Analytical Mechanics — HW#2 — Solutions

Q1 Decay of a Binary Star ☉

[Two ☉ stars of mass m_1 and m_2 move in a circular path around their common center of gravity, diametrically.]

The second star explodes in a spherically symmetric (or merely inversion symmetric) fashion and leaves behind the mass m_2'.

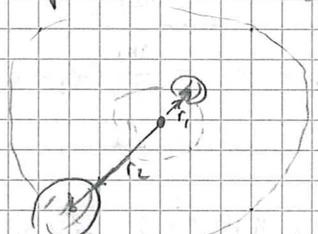
The discarded mass $\Delta m_2 := m_2 - m_2'$ leaves the binary star without delay and without hitting the first star.

Cl.: The binary star remains bounded if $\Delta m_2 < \frac{1}{2}(m_1 + m_2)$.

Pf.:

Cl.: $T_S = -\frac{1}{2} V$

Pf.: Note that because the two masses move in a circular fashion and diametrically, the force on each mass always points towards the center of mass and so is their acceleration.



$r := r_1 + r_2$

Hence their speed and so their kinetic energy is constant.

$\|F_{12}\| = G \frac{m_1 m_2}{r^2}$ and $\|F_{12}\| = m_1 \|a_1\|$

$\|a_i\| = \frac{\|v_i\|^2}{r_i}$ for circular motion,

$$T_i \equiv \frac{1}{2} m_i \|v_i\|^2 = \frac{1}{2} m_i r_i \|a_i\| = \frac{1}{2} m_i r_i \frac{G m_1 m_2}{r^2 m_i}$$

$$= \frac{1}{2} G m_1 m_2 \frac{r_i}{r^2}$$

$$\Rightarrow T_S \equiv T_1 + T_2 = \frac{1}{2} G m_1 m_2 \frac{1}{r}$$

However, $V \equiv -G \frac{m_1 m_2}{r}$

$$\Rightarrow T_S = -\frac{1}{2} V$$

Cl.: If $E < 0$ then the system is bounded.

Pf.: Assume system is unbounded and show $E \geq 0$.

Unbounded system means $\forall R > 0 \; \exists t_R \in \mathbb{R} : r(t_R) \geq R$.

$\Rightarrow V(r)$ gets arbitrarily small. But $E = \frac{1}{2} m \|\dot{x}\|^2 - G m M \frac{1}{\|x\|}$ is const $\Rightarrow \|\dot{x}\| = \sqrt{\frac{2}{m}\left(E + G m M \frac{1}{\|x\|}\right)} \xrightarrow{\|x\| \to \infty} \sqrt{\frac{2}{m} E} = \text{const} \Rightarrow E \geq 0$.

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Cl.: T_S depends only on the difference in velocities of the two particles.

Pf.: The center of mass is given by

$$X \equiv \frac{1}{M}(m_1 x_1 + m_2 x_2)$$

with $M \equiv m_1 + m_2$.

The kinetic energy relative to the center of mass is

$$T_S \equiv \frac{1}{2}\sum_{i=1}^{2} m_i \|\dot{x}_i - \dot{X}\|^2$$

Cl.: $x_1 - X = \frac{m_2}{M}(x_1 - x_2)$

Pf.: $$x_1 - X = \frac{M x_1 - m_1 x_1 - m_2 x_2}{M}$$

$$= \frac{m_2 x_1 - m_2 x_2}{M} = \frac{m_2}{M}(x_1 - x_2)$$

Similarly, $x_2 - X = \frac{m_1}{M}(x_2 - x_1)$

Hence,

$$T_S = \frac{1}{2}\sum_{i=1}^{2} m_i \left\| \frac{m_{i'}}{M}(\dot{x}_i - \dot{x}_{i'}) \right\|^2 =$$

i' is the other index in $\{1,2\}$ not eq to i

$$= \frac{1}{2}\left[m_1 \frac{m_2^2}{M^2}\|\dot{x}_1 - \dot{x}_2\|^2 + m_2 \frac{m_1^2}{M^2}\|\dot{x}_2 - \dot{x}_1\|^2 \right]$$

$\boxed{m := \frac{m_1 m_2}{M}}$ reduced mass

$$= \frac{1}{2}\frac{m_1 m_2^2 + m_2 m_1^2}{M^2}\|\dot{x}_1 - \dot{x}_2\|^2$$

$$= \frac{1}{2}\frac{m_2 m + m_1 m}{M}\|\dot{x}_1 - \dot{x}_2\|^2 = \frac{1}{2} m \|\dot{x}_1 - \dot{x}_2\|^2$$

In the (instantaneous) rest frame of particle 2, with unprimed letters denoting just before the explosion and primed letters denoting right after the explosion, we have:

$$(*)\begin{cases} \dot{x}_1 = \dot{x}_1' & \leftsquigarrow \text{ bcs. the } 1^{st} \text{ mass is not hit by expl.} \\ \dot{x}_2 = \dot{x}_2' = 0 & \leftsquigarrow \text{ inversion symmetry.} \end{cases}$$

We may compute the physical quantities in any inertial frame we choose. In particular that of the 2^{nd} mass's rest.

By the two conditions above ⊛ we have: 2.1

$$\frac{T_s'}{T_s} = \frac{\frac{1}{2}m'\|\dot{x}_1' - \dot{x}_2'\|^2}{\frac{1}{2}m\|\dot{x}_1 - \dot{x}_2\|^2} = \frac{m'}{m} \equiv \frac{\left(\frac{m_1' m_2'}{M'}\right)}{\left(\frac{m_1 m_2}{M}\right)} \overset{m_1'=m_1}{=} \frac{m_2'}{m_2}\frac{M}{M'}$$

$$\frac{V'}{V} = \frac{-G\frac{m_1' m_2'}{r'}}{-G\frac{m_1 m_2}{r}} \overset{r'=r,\ m_1'=m_1}{=} \frac{m_2'}{m_2}$$

Hence, $E' = T_s' + V' = T_s \frac{m_2'}{m_2}\frac{M}{M'} + \frac{m_2'}{m_2}(-2T_s)$

$$= \underbrace{T_s \frac{m_2'}{m_2}}_{\text{positive}}\left(\frac{M}{M'} - 2\right)$$

$\Rightarrow E' < 0 \iff \frac{M}{M'} - 2 < 0 \iff M < 2M' \iff \frac{1}{2}M < M'$

But $M' \equiv m_1' + m_2' = m_1 + m_2 - \Delta m_2 = M - \Delta m_2$

$\Rightarrow$ Want $\frac{1}{2}M < M - \Delta m_2$

$$\iff \boxed{\Delta m_2 < \frac{1}{2}M \equiv \frac{1}{2}(m_1 + m_2)}$$

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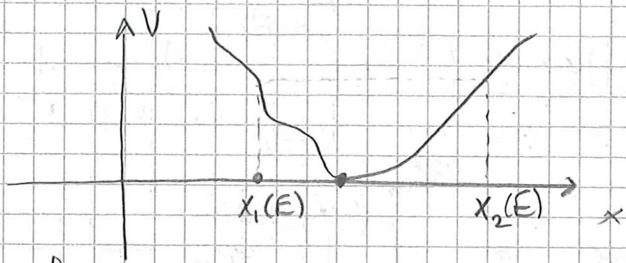
One Dimensional Oscillations

Let $V:\mathbb{R}\to\mathbb{R}$ be (strictly?) monotone from both sides of its minimum zero:



$x_1(E) \equiv$ Solution of $V(x_1(E))=E$ to the left of min.

$x_2(E) \equiv$ Soution of $V(x_2(E))=E$ to the right of min.

Let $\tau(E)$ be the period of movement of energy $E>0$, which is between the two points $x_1(E)<x_2(E)$.

Define $m:=2$ and assume the mass of the particle is m.

a) Cl.: $\tau(E) = \int_{x_1(E)}^{x_2(E)}$?

Pf.: We have $E \equiv T+V$, $T \equiv \frac{1}{2} m v^2 = (\dot{\gamma})^2$

$m \equiv 2$ velocity

where $\gamma:\mathbb{R}\to\mathbb{R}$ is the (time-parametrized) path of the particle. (Hence a priori T depends on time).

$\tau(E)$, being the time period, is equal the integral over time, from the time the particle was at $x_1(E)$ until it reaches $x_2(E)$, times two (to get back to $x_1(E)$).

We assume the initial conditions are s.t. $\begin{cases} \gamma(0)=x_1(E). \\ \gamma(\frac{1}{2}\tau(E)) = x_2(E) \end{cases}$

Hence $\tau(E) = 2\int_0^{\frac{1}{2}\tau(E)} dt$

$T = (\dot{\gamma})^2 \Rightarrow \dot{\gamma} = \sqrt{E-V(\gamma)}$

So we make a change of variable in the integral:

$$y := \gamma(t) \Rightarrow dy = \dot{\gamma}(t)\, dt$$

$$\Rightarrow \tau(E) = 2\int_{x_1(E)}^{x_2(E)} \frac{1}{\dot{\gamma}(t)}\, dy = \int_{x_1(E)}^{x_2(E)} \frac{1}{\sqrt{E-V(y)}}\, dy$$

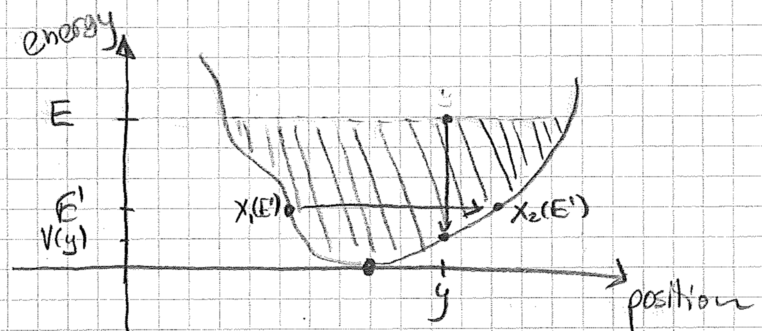
4 b) Cl.: $x_2(E) - x_1(E)$ = something depending on $\tau(E)$

Pf.: We follow the hint by computing the integral:

$$\int_0^E \frac{\tau(E')dE'}{\sqrt{E-E'}} = \int_0^E \left(2\int_{x_1(E)}^{x_2(E)} \frac{1}{\sqrt{E'-V(y)}} dy\right) \frac{1}{\sqrt{E-E'}} dE'$$

$$\overset{\text{integral is linear}}{=} 2\int_0^E \left(\int_{x_1(E')}^{x_2(E')} \frac{1}{\sqrt{E'-V(y)}\sqrt{E-E'}} dy\right) dE'$$

We now have a 2D integral over an area in the x−E (position-energy) plane:



Thus using Fubini's theorem we may exchange the order of integration:

$$\int_0^E \int_{x_1(E')}^{x_2(E')} \bullet \, dy \, dE' = \int_{x_1(E)}^{x_2(E)} \int_{V(y)}^{E} \bullet \, dE' \, dy$$

As a result we find:

$$\int_0^E \frac{\tau(E')dE'}{\sqrt{E-E'}} = 2\int_{x_1(E)}^{x_2(E)} \left(\int_{V(y)}^{E} \frac{1}{\sqrt{(E-V(y))(E-E')}} dE'\right) dy$$

Cl.: $\int_{V(y)}^{E} \frac{1}{\sqrt{(E'-V(y))(E-E')}} dE' = \pi$ (indep. of y!)

Pf.: Define $E' := \underbrace{\left[1-\tfrac{1}{2}(t+1)\right]}_{\frac{1}{2}(1-t)} V(y) + \tfrac{1}{2}(t+1)E$

where $t \in [-1,1]$ so that $E' \in [V(y), E]$

Then $t = \frac{2E'-V(y)-E}{E-V(y)}$ and $dt = \frac{2}{E-V(y)} dE'$

$t(V(y)) = -1 \qquad t(E) = +1 \qquad dE' = \tfrac{1}{2}(E-V(y))dt$

$$(E'-V(y))(E-E') = \left(-\tfrac{1}{2}(1+t)V(y) + \tfrac{1}{2}(t+1)E\right)\left(-\tfrac{1}{2}(1-t)V(y) - \tfrac{1}{2}(t-1)E\right)$$

$$= \left(\tfrac{1}{2}(1+t)(E-V(y))\right)\left(\tfrac{1}{2}(1-t)(E-V(y))\right)$$

$$\Rightarrow \int_{V(y)}^{E} [(E'-V(y))(E-E')]^{-1/2} dE' = \int_{-1}^{1} [(1+t)(1-t)]^{-1/2} dt$$

$$= \int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} dt$$

$$\overset{t:=\sin(\varphi)}{=} \int_{-\pi/2}^{\pi/2} \frac{1}{\cos(\varphi)} \cos(\varphi) d\varphi = \pi$$

$$\Rightarrow \int_0^E \frac{\tau(E')dE'}{\sqrt{E-E'}} = 2\int_{X_1(E)}^{X_2(E)} \pi \, dy = 2\pi [X_2(E) - X_1(E)]$$

$$\Rightarrow X_2(E) - X_1(E) = \frac{1}{2\pi} \int_0^E \frac{\tau(E')dE'}{\sqrt{E-E'}}$$

Cl.: If V is even, then V may be determined via $\tau(E)$

Pf.: First note that V is assumed to be strictly monotone, hence, it is injective on each side of its minimum.
Also, the minimum is $V=0$, so that by being strictly monotone it is also surjective onto $[0,\infty)$, on both its sides.
$\Rightarrow$ $X_1(E)$ and $X_2(E)$ are actually unique (this should have been said @ the beginning).
Thus there are two maps

$$\begin{cases} [0,\infty) \longmapsto [\min(V), \infty) \\ E \longmapsto X_2(E) \end{cases}$$

$$\begin{cases} [0,\infty) \longmapsto (-\infty, \min(V)] \\ E \longmapsto X_1(E) \end{cases}$$

which are the inverse maps of $V|_{[\min(V),\infty)}$ and $V|_{(-\infty,\min(V)]}$ respectively.
When V is even, $\min(V)$ has to be zero, and $X_1(E) = -X_2(E)$ necessarily. Hence we need only determine $V|_{[0,\infty)}$. But we have its inverse map

$$E \longmapsto \frac{1}{4\pi} \int_0^E (E-E')^{-1/2} \tau(E') dE'$$