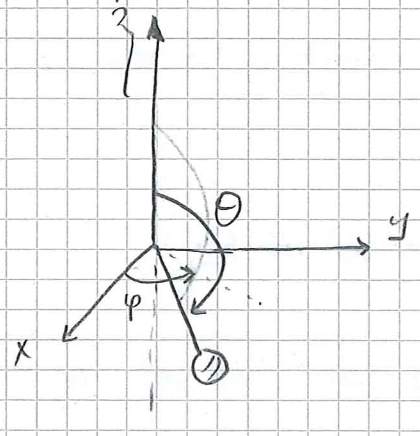
Analytical Mechanics — HW#13 — 20/12/2016

Q1 The Spherical Pendulum

Recall from HW9Q1.3:



Pendulum of length a, with point mass m attached to it.
The system is under the influence of gravity.

Recall the Lagrangian:

$$L(\theta,\varphi,\dot\theta,\dot\varphi) = \frac{1}{2} m a^2\left[\sin(\theta)^2\dot\varphi^2 + \dot\theta^2\right] - mga\cos(\theta)$$

Minus here compared to official sol-ns bcs. their θ is our $\pi-\theta$.

Find the two conjugate momenta:

$$p_\theta(\theta,\varphi,\dot\theta,\dot\varphi) = \frac{\partial L}{\partial\dot\theta}(\theta,\varphi,\dot\theta,\dot\varphi) = ma^2\dot\theta \qquad p_\varphi(\theta,\varphi,\dot\theta,\dot\varphi) \equiv \frac{\partial L}{\partial\dot\varphi}(\theta,\varphi,\dot\theta,\dot\varphi) = ma^2\sin(\theta)^2\dot\varphi$$

Compute the Hamiltonian (via a Legendre transform):

$$\dot\theta(\theta,\varphi,p_\theta,p_\varphi) = \frac{p_\theta}{ma^2} \qquad \dot\varphi(\theta,\varphi,p_\theta,p_\varphi) = \frac{p_\varphi}{ma^2\sin(\theta)^2}$$

$$H(\theta,\varphi,p_\theta,p_\varphi) \equiv p_\theta\,\dot\theta(\theta,\varphi,p_\theta,p_\varphi) + p_\varphi\,\dot\varphi(\theta,\varphi,p_\theta,p_\varphi) -$$

$$- L(\theta,\varphi,\dot\theta(\theta,\varphi,p_\theta,p_\varphi),\dot\varphi(\theta,\varphi,p_\theta,p_\varphi))$$

$$= \frac{p_\theta^2}{ma^2} + \frac{p_\varphi^2}{ma^2\sin(\theta)^2} - \frac{1}{2}ma^2\left[\cancel{\sin(\theta)^2}\frac{p_\varphi^2}{m^2a^4\sin(\theta)^{\cancel{4}2}} + \frac{p_\theta^2}{m^2a^4}\right] +$$

$$+ mga\cos(\theta)$$

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$$\boxed{H(\theta,\varphi,p_\theta,p_\varphi) = \frac{1}{2}\frac{p_\theta^2}{ma^2} + \frac{1}{2}\frac{p_\varphi^2}{ma^2\sin(\theta)^2} + mga\cos(\theta)}$$

The reduced Hamilton-Jacobi equation for a generator of canonical transf. S s.t. in the new coordinates the new Hamiltonian is equal to merely one momentum coordinate ((7.39)) are given by:

$$\frac{1}{2ma^2}\left[\left(\frac{\partial S}{\partial\theta}\right)^2 + \sin(\theta)^{-2}\left(\frac{\partial S}{\partial\varphi}\right)^2\right] + mga\cos(\theta) = E$$

Re-arrange to put all $\left(\frac{\partial S}{\partial\varphi}\right)$ on one side and all $\theta, \left(\frac{\partial S}{\partial\theta}\right)$ on the other side to get:

$$\left(\frac{\partial S}{\partial\varphi}\right)^2 = [\sin(\theta)]^2\left\{2ma^2[E - mga\cos(\theta)] - \left(\frac{\partial S}{\partial\theta}\right)^2\right\}$$

Make a separation Ansatz: $S(\theta,\varphi) = S_\varphi(\varphi) + S_\theta(\theta)$

$$\Rightarrow \frac{\partial S}{\partial\varphi} = \ell \quad \text{for some constant } \ell.$$

$$\Rightarrow \boxed{S_\varphi(\varphi) = \ell\varphi}$$

$$\pm\sqrt{-\ell^2\sin(\theta)^{-2} + 2ma^2[E - mga\cos(\theta)]} = \frac{\partial S}{\partial\theta}$$

$$\Rightarrow \boxed{S_\theta(\theta) = \pm\int^\theta \sqrt{2ma^2[E - mga\cos(\theta')] - \ell^2\sin(\theta')^{-2}}\, d\theta'}$$

We also know that $\frac{\partial S}{\partial\ell} = c_1$ $\frac{\partial S}{\partial E} = c + t$ for two constant c, so that

$$\boxed{\varphi \mp \int^\theta \frac{\ell\sin(\theta')^{-2}}{\sqrt{\cdots}}\, d\theta' = c}$$

$$\boxed{\pm \int \frac{ma^2}{\sqrt{\dots}}\, d\theta' = c + t}$$

The energy E and the angular momentum ℓ are both conserved.

Q2 Applications of Lorentz Transformations

(a) Time Dilations

We measure in units s.t. $\boxed{c=1}$.

Two events, A and B, are measured in an inertial system K to have the coordinates $(t_z, x_z) \in \mathbb{R}^4 \ \forall z \in \{A, B\}$.

Furthermore it is given that $x_A = x_B$.

Cl.: Let K' be a coordinate system moving with velocity v w.r.t. K. Then $t_A' - t_B' > t_A - t_B$.

Pf.: According to the decomposition law, (pp. 98 in script) the relation between K and K' must be via a Lorentz transf. of the form

$$\Lambda = \Lambda_R(R_1)\Lambda_B(v)\Lambda_R(R_2)$$

where Λ_R is a rotation transf. $(R_i \in SO(3))$ and Λ_B is a boost transf.

$$\Lambda_B(v) = \begin{bmatrix} \gamma & -v\gamma \\ -v\gamma & \gamma \end{bmatrix} \oplus \mathbb{1}_{2\times2} \quad \text{where} \quad \gamma \equiv (1-v^2)^{-1/2}$$

Then $\Lambda_B(v)\begin{bmatrix} t \\ x \end{bmatrix} = \begin{bmatrix} \gamma t - v\gamma x_1 \\ -v\gamma t + \gamma x_1 \end{bmatrix} \oplus \begin{bmatrix} x_2 \\ x_3 \end{bmatrix}$

Hence the time difference after a mere boost is:

$$(\gamma t_A - v\gamma x_A) - (\gamma t_B - v\gamma x_B) =$$

$$\overset{x_A = x_B \text{ by assumption}}{=} \gamma(t_A - t_B)$$

<u>Cl.</u>: $\gamma \geq 1 \quad \forall \; v$

<u>Pf.</u>: Assume otherwise $\Rightarrow \quad \gamma < 1$

$\Leftrightarrow \quad (1-v^2)^{-1/2} < 1$

$\Leftrightarrow \quad (1-v^2) > 1$

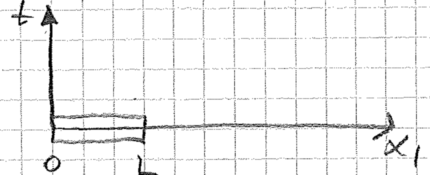
$\Leftrightarrow \quad v^2 < 0 \quad \Rightarrow \quad \boxed{\bot}$

$\Rightarrow$ If Λ were only a boost, the time interval necessarily grows.
But any rotation clearly does <u>not</u> change the time interval.

(b) <u>Length Contraction</u>

<u>Cl.</u>: A ruler of length L_0 in its rest system K has a shorter length in a system K' moving longitudinally w.r.t. K.

<u>Pf.</u>:



Since the ruler is at rest @ K, we may measure its length in K @ different times, t_A and t_B.
Then we have $(t_A, 0)$, (t_B, L_0) as the two measurement events in K.
The transf. must be s.t. in K' the measurements happen at the same time ($t_A' \overset{!}{=} t_B'$).

Thus we transform back from K' to K via $\Lambda(-v)$ to get

$$x_2 - x_1 = (v\gamma t_2' + \gamma x_2') - (v\gamma t_1' + \gamma x_1')$$

$$\overset{t_2' \overset{!}{=} t_1'}{=} \gamma(x_2' - x_1')$$

$$\equiv \gamma L$$

$$\Rightarrow \boxed{L = \frac{L_0}{\gamma} \leq L_0}$$

(c) Def.: A vector $v \in \mathbb{R}^4$ is

time-like iff $(v, v) > 0$

space-like iff $(v, v) < 0$

where $(\cdot, \cdot)$ is the Minkowski form.

Cl.: Two events $(x, y) \in (\mathbb{R}^4)^2$ happen simultaneously in some inertial sys. iff $x-y$ is space-like. They happen at the same space point iff $x-y$ is time-like.

Pf.: WLOG (after transl. and rotations) we may assume $x = 0$ and $y = (y_0, y_1, 0, 0)$ $\exists\ (y_0, y_1) \in \mathbb{R}^2$.

We perform an arbitrary boost along e_1 to get:

$$x' = \begin{bmatrix} \gamma x_0 - v\gamma x_1 \\ -v\gamma x_0 + \gamma x_1 \end{bmatrix} \oplus 0_2 = 0$$

$$y' = \begin{bmatrix} \gamma y_0 - v\gamma y_1 \\ -v\gamma y_0 + \gamma y_1 \end{bmatrix} \oplus 0_2 = \gamma \begin{bmatrix} y_0 - v y_1 \\ -v y_0 + y_1 \end{bmatrix} \oplus 0_2$$

Furthermore, the (invariant) length of $x-y$ is:

$(x-y, x-y) = (y, y) \equiv (y_0)^2 - (y_1)^2$

Assume $x-y$ is time-like $\Rightarrow (x-y, x-y) > 0 \Rightarrow (y_0)^2 > (y_1)^2$

$$\Leftrightarrow \left|\frac{y_1}{y_0}\right| < 1$$

So pick the transf. s.t. $v = y_1/y_0 < 1$. Then $y' = \gamma\left(y_0 - \frac{(y_1)^2}{y_0}\right) \oplus 0_3$

$\Rightarrow (y')_1 = (x')_1 \Leftrightarrow$ Events happen at same spatial location.

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If $x-y$ is space-like, $\left|\frac{y_0}{y_1}\right| < 1$. So if we pick

$$v := \frac{y_0}{y_1} < 1 \quad \text{we find}$$

$$y' = 0 \oplus \gamma\left(-\frac{(y_0)^2}{y_1} + y_1\right) \oplus 0_2$$

$\Rightarrow (x')_0 = (y')_0 \Rightarrow$ Events happen simultaneously.

Conversely, if the events happen at the same spatial location, then (in an appropriate system — and hence in all systems) $(x-y, x-y) = [(x-y)_0]^2 > 0 \Rightarrow$ time-like.

If the events happen at the same time,

$$(x-y, x-y) = -[(x-y)_1]^2 - [(x-y)_2]^2 - [(x-y)_3]^2 < 0$$

$\Rightarrow$ space-like.

Q3 Relativistic Rocket

A rocket of mass m_0 starts from rest in an inertial system. It then accelerates via expulsion of mass downwards with constant velocity w in its instantaneous rest frame.

(a) Claim: A change of dm (<0) in the rocket's mass results in a change of its velocity (in its instantaneous rest frame) of $dv' = -\left(\frac{w}{m}\right)dm$.

Proof: Before expulsion of dm, the momentum in the rest system is $p = (m, 0_3)$
(because $p \equiv (E, mv)$ and $E \equiv m$)

After the expulsion the momentum will be the sum of the momentum of the expelled mass $d\mu$ and what remains in the rocket. (Note $d\mu \neq dm \Leftrightarrow$ No conservation of mass!)

The momentum of the expelled mass in its own rest frame is given by

$$(d\mu, 0_3)$$

We need to transform this to the instantaneous rest frame ~~of~~ the rocket, hence, boost by ω, to get:

$$\begin{bmatrix} \gamma(\omega) & -\omega\gamma(\omega) \\ -\omega\gamma(\omega) & \gamma(\omega) \end{bmatrix} \begin{bmatrix} d\mu \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma(\omega)\, d\mu \\ -\omega\gamma(\omega)\, d\mu \end{bmatrix}$$

where $\gamma(\omega) \equiv (1-\omega^2)^{-1/2}$

The momentum of the rocket after expulsion is:

$$\begin{bmatrix} m+dm \\ m\,dv' \end{bmatrix}$$

where dv' is the gain in velocity.

Cons. of mom.

$$\Rightarrow \begin{bmatrix} m \\ 0 \end{bmatrix} = \gamma(\omega)\,d\mu \begin{bmatrix} 1 \\ -\omega \end{bmatrix} + \begin{bmatrix} m+dm \\ m\,dv' \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} \cancel{m} = \gamma(\omega)\, d\mu + \cancel{m} + dm & \Rightarrow d\mu = -\gamma(\omega)^{-1} dm \\ 0 = -\gamma(\omega)\, d\mu\, \omega + m\, dv' & \Rightarrow \boxed{dv' = -\left(\frac{\omega}{m}\right) dm} \end{cases}$$

(b) <u>Claim</u>: $m(v) = m_0 \left(\frac{1+v}{1-v}\right)^{-\frac{1}{2\omega}}$

<u>Proof</u>: We need to transform dv' back to the initial system, i.e., find dv. We use the velocity addition formula to get:

relativistically

$$v + dv = v \boxed{+} dv' \equiv \frac{v+dv'}{1+v\,dv'} = v + (1-v^2)\,dv' + O(dv'^2)$$

$$= v + \gamma(v)^{-2}\, dv' + O(dv'^2)$$

$$\Rightarrow \quad \gamma(v)^2\, dv = -\left(\frac{\omega}{m}\right) dm$$

Integrate from velocity zero to v to get:

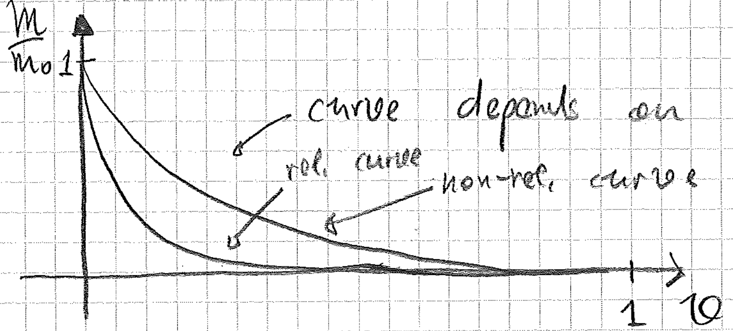
$$\int_0^v \frac{dv'}{1-(v')^2} = -\omega \int_{m_0}^{m} \frac{dm'}{m'} = -\omega \log(m')\Big|_{m_0}^{m} = -\omega \log\left(\frac{m}{m_0}\right)$$

Note $\int \frac{dv}{1-v^2} = \text{arctgh}(v) + C_1$

$$\Rightarrow \quad \text{arctgh}(v) - \text{arctgh}(0) = -w \log(m/m_0)$$

Recall $\text{arctgh}(v) = \frac{1}{2}\log\left(\frac{1+v}{1-v}\right) \Rightarrow \text{arctgh}(0) = 0$

and $\boxed{\left(\frac{1+v}{1-v}\right)^{-1/2w} = \frac{m}{m_0}}$



How to relate this to HW3?

In the non-rel. case, $v \ll 1$, so that:

$$\frac{m}{m_0} = \left(\frac{1+v}{1-v}\right)^{-1/2w} = \exp \log\left(\frac{1+v}{1-v}\right)^{-1/2w}$$

$$= \exp\left[-\frac{1}{2w}\log\left(\frac{1+v}{1-v}\right)\right]$$

$$\boxed{\log\left(\frac{1+v}{1-v}\right) = 2\,\text{arctgh}(v) = 2v + O(v^3)}$$

$$\approx \exp\left[-\frac{1}{2w}2v\right] = e^{-v/w}$$

which is the result we had then.