# Analytical Mechanics HW#1 — 26/9/2016

– Ask Q's on Moodle. (Anon.?)
– Sketch for each session.



## Q1 Galilean Invariance

Let a mechanical system of $N$ point masses in $\mathbb{R}^3$ be given, whose Galilean invariant force law is of the form

$$m_i \ddot{\underline{\gamma}}_i = -\underline{\nabla}_i V(\underline{\gamma}_1, \dots, \underline{\gamma}_N)$$

with $\underline{\gamma}_i : \mathbb{R} \to \mathbb{R}^3$ being the orbit of the $i^{th}$ particle.

(a) **Cl.:** Let $N \in \{2,3\}$. Then if at zero time the masses are at rest, their motion remains in the line $(N=2)$ or plane $(N=3)$ which is defined by their initial positions.

**Pf.:** Because of our assumption on the form of the force on the particles, a rotated orbit obeys the same EoM as the original orbit. The same holds if a rotation is replaced with reflection:

$$m_i \ddot{\underline{\gamma}}_i = -(\underline{\nabla}_i V)(\underline{\gamma}_1, \dots, \underline{\gamma}_N)$$

is the original EoM, $\underline{\lambda}_i := R\underline{\gamma}_i$ for some $R \in O(3)$. Then $m_i \ddot{\underline{\lambda}}_i = -(\underline{\nabla}_i V)(\underline{\lambda}_1, \dots, \underline{\lambda}_N)$ is also satisfied.

$\boxed{N=2}$ The initial data $\underline{\gamma}_1(0)$ and $\underline{\gamma}_2(0)$ define a straight line in $\mathbb{R}^3$.

This data is invariant under rotations about the axis passing through $\underline{\gamma}_1(0)$ and $\underline{\gamma}_2(0)$. (Note this would not have been generically the case if we hadn't had $\dot{\underline{\gamma}}_i(0) = 0$).

$\{\underline{\gamma}_i\}_{i=1}^2$ lies on the straight line passing through $\underline{\gamma}_1(0)$ and $\underline{\gamma}_2(0)$ iff it is invariant under rotations about that line. Let $R$ be any such rotation, and def. $\underline{\lambda}_i := R\underline{\gamma}_i \ \forall i \in \{1,2\}$. Then $\{\underline{\lambda}_i\}_{i=1}^2$ satisfies the same EoM and has the same initial data $\Rightarrow$ By uniqueness of solutions to ODEs, $\{\underline{\gamma}_i\}_{i=1}^2 = \{\underline{\lambda}_i\}_{i=1}^2$.

But $\{\underline{\lambda}_i\}_{i=1}^2$ is the rotated orbit.

$\boxed{N=3}$ $\underline{\gamma}_1(0), \underline{\gamma}_2(0), \underline{\gamma}_3(0)$ define a plane in $\mathbb{R}^3$.

The initial data is invariant under reflections about this

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plane. (This would not have been the case if the initial velocities were not zero).
We follow the same argument as before, using now $R$ as reflection about said plane, and the criteria that an orbit lies on a plane iff it is invariant to reflections about that plane.

(B) Cl.: Let $N=2$. Then the motion of the masses remains on a plane.

Pf.: Cl.: The following Galilean transf. makes the second mass start at rest:

$$\begin{bmatrix} t \\ \underline{x} \end{bmatrix} \longmapsto \begin{bmatrix} t \\ \underline{x} - \dot{\underline{\gamma}}_2(0)t \end{bmatrix} \quad \forall \begin{bmatrix} t \\ \underline{x} \end{bmatrix} \in \mathbb{R}^4$$

Pf.: We would have:

$$\underline{\gamma}_2(t) \longmapsto \underline{\gamma}(t) - \dot{\underline{\gamma}}_2(0)t =: \underline{\gamma}_2'(t)$$

Hence

$$\dot{\underline{\gamma}}_2'(0) = \dot{\underline{\gamma}}_2(0) - \dot{\underline{\gamma}}_2(0) = 0$$

Because the force law is Galilean invariant, we could (WLOGT) assume that in our initial data, the second mass (WLOG) starts at rest.
These initial data again define a plane, which passes through:

$\underline{\gamma}_1(0)$, $\underline{\gamma}_2(0)$, $\dot{\underline{\gamma}}_1(0)$

and these data are invariant under rotations about this plane.
Using the same argument as above the reflected orbit then is identical to the original one, and hence the original one lies on that plane.

## Q2 Time-Dependent Rotations

Let $R: \mathbb{R} \longrightarrow O(3)$ be given. ($O(3)$ preserves inner product in $\mathbb{R}^3$)
lengths & angles
$\det = \pm 1$, $R^T = R^{-1}$

(a) Cl.: Let $\underline{y} \in \mathbb{R}^3$ be given, and define $\underline{\gamma}: \mathbb{R} \to \mathbb{R}^3$ by $\underline{\gamma}(t) := R(t)\underline{y}$.

Then $\exists\ \underline{\omega}: \mathbb{R} \to \mathbb{R}^3$ s.t. $\dot{\underline{\gamma}} = \underline{\omega} \times \underline{\gamma}$.
$\underline{\omega}$ is called the angular velocity.

Pf.: We have $\underline{\gamma} = R\underline{y}$ so $\dot{\underline{\gamma}} = \dot{R}\underline{y} = \dot{R}R^{-1}\underline{\gamma} \overset{R(t)\in O(3)\ \forall t}{=} \dot{R}R^T\underline{\gamma}$

Next by orthogonality again we have:

$$\mathbb{1} = RR^T$$

differentiate both sides to get:

$$\begin{aligned} 0 &= (\dot{RR^T}) \\ &= \dot{R}R^T + R(\dot{R^T}) \\ &= \dot{R}R^T + R(\dot{R})^T \\ &= \dot{R}R^T + (\dot{R}R^T)^T \end{aligned}$$

Define $\Omega := \dot{R}R^T$. Hence we have found $\boxed{\Omega = -\Omega^T}$.

$$\Rightarrow \quad \Omega = \begin{bmatrix} 0 & \tilde{\omega}_1 & \tilde{\omega}_2 \\ -\tilde{\omega}_1 & 0 & \tilde{\omega}_3 \\ -\tilde{\omega}_2 & -\tilde{\omega}_3 & 0 \end{bmatrix} \quad \exists \ (\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3) \in \mathbb{R}^3$$

Then

$$\Omega \underline{\gamma} = \begin{bmatrix} 0 & \tilde{\omega}_1 & \tilde{\omega}_2 \\ -\tilde{\omega}_1 & 0 & \tilde{\omega}_3 \\ -\tilde{\omega}_2 & -\tilde{\omega}_3 & 0 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix} = \begin{bmatrix} \tilde{\omega}_1 \gamma_2 + \tilde{\omega}_2 \gamma_3 \\ -\tilde{\omega}_1 \gamma_1 + \tilde{\omega}_3 \gamma_3 \\ -\tilde{\omega}_2 \gamma_1 - \tilde{\omega}_3 \gamma_3 \end{bmatrix}$$

$$\overset{!}{=} \begin{bmatrix} \omega_2 \gamma_3 - \omega_3 \gamma_2 \\ \omega_3 \gamma_1 - \omega_1 \gamma_3 \\ \omega_1 \gamma_2 - \omega_2 \gamma_1 \end{bmatrix} \Longrightarrow \underline{\omega} := \begin{bmatrix} -\tilde{\omega}_3 \\ \tilde{\omega}_2 \\ -\tilde{\omega}_1 \end{bmatrix} = \begin{bmatrix} (\dot{R}R^T)_{3,2} \\ (\dot{R}R^T)_{1,3} \\ (\dot{R}R^T)_{2,1} \end{bmatrix}$$

and then

$$\Omega = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ +\omega_3 & 0 & -\omega_1 \\ -\omega_2 & +\omega_1 & 0 \end{bmatrix} \quad \text{and} \quad \dot{\underline{\gamma}} = \underline{\omega} \times \underline{\gamma}$$

(b) Cl.: If $\underline{\omega}$ is constant then $R(t) = \mathbb{1}_{3\times3} + \frac{1}{\|\underline{\omega}\|}\sin(\|\underline{\omega}\|t)\Omega + \frac{1}{\|\underline{\omega}\|^2}[1-\cos(\|\underline{\omega}\|t)]\Omega^2$

Pf.: We have $\Omega \equiv \dot{R}R^T \iff \dot{R} = \Omega R$ by orthogonality.

The solution to this matrix differential equation (if $\Omega$ is const, which it is as $\underline{\omega}$ is const.) is $R(t) = \exp(\Omega t) \ \forall \ t$.

For this we need:

Cl.:

$$\left.\begin{aligned} \Omega^{2n+2} &\overset{①}{=} (-1)^n \|\underline{\omega}\|^{2n} \Omega^2 \quad \text{and} \\ \Omega^{2n+1} &\overset{②}{=} (-1)^n \|\underline{\omega}\|^{2n} \Omega \end{aligned}\right\} \forall \ n \in \mathbb{N}_{\geq 0}$$

Pf.: We proceed by induction.

For $n=0$, ② is trivially true and so is ①.

Assume the result holds $\forall \ n \leq n_0$ for some $n_0 \in \mathbb{N}_{\geq 0}$.

Check $n_0+1$:

$$\Omega^{2(n_0+1)+2} = \Omega^{2n_0+4} = \Omega^{2n_0+2}\Omega^2 = (-1)^{n_0}\|\underline{\omega}\|^{2n_0}\Omega^4$$

$$\Omega^{2(n_0+1)+1} = \Omega^{2n_0+3} = \Omega^{2n_0+1}\Omega^2 = (-1)^{n_0}\|\underline{\omega}\|^{2n_0}\Omega^3$$

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$$\Omega^2 \equiv \underline{\omega} \times (\underline{\omega} \times \cdot\,)$$

That is, $\forall\ \underline{x}$,

$$\Omega^2 = \underline{\omega} \times (\underline{\omega} \times \underline{x})$$

$$= \left(\varepsilon_{ijk}\, w_j\, (\underline{w} \times \underline{x})_k\right)_i$$

$$= \left(\varepsilon_{ijk} w_j\, \varepsilon_{klm}\, w_l\, x_m\right)_i$$

$$= \big(\underbrace{\varepsilon_{ijk}\varepsilon_{klm}}_{\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}}\, w_j\, w_l\, x_m\big)_i$$

$$= \left(w_m x_m w_i - w_m w_m x_i\right)_i$$

$$= (\underline{w} \cdot \underline{x})\, \underline{w} - \|\underline{w}\|^2 \underline{x}$$

$$\Omega^3 \underline{x} = \Omega\, \Omega^2 \underline{x} = \underline{\omega} \times (\Omega^2 \underline{x})$$

$$= \underline{\omega} \times \left( (\underline{\omega} \cdot \underline{x})\, \underline{\omega} - \|\underline{\omega}\|^2 \underline{x} \right)$$

$$= -\|\underline{\omega}\|^2\, \underline{\omega} \times \underline{x}$$

$$= -\|\underline{\omega}\|^2\, \Omega\ \underline{x}$$

$$\Rightarrow \Omega^4 = \Omega\, \Omega^3 = \Omega(-\|\underline{\omega}\|^2 \Omega) = -\|\underline{\omega}\|^2 \Omega^2$$

We find:

$$\Omega^{2(n_0+1)+2} = (-1)^{n_0}\, \|\underline{\omega}\|^{2n_0} \left(-\|\underline{\omega}\|^2 \Omega^2\right)$$

$$= (-1)^{n_0+1}\, \|\underline{\omega}\|^{2n_0+2}\, \Omega^2 \quad \checkmark$$

$$\Omega^{2(n_0+1)+1} = (-1)^{n_0}\, \|\underline{\omega}\|^{2n_0} \left(-\|\underline{\omega}\|^2 \Omega\right)$$

$$= (-1)^{n_0+1}\, \|\underline{\omega}\|^{2n_0+2}\, \Omega \quad \checkmark$$

Now we may calculate the series:

$$R(t) = \exp(\Omega t) = \sum_{n=0}^{\infty} \frac{1}{n!} (\Omega t)^n$$

abs. conv. allows rearranging

$$= \sum_{n \in 2\mathbb{N}_{\geq 0}} \frac{1}{n!} (\Omega t)^n + \sum_{n \in 2\mathbb{N}_{\geq 0}+1} \frac{1}{n!} (\Omega t)^n$$

change of var.

$$= \mathbb{1}_{3\times3} + \sum_{n=0}^{\infty} \frac{1}{(2n+2)!} (\Omega t)^{2n+2} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} (\Omega t)^{2n+1}$$

$$= \mathbb{1}_{3\times3} + \sum_{n=0}^{\infty} \frac{1}{(2n+2)!}\, t^{2n+2} (-1)^n \|\underline{\omega}\|^{2n}\, \Omega^2 + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\, t^{2n+1} (-1)^n \|\underline{\omega}\|^{2n}\, \Omega$$

$$= \mathbb{1}_{3\times3} + \underbrace{\left[\frac{1}{\|\underline{\omega}\|^2} \sum_{n=0}^{\infty} \frac{1}{(2n+2)!} (-1)^n (t\|\underline{\omega}\|)^{2n+2}\right]}_{1-\cos(\|\underline{\omega}\| t)} \Omega^2 + \underbrace{\left[\frac{1}{\|\underline{\omega}\|} \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} (-1)^n (t\|\underline{\omega}\|)^{2n+1}\right]}_{\frac{1}{\|\underline{\omega}\|} \sin(\|\underline{\omega}\| t)} \Omega$$

# Q3 Angular Velocity

**Cl.:** The angular velocity "behaves as a sort-of vector" under coordinate transformations.

**Pf.:** A Galilean coordinate transf. is given (for the purpose of this exercise) by a matrix $S \in O(3)$:

$$\begin{bmatrix} t \\ \underline{x} \end{bmatrix} \longmapsto \begin{bmatrix} t \\ S\underline{x} \end{bmatrix} \quad \forall \begin{bmatrix} t \\ \underline{x} \end{bmatrix} \in \mathbb{R}^4$$

Then we have

$$\begin{aligned} \dot{\underline{\gamma}} \longmapsto S\dot{\underline{\gamma}} &= \\ &= S\dot{R}\underline{y} \\ &= S\underbrace{\dot{R}R^T}_{=\Omega}\underline{\gamma} \\ &= S\Omega S^{-1}\underline{\gamma}' \\ &= S\Omega S^T\underline{\gamma}' \end{aligned}$$

(*) $M_{ji}M_{kk'}M_{\ell\ell'}\varepsilon_{jk\ell} =: T_{ik'\ell'}$

Note: $T_{ik'\ell'}$ has the following properties:

1. A.S. under exchange of any indices.
2. Zero if any two indices are equal.

$\Rightarrow$ It is proportional to $\varepsilon_{ik'\ell'}$.
Just verify on $(i,k',\ell')=(1,2,3)$.

Compare that with $\dot{\underline{\gamma}} = \Omega\underline{\gamma}$.

$$\Longrightarrow \boxed{\Omega \longmapsto S\Omega S^T}$$

Thus

$$\underline{w}' \times \underline{\gamma}' = \Omega'\underline{\gamma}' = S\Omega S^T\underline{\gamma}' = S\Omega\underline{\gamma} = S(\underline{w}\times\underline{\gamma})$$

**Cl.:** $(M\underline{a})\times(M\underline{b}) = \overbrace{\det(M)(M^T)^{-1}}^{\mathrm{cof}(M)}(\underline{a}\times\underline{b})$ if $\det(M)\neq 0$.

**Pf.:** $\det(M) = \varepsilon_{ijk}M_{1i}M_{2j}M_{3k}$

$$\begin{aligned} [M^T(M\underline{a})\times(M\underline{b})]_i &= (M^T)_{ij}\,\varepsilon_{jk\ell}\,M_{kk'}a_{k'}M_{\ell\ell'}b_{\ell'} \\ &= \underbrace{M_{ji}M_{kk'}M_{\ell\ell'}\varepsilon_{jk\ell}}_{\underbrace{M_{j1}M_{k2}M_{\ell3}\varepsilon_{jk\ell}}_{\det(M)}\varepsilon_{ik'\ell'}}\,a_{k'}b_{\ell'} \quad (*) \\ &= [\det(M)\,\underline{a}\times\underline{b}]_i \end{aligned}$$

Note that for $S\in O(3)$, $S^T=S^{-1} \Rightarrow (S^T)^{-1} = S$ & $\det(S)=\pm 1$.

Hence we find:

$$S(\underline{\omega} \times \underline{\gamma}) = \det(S)\,(S\underline{\omega}) \times (S\underline{\gamma})$$

$$\equiv \underbrace{\det(S)\,(S\underline{\omega})}_{\underline{\omega}'} \times \underline{\gamma}'$$

So that

$$\boxed{\underline{\omega} \longmapsto \det(S)\,S\underline{\omega}}$$

$\Rightarrow$ The angular velocity is an axial vector, a vector transforms as $\underline{\omega} \mapsto -\underline{\omega}$ under reflections!

Cl.: Angular velocity is additive vectorially.

Pf.: We apply two consecutive rotations $R_2$ then $R_1$. Since $O(3)$ is a group, the result is also a rotation. Our goal is to relate $\underline{\omega}$ of the total rotation to $\underline{\omega}_1$ and $\underline{\omega}_2$.

$$\begin{aligned}
\Omega &\equiv \dot{R} R^T \\
&= \dot{(R_1 R_2)} (R_1 R_2)^T \\
&= (\dot{R}_1 R_2 + R_1 \dot{R}_2)(R_2^T R_1^T) \\
&= \dot{R}_1 \underbrace{R_2 R_2^T}_{=\mathbb{1}} R_1^T + R_1 \underbrace{\dot{R}_2 R_2^T}_{\Omega_2} R_1^T \\
&= \Omega_1 + R_1 \Omega_2 R_1^T \\
&= \underline{\omega}_1 \times \cdot \;+ \det(R_1)(R_1 \underline{\omega}_2) \times \cdot \\
&= \underbrace{(\underline{\omega}_1 + \det(R_1)\, R_1 \underline{\omega}_2)}_{\underline{\omega}} \times \cdot
\end{aligned}$$

by previous claim