambda_1+\lambda_2} q, e^{-\lambda_1-\lambda_2} p)$$

$$(\varphi(\lambda_1))((\varphi(\lambda_2))(q,p)) \equiv (\varphi(\lambda_1))(e^{\lambda_2} q, e^{-\lambda_2} p)$$

$$\equiv (e^{\lambda_1} e^{\lambda_2} q, e^{-\lambda_1} e^{-\lambda_2} p)$$

$$= (\varphi(\lambda_1+\lambda_2))(q,p).$$

Also, since mul. is smooth, $\varphi(\lambda)$ is smooth $\forall \lambda$. ✓

<u>Claim</u>: F is the generating function of φ (and hence in particular φ is a canonical flow).

<u>Proof</u>: We need to show that φ satisfies the differential eq-n (for any fixed $x \in \mathbb{R}^{2f}$)

$$\Omega\big((\partial_\lambda \varphi)(\lambda)\big)(x) = (\nabla F) \circ \big(\varphi(\lambda)\big)(x)$$

(the unknown is the map $\lambda \longmapsto (\varphi(\lambda))(x)$ where x is kept fixed, so that really we have a family of differential eq-ns parametrized by x)

with initial cond $\quad (\varphi(0))(x) \overset{!}{=} x$.

(This cond. is the thm. on bottom of pp. 73 in the script)

$$\nabla F = e_i \partial_i F = e_i \partial_i x_{j+f} x_j = e_i \delta_{i,j+f} x_j + e_i x_{j+f} \delta_{i,j}$$

$$= e_{j+f} x_j + e_j x_{j+f}$$

$$\Rightarrow (\nabla F)(q,p) = (p,q)$$

Write $\quad (\varphi(\lambda))(x) = \begin{bmatrix} (\varphi(\lambda))(x)_q \\ (\varphi(\lambda))(x)_p \end{bmatrix}.$

Then $\quad \Omega\big((\partial_\lambda \varphi)(\lambda)\big)(x) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} (\varphi'(\lambda))(x)_q \\ (\varphi'(\lambda))(x)_p \end{bmatrix} =$

$$= \begin{bmatrix} -(\varphi'(\lambda))(x)_p \\ (\varphi'(\lambda))(x)_q \end{bmatrix} \overset{!}{=} \begin{bmatrix} (\varphi(\lambda))(x)_p \\ (\varphi(\lambda))(x)_q \end{bmatrix}$$

$$\begin{cases}(\varphi'(\lambda))(x)_p = -(\varphi(\lambda))(x)_p \\ (\varphi'(\lambda))(x)_q = (\varphi(\lambda))(x)_q\end{cases}$$

Solve D-E $\Rightarrow$

$$\begin{cases}(\varphi(\lambda))(x)_q = e^{\lambda}(\varphi(0))(x)_q \\ (\varphi(\lambda))(x)_p = e^{-\lambda}(\varphi(0))(x)_p\end{cases}$$

Apply initial condition: $(\varphi(0))(x) = x \iff \begin{cases}(\varphi(0))(x)_q = (x)_q \\ (\varphi(0))(x)_p = (x)_p\end{cases}$

$$\Rightarrow \quad (\varphi(\lambda))(x) = \begin{bmatrix} e^{\lambda}(x)_q \\ e^{-\lambda}(x)_p \end{bmatrix} = \begin{bmatrix} e^{\lambda} & 0 \\ 0 & e^{-\lambda}\end{bmatrix}\begin{bmatrix}(x)_q \\ (x)_p\end{bmatrix}$$

$$\Rightarrow \quad \varphi(\lambda) = \begin{bmatrix} e^{\lambda} & 0 \\ 0 & e^{-\lambda}\end{bmatrix}$$ as desired. ✓

(b) Claim: $\{F, H\} = -2T - nV$

where $T : \mathbb{R}^f \to \mathbb{R}$ is $p \longmapsto \frac{1}{2}[p \cdot p]$

Proof: Recall $\{H, F\} \equiv \langle \nabla H, \Omega \nabla F\rangle_{\mathbb{R}^{2f}}$

Also recall $\{H, F\} \circ \gamma = \partial_t (F \circ \gamma)$ for any path $\gamma : \mathbb{R} \to \mathbb{R}^{2f}$ which obeys the EoM $\Omega\dot{\gamma} = \nabla H \circ \gamma$.

Hence $\{F, H\} \circ \gamma = -\partial_t(F \circ \gamma)$.

But we can also apply the formalism in the other way: $\{F, H\} \circ \varphi = -\partial_\lambda (H \circ \varphi)$

$$H \circ (\varphi(\lambda))(q, p) = H(e^{\lambda} q, e^{-\lambda} p) = T(e^{-\lambda} p) + V(e^{\lambda} q)$$

$$\partial_\lambda V(e^{\lambda} q) = \partial_\lambda (e^{\lambda})^{-n} V(q) = -n(e^{\lambda})^{-n-1} e^{\lambda} V(q)$$

4

$$= -n\,(e^{\lambda})^{-n}\, V(q) = -n\, V(e^{\lambda} q)$$

$$\partial_\lambda T(e^{-\lambda} p) = \partial_\lambda \tfrac{1}{2}[e^{-\lambda} p, e^{-\lambda} p]$$

$$= \partial_\lambda e^{-2\lambda}\, T(p)$$

$$= -2\, e^{-2\lambda}\, T(p)$$

$$= -2\, T(e^{-\lambda} p)$$

$$\Rightarrow \partial_\lambda \big(H \circ ((\varphi(\lambda))(q,p))\big) = -2\, T\big(((\varphi(\lambda))(q,p))_p\big) - n\, V\big(((\varphi(\lambda))(q,p))_q\big)$$

$$\Rightarrow \{F, H\} \circ ((\varphi(\lambda))(q,p)) = \ldots$$

Since this holds $\forall\, \lambda$, evaluate @ $\lambda = 0$:

$$\boxed{\{F, H\}(q,p) = -2\, T(p) - n\, V(q)}$$

(c) Define a trajectory $\gamma : \mathbb{R} \to \mathbb{R}^{2f}$ in phase sp. to be **bounded** iff γ is bounded.

$\forall$ scalar $A : \mathbb{R}^{2f} \to \mathbb{R}$, and any trajectory $\gamma : \mathbb{R} \to \mathbb{R}^{2f}$ define the mean of A along γ,

$$\boxed{\langle A \rangle_\gamma := \lim_{t \to \infty} \frac{1}{2t} \int_{-t}^{t} A \circ \gamma}$$

<u>Claim</u>: If $\gamma : \mathbb{R} \to \mathbb{R}^{2f}$ is bounded then

$$2 \langle T \rangle_\gamma + n \langle V \rangle_\gamma = 0$$

<u>Proof</u>:

by linearity of limits & integrals

$$2\langle T \rangle_\gamma + n \langle V \rangle_\gamma \overset{}{=} -\langle -2T - nV \rangle_\gamma$$

$$\overset{(b)}{=} -\langle \{F, H\} \rangle_\gamma$$

$$\overset{(7.25)}{=} \langle \{H, F\} \rangle_\gamma$$

$$\equiv \lim_{t\to\infty} \frac{1}{2t} \int_{-t}^{t} \{H, F\} \circ \gamma$$

$$\overset{(7.23)}{=} \lim_{t\to\infty} \frac{1}{2t} \int_{-t}^{t} \partial(F\circ\gamma)$$

$$\overset{\text{Fund. Thm. of Calc.}}{=} \lim_{t\to\infty} \frac{1}{2t} \left[F(\gamma(t)) - F(\gamma(-t))\right]$$

$$\overset{\text{def. of } F}{=} \lim_{t\to\infty} \frac{1}{2t} \left[\langle \gamma(t)_p, \gamma(t)_q \rangle_{\mathbb{R}^p} - \langle \gamma(-t)_p, \gamma(-t)_q \rangle_{\mathbb{R}^p}\right]$$

$$= 0$$

where the last line follows by the fact γ is bounded yet $t \to \infty$.

(d) Claim: If $n=2$ and the trajectory γ is bdd. then $E=0$.

If $n \neq 2$ and the trajectory γ is bdd. then

$$\langle V \rangle_\gamma = \frac{2}{2-n} E$$

$$\langle T \rangle_\gamma = -\frac{n}{2-n} E$$

Proof: Since the energy is conserved, $H\circ\gamma = E$ (the energy of the system) and this is a time-indep. const.

$$\Rightarrow \quad \langle T \rangle_\gamma + \langle V \rangle_\gamma = \langle H \rangle_\gamma = E$$

By (c) we have $2\langle T \rangle_\gamma + n \langle V \rangle_\gamma = 0$.

$$\Rightarrow \begin{bmatrix} 2 & n \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \langle T \rangle_\gamma \\ \langle V \rangle_\gamma \end{bmatrix} = \begin{bmatrix} 0 \\ E \end{bmatrix}$$

6

When $n=2$ this matrix is not invertible so that E must be zero.

When $n\neq 2$ we get: $\begin{bmatrix} 2 & n \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1/(2-n) & \frac{n}{-2+n} \\ \frac{1}{-2+n} & -\frac{2}{-2+n} \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} \langle T\rangle_\gamma \\ \langle V\rangle_\gamma \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix} \begin{bmatrix} 0 \\ E \end{bmatrix} = \begin{bmatrix} \frac{n}{-2+n}E \\ -\frac{2}{-2+n}E \end{bmatrix}$$

(e) Let $n\in(0,2)$.

Claim: If γ is a bounded trajectory then $E\leq 0$.

Proof: Note by hypothesis $T\geq 0$.

$$\Rightarrow \langle T\rangle_\gamma \geq 0$$

But $n\in(0,2)$. $\Rightarrow \frac{n}{-2+n} < 0$

But $\langle T\rangle_\gamma = \frac{n}{-2+n}E \Rightarrow \boxed{E\leq 0}$.

Claim: If $E>0$ then $[\gamma(t)_q, \gamma(t)_q] > n E t^2 + O(t)$ as $t\to\pm\infty$.

Proof: Let the symmetric matrix corresponding to $[\cdot,\cdot]$ be given by $g \in Mat_{f\times f}(\mathbb{R})$.

Then $T(p) = \frac{1}{2}p^T g\, p \quad \forall\ p\in\mathbb{R}^f$.

We also know by the canonical Hamilton's E.o.M.:

$$\begin{cases} (\partial_{i+f}H)\circ\gamma = \dot{\gamma}_i \\ (\partial_i H)\circ\gamma = -\dot{\gamma}_{i+f} \end{cases} \quad \forall\ i\in\{1,\dots,f\}$$

V has no p dependence.

$$\Rightarrow (\partial_{i+f}H)(q,p) = (\partial_{p_i}H)(q,p) = \frac{1}{2}\partial_{p_i}(p^T g\, p) =$$

$$= \frac{1}{2}\partial_{p_i} \; p_j \, g_{jk} \, p_k = \frac{1}{2}\left(\delta_{ij} \, g_{jk} \, p_k + p_j \, g_{jk} \, \delta_{ik}\right)$$

$$= \frac{1}{2}\left(g_{ik} \, p_k + p_j \, g_{ji}\right)$$

$$\overset{g \text{ symmetric}}{=} g_{ij} \, p_j$$

$$\Longrightarrow \boxed{\dot{\gamma}_i = g_{ij} \, \gamma_{j+f}}$$

g is invertible bcs. $[\circ, \circ]$ is pos. def.

$$\Longrightarrow \underbrace{\gamma_{j+f}}_{\substack{j^{th} \text{ momentum} \\ \text{component of} \\ \text{trajectory}}} = (g^{-1})_{jk} \underbrace{\dot{\gamma}_k}_{\substack{k^{th} \text{ velocity} \\ \text{component of traj.}}}$$

We thus find:

by (7.7) here g^{-1}!

$$\partial_t [\gamma(t)_q, \gamma(t)_q] = \partial_t \sum_{\substack{i=1 \\ j=1}}^{f} \gamma(t)_i \, (g^{-1})_{ij} \, \gamma(t)_j$$

$$= \sum_{i,j=1}^{f} \left[\dot{\gamma}(t)_i (g^{-1})_{ij} \gamma(t)_j + \underbrace{\gamma(t)_i (g^{-1})_{ij} \dot{\gamma}(t)_j}_{\gamma(t)_i (g^{-1})_{ij} \dot{\gamma}(t)_j} \right]$$

$g^{-1} = g^{-1T}$

$$= 2 \sum_{i,j=1}^{f} \dot{\gamma}(t)_i (g^{-1})_{ij} \gamma(t)_j = 2 \sum_{i,j,l=1}^{f} \underbrace{g_{il} \gamma(t)_{l+f} (g^{-1})_{ij} \gamma(t)_j}$$

$$= 2 \sum_{i=1}^{f} \gamma(t)_{i+f} \, \gamma(t)_i = 2 (F \circ \gamma)(t)$$

$$\Rightarrow \partial_t^2 [\gamma(t)_q, \gamma(t)_q] = 2 \partial_t (F \circ \gamma)(t) \overset{(b)}{=} 2\left(2 T \circ \gamma_p + n V \circ \gamma_q\right)$$

$$= 2\left[(2-n) T \circ \gamma_p + n (T+V) \circ \gamma\right] =$$

$$= 2\Big[\underbrace{(2-n)}_{\geq 0}\underbrace{T\circ\gamma}_{\geq 0} + nE\Big] \geq 2nE$$

$n\in(0,2) \Rightarrow 2-n\geq 0$

$$\Longrightarrow \boxed{\partial_t^2 \,[\gamma_q, \gamma_q] \geq 2nE}$$

$$\Longrightarrow [\gamma_q, \gamma_q] \geq nEt^2 + Ct + D$$

$$\overset{t\to\pm\infty}{\Longrightarrow} [\gamma_q, \gamma_q] \geq nEt^2 + o(t)$$

$\Longrightarrow$ for $E>0$, $\|\gamma_q\| \sim \sqrt{[\gamma_q,\gamma_q]} \sim \sqrt{nE}\,t$

linear growth, as a free particle

$\Longleftrightarrow$ Scattering trajectory.

(f) Claim: For a conservative oscillating system, $n=-2$.

Proof: Recall from chapter 4 (pp. 35) that we have V also a quadratic form, so that in particular $V(\alpha x) = \alpha^2 V(x)$.

(E.g. spring: $V(x) = \frac{1}{2}m\omega^2 x^2$.)

$\Longrightarrow$ For such systems w/ bounded orbits,

$$\langle T\rangle_\gamma = -\frac{-2}{2-(-2)}E = \frac{E}{2}$$

$$\langle V\rangle_\gamma = \frac{2}{2-(-2)}E = \frac{E}{2}$$

Claim: For an N-particle system which interacts w/ gravity we have $n=1$

Proof: $V(x) = -G\dfrac{m_1 m_2}{\|r_1 - r_2\|}$ for grav.

$\Longrightarrow$ for bounded orbits of such systems,

$$\langle T \rangle_\gamma = -E$$

$$\langle V \rangle_\gamma = 2E$$

Q2 Fictional Forces in the Hamiltonian Formalism

Let $R : \mathbb{R} \to SO(3)$, $b : \mathbb{R} \to \mathbb{R}^3$ be given.

Define a map $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ by

$$(x,t) \longmapsto R(t)x + b(t)$$

A particle's trajectory in a stationary system is

$\gamma : \mathbb{R} \to \mathbb{R}^3$

and its trajectory in the accelerating frame is

$\tilde{\gamma} : \mathbb{R} \to \mathbb{R}^3$

The two are related via:

$$\boxed{\gamma(t) = f(\tilde{\gamma}(t), t)}$$

(a) Claim: f is a (time-dep.) canonical transf.

Proof: Head against the wall: Compute Jacobian, show it is symplectic.

Need to figure out how f acts on momentum sp.:

$$(p,t) \longmapsto R(t)\,p$$

$\Longrightarrow$ General Jacobian is $R(t) \oplus R(t)$.

$$\underbrace{(R(t)\oplus R(t))^T}_{\begin{bmatrix} R(t)^{-1} & 0 \\ 0 & R(t)^{-1} \end{bmatrix}} \underbrace{\begin{bmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix} R(t)\oplus R(t)}_{\begin{bmatrix} 0 & -R(t) \\ R(t) & 0 \end{bmatrix}} = \begin{bmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix} = \Omega \checkmark$$

Alternatively: Via a generator for the transf.:

$$S(x,\tilde p,t) := \langle \tilde p, \underbrace{R(t)^T(x-b)}_{\tilde x}\rangle$$

$$\Rightarrow \frac{\partial S}{\partial \tilde p} = \tilde x \checkmark$$

$$\frac{\partial S}{\partial x} = R(t)\tilde p = p \checkmark$$

$\Rightarrow$ S is a generating function!

$\Rightarrow$ The transf. is canonical:

$$\tilde p = R(t)^T p$$
$$\tilde x = R(t)^T (x - b(t))$$

b) Claim: $\tilde H = \frac{\tilde p^2}{2m} + (\tilde\omega \times \tilde p)\tilde x - R\,\tilde p \cdot \dot b$ where $\tilde\omega\times\tilde p = R^T\dot R\tilde p$.

Proof: We know $\partial_t S = \tilde H - H$, and $H = \frac{p^2}{2m}$.

$$\Rightarrow \tilde H = \frac{p^2}{2m} + \partial_t S$$

$$\partial_t S = \langle \dot R \tilde p, x - b\rangle - \langle \dot R\tilde p, \dot b\rangle$$

$$= \langle \underbrace{R^T \dot R \tilde p}_{\tilde\omega\times\tilde p}, \tilde x\rangle - \langle R\,\tilde p, \dot b\rangle$$

$\dot R^T(b+\dot b) = R^T\dot b$

$$-\langle \dot R\tilde p, b\rangle - \langle \dot R\tilde p, \dot b\rangle = -\langle R\tilde p, \dot b\rangle$$

(c) Claim: The fictitious forces are given by (1,26) with $K=0$.