Analytical Mechanics HW#3 — 10/10/2016

Q1 The Tsiolkovsky Rocket Equation

A rocket of initial mass m_0 launches from rest.
We neglect the friction of air.
The exhaust speed relative to the rocket is constant and equal to $v_0 > 0$.

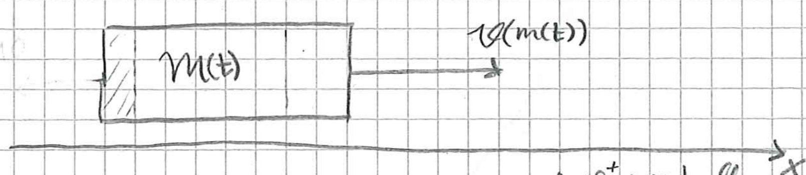
a) Let there be no gravitational field.
Assume at any given moment, the speed of the rocket depends on time only via the remaining mass m. In particular, that speed does not depend explicitly on $\dot{m}$, as long as the mass loss obeys

$$\int_0^t \dot{m} = m_0 - m$$

Cl.: $v(m) = v_0 \log\left(\frac{m_0}{m}\right)$

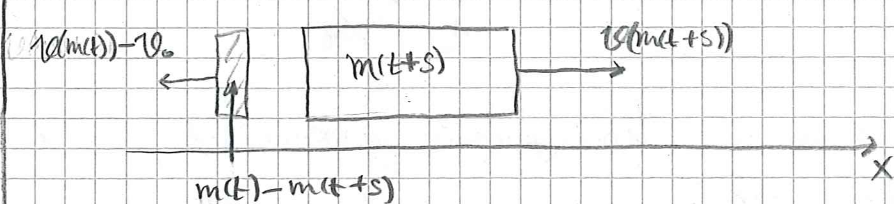
Pf.: Let $m: \mathbb{R} \to [0,\infty)$ be the function which gives the dependence of the rocket's mass on time.
$v: [0,\infty) \to \mathbb{R}$ dependence of rocket's velocity (1D) on mass.

At time t we have:



After some time interval $s > 0$ ($s \to 0^+$ eventually,) (at time $t+s$) we have:

We assume s is so small so that at t the exhaust was immediately expelled and then nothing more happened until $t+s$. That is why its speed is $v(m(t)) - v_0$



As $\nexists$ external forces, the total momentum P is constant.

$$P(t) = m(t)\, v(m(t))$$

2

$$P(t+s) = m(t+s)\, v(m(t+s)) + [m(t) - m(t+s)][v(m(t)) - v_0]$$
$$= m(t) v(m(t)) + m(t+s)[v(m(t+s)) - v(m(t))] - v_0 [m(t) - m(t+s)]$$

Since P is constant we obtain:

$$m(t+s)[v(m(t+s)) - v(m(t))] = v_0 [m(t) - m(t+s)]$$

$$\Rightarrow \frac{v(m(t+s)) - v(m(t))}{m(t+s) - m(t)} = -\frac{v_0}{m(t+s)}$$

and assuming m is cont.

$$\overset{s \to 0^+}{\Longrightarrow} (\partial_m v)(m(t)) = -v_0 [m(t)]^{-1} \quad \forall t \in \mathbb{R}.$$

Solve differential eq-n for v:

$$v' = -v_0 [\cdot]^{-1}$$
$$\Downarrow$$
$$v = -v_0 \log + C \quad \text{(function eq-n)}$$

Const (indep. of m)

To find C, employ initial cond.:

$$v(m_0) \overset{!}{=} 0$$

$$\Rightarrow -v_0 \log(m_0) + C = 0 \Rightarrow C = v_0 \log(m_0)$$

$$\Rightarrow \boxed{\begin{aligned} v(m) &= -v_0 \log(m) + v_0 \log(m_0) \\ &= -v_0 \log(m/m_0) = v_0 \log\left(\frac{m_0}{m}\right) \end{aligned}}$$

b) Now we assume a homogeneous gravitational field $g > 0$ in the direction $-\hat{x}$.

Cl.: $v(t) = v_0 \log\left(\frac{m_0}{m(t)}\right) - gt$

Pf.: We def. a new coordinate system S' which is "free falling" w.r.t. the initial coord. sys. S and s.t. at $t=0$ S' is at rest compared with S. That means:

As in eq-n (1,24) in the script with $R(t)=\mathbb{1}$, $b(t)=+\frac{1}{2}gt^2$:

$$x'(t) = x(t) + \frac{1}{2}gt^2$$
$$\dot{x}'(t) = \dot{x}(t) + gt$$
$$\ddot{x}'(t) = \ddot{x}(t) + g$$
$$= \frac{1}{m}F + g$$

Since $F=-mg$ we get $\ddot{x}'(t)=0$.

$\Rightarrow$ The grav. field is transformed away in S'.

If we repeat the derivation of part a) in S' (where $\nexists$ grav. field) we find

$$v' = v_0 \log\left(\frac{m_0}{m}\right)$$

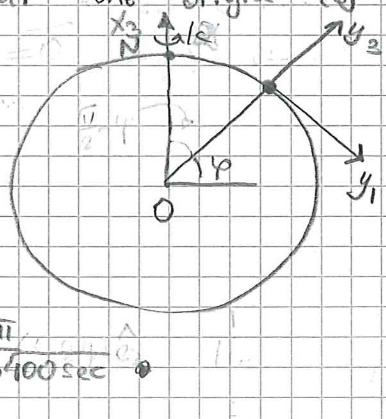
Now all that is left is to transform this result back to S:

$$v'(t) = v(t) + gt$$

$$\Rightarrow \boxed{v(t) = v_0 \log\left(\frac{m_0}{m(t)}\right) - gt}$$

Foucault's Pendulum

Q2 a) A point mass is suspended via a massless thread of length l which is fastened at the origin (of the free falling coordinate system S').



We have $k = \frac{2\pi}{\text{day}} = \frac{2\pi}{86400\,\text{sec}}$.

To write $R(t)$ (to use eq-n (1,24)) is too complicated to do directly. Hence we use the angular velocity vector.

It is given by
$$\underset{\text{in } y}{\omega} = \begin{bmatrix} -k\cos(\varphi) \\ 0 \\ k\sin(\varphi) \end{bmatrix}$$

Note $\dot{\omega}=0$. Plugging this into eq-n (1,25) in the script gives:

$$m\ddot{y} = K \underbrace{-2m(\omega\times\dot{y})}_{\text{Coriolis force}} \underbrace{-m\omega\times(\omega\times y)}_{\text{Centrifugal force}} - ma$$

Where K is the force as measured in S'

a is the acceleration of the origin of S' as measured in S.

acceleration from rotation of origin of S'

For us, $K = \underbrace{-\lambda m y}_{\substack{\text{tension} \\ \text{of thread}}} + m g'$ grav. field as measured in S'

λ to be determined

Cl.: We have $mg' - ma = m \begin{bmatrix} 0 \\ 0 \\ -c \end{bmatrix}$ with $c \doteq 9.8\, m/s^2$

Pf.: This amounts to saying $\omega^2 R$ is negligible compared to g.

As a result, the EoM for the pt. mass is:

$$\ddot{y} = \underbrace{-\lambda y}_{\text{tension}} \underbrace{- c\hat{e}_3'}_{\text{gravity}} \underbrace{-2(\omega \times \dot{y})}_{\text{Coriolis}} \underbrace{- \omega \times (\omega \times y)}_{\text{centrifugal}}$$

Cl.: If k is small ($k << \sqrt{c/\ell}$) then the centrifugal term is negligible compared to the Coriolis term.

Pf.: $k \equiv \frac{2\pi}{86'400 \sec} \approx 7.3 \cdot 10^{-5} \frac{1}{\sec}$

$$k << \sqrt{c/\ell} \iff k^2 << c/\ell \iff \ell << \frac{c}{k^2}$$

$$\frac{c}{k^2} \cong \frac{9.8\, m/s^2}{((7.3 \cdot 10^{-5})\, 1/\sec)^2} = 1.85 \cdot 10^9\, m \approx \text{diameter of the sun.}$$

$\Rightarrow$ Any pendulum on earth obeys this condition!

If $\omega = 0$ we'd have an "ideal" pendulum, whose time period is known to be $T_0 = 2\pi \sqrt{\ell/c}$:

$$\ddot{\theta} + \frac{g}{\ell} \sin(\theta) = 0 \Rightarrow \theta(t) = \theta_0 \cos(\sqrt{\tfrac{g}{\ell}}\, t)$$

$$\Rightarrow y_1 \approx \ell\theta \Rightarrow \dot{y}_1 \approx \ell\dot{\theta} = -\ell \sqrt{\tfrac{g}{\ell}}\, \theta_0 \sin(\sqrt{\tfrac{g}{\ell}}\, t)$$

$$\Rightarrow \mathcal{O}(|\dot{y}_1|) = \ell\sqrt{g/\ell}$$

$$\Rightarrow \mathcal{O}(\text{Coriolis}) = \mathcal{O}(\omega \times \dot{y}) = \omega \ell \sqrt{g/\ell}$$

But $\mathcal{O}(\text{Centrifugal}) = \mathcal{O}(\omega \times (\omega \times y)) = \omega^2 \ell$

Thus we find:

$$\boxed{\ddot{y} = -\lambda y - c\hat{e}_3' - 2(\omega \times \dot{y})}$$

In components:

$$\begin{cases} \ddot{y}_1 = -\lambda y_1 + 2k\sin(\varphi)\dot{y}_2 \\ \ddot{y}_2 = -\lambda y_2 - 2k\cos(\varphi)\dot{y}_3 - 2k\sin(\varphi)\dot{y}_1 \\ \ddot{y}_3 = -\lambda y_3 - c + 2k\sin(\varphi)\dot{y}_2 \end{cases}$$

According to our approximation $\left|\frac{y_1}{\ell}\right| << 1$ and $\left|\frac{y_2}{\ell}\right| << 1$.

Thus we neglect anything of order $\left(\frac{y_i}{\ell}\right)^2$, $i \in \{1,2\}$.

$a := \max\{\mathcal{O}(|\frac{y_1}{\ell}|), \mathcal{O}(|\frac{y_2}{\ell}|)\}$)

That is, neglect anything of order a^2.

Recall the pendulum is constrained by $\|y\| = \ell$.

$$\Rightarrow \sqrt{y_1^2 + y_2^2 + y_3^2} = \ell \iff y_3 = -\ell\sqrt{1 - \left(\frac{y_1}{\ell}\right)^2 - \left(\frac{y_2}{\ell}\right)^2}$$

We know $y_3 < 0$

$$\Rightarrow y_3 \approx -\ell\sqrt{1 - 2a^2} \approx -\ell(1 - a^2) \approx -\ell$$

$$\Rightarrow \ddot{y}_3 = 0 \text{ up to } \mathcal{O}(a^2)$$

Thus the 3rd component of the EoM is:

$$0 = \underbrace{+\lambda\ell - d}_{\text{const}} + \underbrace{2k\sin(\varphi)\dot{y}_2}_{\text{depends on time}} \qquad \forall \text{ times}$$

Two possibilities: ① $\mathcal{O}(k\dot{y}_2) \approx a^2$ and $\boxed{\lambda \cong +\frac{d}{\ell}}$

② y_2 is linear in time $\Rightarrow$ unphysical.

λ also depends on a, s.t. it exactly cancels the y_2 part.

Plugging this into the first two components of the EoM we get:

$$\boxed{\begin{cases} \ddot{y}_1 = -\frac{d}{\ell}y_1 + 2k\sin(\varphi)\dot{y}_2 \\ \ddot{y}_2 = -\frac{d}{\ell}y_2 - 2k\sin(\varphi)\dot{y}_1 \end{cases}}$$

(8) Assume now that $\dot{y}(0) = 0$

Cl.: The period $T = \frac{2\pi}{\sqrt{k^2\sin^2(\varphi) + d/\ell}}$

Pf.: Recall the period is def. as the smallest positive time t s.t. $\dot{y}(t) = 0$.

Define $z : \mathbb{R} \to \mathbb{C}$ by $z := y_1 + iy_2$.

The EoM for z is:

$$\ddot{z} = \ddot{y}_1 + i\ddot{y}_2 = -\frac{d}{\ell}y_1 + 2k\sin(\varphi)\dot{y}_2 - i\frac{d}{\ell}y_2 - 2k\sin(\varphi)i\dot{y}_1$$
$$= -\frac{d}{\ell}z - i2k\sin(\varphi)\dot{z}$$

$$\Rightarrow \boxed{\ddot{z} + \frac{c}{e} z + 2ik\sin(\varphi)\dot{z} = 0}$$

Solve w/ the Ansatz:

$$z(t) := a e^{i\alpha t}$$
$$\dot{z}(t) = i\alpha a e^{i\alpha t}$$
$$\ddot{z}(t) = -\alpha^2 a e^{i\alpha t}$$

$$-\alpha^2 + \frac{c}{e} + 2ik\sin(\varphi)\, i\alpha = 0$$

$$\alpha^2 + 2k\sin(\varphi)\alpha - \frac{c}{e} = 0$$

$$\alpha_{1,2} = \frac{1}{2}\left(-2k\sin(\varphi) \pm \sqrt{4k^2\sin^2(\varphi) + 4\, c/e}\right)$$
$$= -k\sin(\varphi) \pm \underbrace{\sqrt{k^2\sin^2(\varphi) + c/e}}_{=: w}$$

$$\boxed{\begin{aligned}\alpha_1 &= -k\sin(\varphi) + w \\ \alpha_2 &= -k\sin(\varphi) - w\end{aligned}}$$

$$\Rightarrow z(t) = a_1 e^{i\alpha_1 t} + a_2 e^{i\alpha_2 t} \qquad \dot{z}(t) = i\alpha_1 a_1 e^{i\alpha_1 t} + i\alpha_2 a_2 e^{i\alpha_2 t}$$

Now apply B.C. to find $a_{1,2}$:

$$z(0) = a_1 + a_2$$
$$\dot{z}(0) = i\alpha_1 a_1 + i\alpha_2 a_2 \overset{!}{=} 0$$

$$\Rightarrow a_1 = -\frac{\alpha_2}{\alpha_1} a_2$$

$$\Rightarrow z(0) = \left(1 - \frac{\alpha_2}{\alpha_1}\right) a_2 \Rightarrow a_2 = \frac{\alpha_1 z(0)}{\alpha_1 - \alpha_2} \qquad a_1 = \frac{\alpha_2 z(0)}{\alpha_2 - \alpha_1}$$

$$\alpha_1 a_1 = -\alpha_2 a_2$$

$$\Rightarrow \dot{z}(t) = i\alpha_1 a_1 \left(e^{i\alpha_1 t} - e^{i\alpha_2 t}\right)$$

$$\alpha_1 - \alpha_2 = 2w$$

$$\frac{\alpha_1}{\alpha_1 - \alpha_2} = \frac{-k\sin(\varphi) + w}{2w} = \frac{-k\sin(\varphi)}{2w} + \frac{1}{2}$$

$$\frac{\alpha_2}{\alpha_2 - \alpha_1} = \frac{-k\sin(\varphi) - w}{-2w} = \frac{k\sin(\varphi)}{2w} + \frac{1}{2}$$

$$\dot{z}(t) = i\alpha_1 a_1 \left(e^{i(-k\sin(\varphi)+w)t} - e^{i(-k\sin(\varphi)-w)t}\right)$$
$$= i\alpha_1 a_1 e^{-ik\sin(\varphi)t}\, 2i\sin(wt)$$

Vieta's formula: $\alpha_1\alpha_2 = -c/e$

$$= -\frac{-c/e}{-2w} z(0)\, e^{-ik\sin(\varphi)t} \sin(wt)$$

$$\boxed{\dot{z}(t) = \frac{c}{we} z(0)\, e^{-ik\sin(\varphi)t} \sin(wt)}$$

This expression is zero by sinus

$$\Rightarrow \boxed{T = \frac{2\pi}{W}}$$

Note $W \equiv \sqrt{k^2 \sin^2(\varphi) + c/e}$

$\alpha := k\sqrt{\frac{e}{c}}$

$$= \sqrt{\alpha^2 \frac{c}{e} \sin^2(\varphi) + c/e}$$

$$= \sqrt{\frac{c}{e}}\sqrt{\alpha^2 \sin^2(\varphi) + 1}$$

$$\approx \sqrt{\frac{c}{e}}\left(1 + \alpha^2 \sin^2(\varphi)\right)$$

$$= \sqrt{\frac{c}{e}}\left(1 + O(\alpha^2)\right)$$

$$\Rightarrow T = \underbrace{2\pi\sqrt{\frac{e}{c}}}_{T_0}\left(1 + O(k^2\, e/c)\right)$$

(c) Assume the same initial cond. as in (b).

Cl.: The oscillations plane spins with angular velocity $-k\sin(\varphi)$.

Pf.: From the explicit formula for $\dot{z}$ (which is of the form $\dot{z}(t) = \underbrace{\underbrace{r(t)}_{\text{real}}}_{\substack{\text{radial part}\\ \text{of velocity}\\ \text{vector}}} \underbrace{\exp(-i\, \underbrace{k\sin(\varphi)t}_{})}_{\substack{\text{angular part of}\\ \text{velocity vector}}}$

$\Rightarrow$ Angular velocity is $\boxed{-k\sin(\varphi)}$.

$\Rightarrow$ On the equator ($\varphi = 0$) $\nexists$ effect.

On the poles, the effect is maximal.

This is opposite behavior compared with deflection in free fall.