Analytical Mechanics — HW #4 — 17/10/2016

Q1 Perihelion Precession

A potential is given by:

$$V(r) := -GMmr^{-1} + \alpha r^{-3}$$

α is a small number parametrizing the perturbation from the initial Kepler problem.
Choose the physical units so that $G = m = 1$.
Define $u := r^{-1}$
The goal is to determine the curve of the trajectory, that is, the map $[0, 2\pi) \ni \varphi \longmapsto u(\varphi) \in [0, \infty)$

i) **Cl.:** The map $u: \varphi \mapsto u(\varphi)$ obeys the following diff. eq-n:

$$u'' + u = \ell^{-2}(M - 3\alpha u^2)$$

Pf.: Following the script, we have from (2.3)

$$U(r) \equiv \frac{\ell^2}{2mr^2} + V(r)$$

$$= \frac{1}{2}\ell^2 u^2 - Mu + \alpha u^3 = \tilde{U}(u)$$

$$\Rightarrow U'(u) = \ell^2 u - M + 3\alpha u^2$$

And so, placing $U'(u)$ into (2.7) we obtain:

$$u'' + \ell^{-2} U'(u) = 0$$

$$\Longleftrightarrow \quad u'' + \ell^{-2}(\ell^2 u - M + 3\alpha u^2) = 0$$

$$\Longleftrightarrow \quad \boxed{u'' + u = \ell^{-2}(M - 3\alpha u^2)}$$

ii). When $\alpha = 0$ we obtain (eq-n (2.14) in the script)

$$u'' + u = \ell^{-2} M$$

We assume u is parametrized in such a way so that at $\varphi = 0$, u is at an extremal point: (extremal pt. $\equiv$ perihelion $\overset{\text{min } r}{}$ or aphelion) $\underset{\text{max } r}{}$

$$u'(0) = 0$$

Thus we define $u_e := u(0)$ (that extremal length^{-1}).

With these B.C. we try to solve $u''+u=\ell^{-2}M$ w/ the Ansatz:

$$u(\varphi)=A\cos(\varphi)+B\sin(\varphi)+C$$
$$u''(\varphi)=-u(\varphi)+C$$

$\Rightarrow$ Solved if $C=\ell^{-2}M$.

Apply B.C. to find B and A:

$$u(0)=A+\ell^{-2}M\overset{!}{=}u_e \quad\Rightarrow\quad A=u_e-\ell^{-2}M$$
$$u'(0)=-B\overset{!}{=}0 \quad\Rightarrow\quad B=0$$

Thus the solution is:

$$\begin{aligned} u(\varphi)&=(u_e-\ell^{2}M)\cos(\varphi)+\ell^{-2}M \\ &=\ell^{-2}M[1-\cos(\varphi)]+u_e\cos(\varphi)\end{aligned}$$

We'd like to express u_e in terms of E and ℓ.
To that end, use eq-n (2.6) at $\varphi=0$. Its left hand side will be zero so we get

$$E=\tfrac{1}{2}\ell u_e^2-Mu_e \qquad \text{(we are now in the } \alpha=0 \text{ case)}$$

$$\Rightarrow u_e=\frac{M\pm\sqrt{2E\ell^2+M^2}}{\ell^2}=\underbrace{\ell^{-2}M}_{=:d^{-1}}\Big(1\pm\underbrace{\sqrt{2E(\ell^{-2}M^2)^{-1}+1}}_{=:\varepsilon}\Big)$$

eccentricity

$$=\frac{1}{d}(1\pm\varepsilon)$$ Pick $u_e=d^{-1}(1+\varepsilon)$ to get $\underbrace{\text{min. pt.}}$
(u_e max $\Leftrightarrow r_e$ min)

$$\Rightarrow u(\varphi)=d^{-1}[1-\cos(\varphi)]+d^{-1}(1+\varepsilon)\cos(\varphi)$$

$$\boxed{u(\varphi)=d^{-1}[1+\varepsilon\cos(\varphi)]}$$ for $\alpha=0$. Starts @ perihelion.

Now back to the $\alpha>0$ case:

Assume $\varepsilon\in(0,1)$ ($\Rightarrow$ Corresponds to ellipse in $\alpha=0$ case).

Cl.: The shift per orbit in the perihelion $\Delta\varphi$ as a function of the perturbation α, the semi-major axis $a\equiv d(1-\varepsilon^2)^{-1}$ and the eccentricity ε, to first order in α, is given by:

$$\boxed{\Delta\varphi=-\frac{6\pi}{Ma^2(1-\varepsilon^2)^2}\,\alpha}+\mathcal{O}(\alpha^2)$$

Pf.: We want to solve $u''+u=(d^{-1}-3\ell^{-2}\alpha u^2)$ with initial conditions:
$$\begin{cases} u'(0)=0 & \text{(also starts at perihelion)} \\ u(0)=d^{-1}(1+\varepsilon) & \text{(perihelion for } \alpha=0 \text{ case)}\end{cases}$$

If α were equal to zero we'd have periodic motion on an ellipse so that at $\varphi=2\pi$ we'd again have the perihelion.

Since $\alpha \neq 0$, we label by $\Delta\varphi$ the difference so that now the perihelion after one orbit will come at

$$\varphi_0 = 2\pi + \Delta\varphi$$

Recall that the perihelion pt. φ_0 satisfies $u'(\varphi_0) = 0$.

$$\Rightarrow \quad \underbrace{u'(\varphi_0)}_{=0} = u'(2\pi + \Delta\varphi) \approx u'(2\pi) + u''(2\pi)\Delta\varphi + \dots$$

($\Delta\varphi$ is very small)

$$\Rightarrow \quad \Delta\varphi = -\frac{u'(2\pi)}{u''(2\pi)} \quad \text{up to linear order}$$

Define a new map $[0, 2\pi) \ni \varphi \longmapsto v(\varphi) \in [0, \infty)$ by

$$v := u - u_0$$

where u is the solution we seek and u_0 is the solution with $\alpha = 0$. Thus: $v(0) = u(0) - u_0(0) = 0$

$$v'(0) = u'(0) - u_0'(0) = 0$$

and
$$\begin{aligned} v'' + v &= u'' - u_0'' + u - u_0 \\ &= u'' + u - u_0'' - u_0 \quad \text{apply diff. eq-ns} \\ &= (d^{-1} - 3\ell^{-2}\alpha u^2 - d^{-1}) \\ &= -3\ell^{-2}\alpha u^2 \\ &= -3\ell^{-2}\alpha(v - u_0)^2 \end{aligned}$$

Because when $\alpha = 0$, $v = 0$, v is at least of order α.

$\Rightarrow$ To linear order in α we obtain:

$$\begin{aligned} v'' + v &= -3\ell^{-2}\alpha u_0^2 \\ &= -3\ell^{-2}\alpha d^{-2}(1 + \varepsilon\cos)^2 \\ &= -3\ell^{-2}\alpha d^{-2}(1 + 2\varepsilon\cos + \varepsilon^2\cos^2) \\ &= \underbrace{-3\ell^{-2}\alpha d^{-2}}_{=:A_1} \underbrace{- 6\ell^{-2}\alpha d^{-2}\varepsilon}_{=:A_2}\cos \underbrace{- 3\ell^{-2}\alpha d^{-2}\varepsilon^2}_{=:A_3}\cos^2 \end{aligned}$$

$$\boxed{v'' + v = A_1 + A_2\cos + A_3\cos^2} \quad (*)$$

Once we obtain v, we may compute:

$$\Delta\varphi = -\frac{u'(2\pi)}{u''(2\pi)} = -\frac{v'(2\pi) + u_0'(2\pi)}{v''(2\pi) + u_0''(2\pi)}$$

Note $u_0'(2\pi) = 0$ by the fact u_0 is periodic and $u_0'(0) = 0$.

v is at least of order α

$$\Rightarrow \quad v'(2\pi) \approx \boxed{}\,\alpha + O(\alpha^2) \quad \text{and} \quad u_0''(2\pi) = -\varepsilon d^{-1}$$

$$v''(2\pi) \approx \boxed{}\,\alpha + O(\alpha^2)$$

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Thus $$\Delta\varphi = -\frac{v'(2\pi)}{v''(2\pi) - \varepsilon d^{-1}}$$

$$\approx -\frac{\square\alpha + O(\alpha^2)}{\diamond\alpha + O(\alpha^2) - \varepsilon d^{-1}}$$

$$\approx -\frac{\square\alpha}{-\varepsilon d^{-1}} + O(\alpha^2)$$

$$\approx -\frac{v'(2\pi)}{-\varepsilon d^{-1}} + O(\alpha^2)$$

$$v'(2\pi) = v'_{per.}(2\pi) + v'_{non\text{-}per.}(2\pi)$$
$$\underset{v_{n.p}(0)=0}{=} v_{per.}'(0) + v'_{non\text{-}per.}(2\pi)$$
$$= v'(0) + v'_{non\text{-}per.}(2\pi)$$
$$\underset{v'(0)=0}{=} v'_{non\text{-}per.}(2\pi)$$

$\Rightarrow$ We only care about $v'(2\pi)$ up to ~~linear~~ linear order in α.
But recall that $v'(0)=0$ is one of our B.C.
$\Rightarrow$ The periodic part of v' will not contribute to $\Delta\varphi$, because $v_{per}'(2\pi) = v_{per}'(0) = 0$, and $v_{non\text{-}per}(0)=0$ (via hints).
We thus discard the homogeneous solution to ⊛ (as it is of the form $\square\cos + \diamond\sin$, which is periodic) and using the hint on the exercise sheet, in eq-n (4), we only need to consider the part proportional to A_2 as it is <u>not</u> periodic.

$\Rightarrow \quad v_{non\text{-}per}(\varphi) = \frac{1}{2}A_2\,\varphi\sin(\varphi)$

$$\left.\begin{array}{ll} v_{non\text{-}per}(0) = 0 & \checkmark \\ v_{non\text{-}per}'(0) = 0 & \checkmark \end{array}\right\} \Rightarrow \text{Obeys B.C.}$$

$$v_{non\text{-}per}'(2\pi) = \tfrac{1}{2}A_2\, 2\pi = \pi A_2 = \pi\left(-6\ell^{-2}\alpha d^{-2}\varepsilon\right)$$

$$\Rightarrow \quad \Delta\varphi = -\frac{\pi\, 6\,\ell^{-2} d^{\cancel{-2}\,-1}\cancel{\varepsilon}}{\cancel{\varepsilon d^{-1}}}\alpha + O(\alpha^2)$$

$$= -6\pi M \ell^{-4}\alpha + O(\alpha^2)$$

Next note that $M\ell^{-4} = \left[Ma^2(1-\varepsilon^2)^2\right]^{-1}$. Indeed,
$$\left[Ma^2(1-\varepsilon)^2\right]^{-1} = \frac{1}{M\left(\ell^2M^{-1}(1-\varepsilon^2)^{-1}\right)^2(1-\varepsilon^2)^2} = M\ell^{-4}.$$

Two Applications of the Above Calculation

iii) We now want to analyze the motion of a satellite around the earth. The precession will come from the effect of the earth not being flat. In order to account for the deviation from a perfect sphere, we assume a perturbation from the point-like gravitational potential by:

$$V(r) = \underbrace{-Mr^{-1}}_{\text{unperturbed potential}} \underbrace{-J\frac{M}{2r^3}\left(\frac{R}{r}\right)^2(r^2-3x_3{}^2)}_{\text{perturbation due to earth's flattening}}$$

where $R \equiv$ Earth's averaged radius
$M \equiv$ Earth's mass (should be total mass, but approx. equal)
$J \equiv$ undetermined constant, measure of earth's oblateness.

and we assume the z-axis points toward north
This perturbation corresponds to a quadrupole field.

Cl.: If a satellite is orbiting earth on the equitorial plane then

$$J = \frac{(\Delta\varphi)a^2(1-\varepsilon^2)^2}{3\pi R^2}$$

where $\Delta\varphi$ is the satellite's perihelion precession,
a, ε are its semi-major axis and its eccentricity resp.

Pf.: Cl.: For equitorial orbits, we may use the potential as

$$V(r) = -Mr^{-1} - \tfrac{1}{2}JMR^2r^{-3}$$

Pf.: The actual potential is of the form

$$V(r) = -Mr^{-1} - \tfrac{1}{2}JMR^2r^{-3} + \tfrac{3}{2}JMR^2\frac{x_3{}^2}{r^5}$$

So our goal is to show the last term has no effect for equitorial orbits.
Note that the force on the satellite is given by:

$$-(\nabla V)(r)$$

Hence the force corresponding to the last term is proportional to:

$$\nabla\left(\frac{x_3{}^2}{r^5}\right) = \partial_i\left(\frac{x_3{}^2}{r^5}\right)e_i \qquad \text{(sum over } i \text{ implied)}$$

$$= \frac{2x_3\delta_{i3}r^5 - x_3{}^2\, 5r^4\frac{x_i}{r}}{r^{10}} = x_3\cdot(\text{something})$$

$$\Rightarrow \nabla\left(\frac{x_3{}^2}{r^5}\right)\Bigg|_{x_3=0} = 0$$

But equitorial orbits precisely have $x_3=0$ for their trajectories.
$\Rightarrow$ The last term does not contribute any force, and so may be disregarded.

As a result of the above claim, we can use the first part of

[6] the question with $\alpha = -\frac{1}{2} J M R^2$ to obtain:

$$\Delta\varphi = -\frac{6\pi}{Ma^2(1-\varepsilon^2)^2}\,\alpha$$

$$= + \frac{\overset{3}{\cancel{6}}\pi}{\cancel{M}a^2(1-\varepsilon^2)^2}\,\frac{1}{\cancel{2}}\,J\cancel{M}R^2$$

$$\Rightarrow \boxed{J = \frac{\Delta\varphi\, a^2 (1-\varepsilon^2)^2}{3\pi R^2}}$$

For example, with the following observational data we may "predict" J:

$$\Delta\varphi = 3.08\cdot 10^{-3}$$
$$a = 2R \qquad (R \equiv \text{earth's radius})$$
$$\varepsilon = 0.3$$

$$J = \frac{3.8\cdot 10^{-3}\;4\;(1-0.3^2)^2}{3\pi} = 1.08\cdot 10^{-3}$$

ib) According to Einstein's general theory of relativity, the perturbation from Newton's usual gravitational law follows the same form which we've calculated, where one has to take α equal to:

$$\alpha = -M\ell^2$$

Hence we find that according to G.R.,

$$\Delta\varphi = -6\pi M \ell^{-4}\alpha \qquad \text{(intermediate expr. for } \Delta\varphi \text{ found b4)}$$
$$= -6\pi M \ell^{-4}(-M\ell^2)$$
$$= 6\pi M^2 \ell^{-2}$$
$$= 6\pi M \frac{1}{a(1-\varepsilon^2)} \qquad M\ell^{-2} = [a(1-\varepsilon^2)]^{-1}$$

Cl.: If we restore the constants (G, c, m) we get:

$$\boxed{\Delta\varphi = \frac{6\pi G M}{c^2 a (1-\varepsilon^2)}}$$

TODO: Show m drops out of the calculations.

Pf.: Note $[\Delta\varphi] = [\text{radians}] = 1 \Rightarrow$ There should be no units left in the end. In the formula above we have:

$$[\Delta\varphi] = [M][a^{-1}] = \text{mass}\times\text{length}^{-1} \neq 1$$

However, $[G][c^{-2}] = \underbrace{\text{length}^3 \times \text{mass}^{-1} \times \text{time}^{-2}}_{[G]} \times \underbrace{\text{length}^{-2} \times \text{time}^2}_{[c^{-2}]}$

$$= \text{length} \times \text{mass}^{-1}$$

$$\Rightarrow [G][c^{-2}][M][a^{-1}] = 1$$

So apparently Gc^{-2} is precisely the constant with the correct units, and since $\nexists$ other natural constants, this must be the appropriate combination.
One could also go back and ask what it means to have $G = c = 1$ (i.e., it means to measure in units s.t. mass × length^{-1} is measured in units of $G^{-1}c^2$)

Note that the perihelion precession will be strongest when a is minimal $\Rightarrow$ Look at Mercury, which is the closest planet to the sun.

$$\Delta\varphi^{G.R.}_{Mercury} = \frac{6\pi \; 6.67\cdot 10^{-11}\, m^3\, kg^{-1} s^{-2} \cdot \overbrace{1.98\cdot 10^{30}\, kg}^{\text{Mass of sun}}}{(2.99\cdot 10^{8}\, m/s)^2 \; \underbrace{58\cdot 10^{9}\, m}_{\text{Semimajor axis for Mercury}} \; (1-\underbrace{0.205}_{\varepsilon \text{ eccentricity for Mercury}}{}^2)} =$$

$$\cong \boxed{5.01\cdot 10^{-7}} \quad \text{(per period)}$$

✓

$$\frac{2\pi}{\Delta\varphi^{G.R.}_{Mercury}} = 12.5\cdot 10^{7}$$ orbits to make a full rotation of the orbit's perihelion.

By Kepler's 3rd law (eq-n (2.15) in the script),

$$T = \frac{2\pi}{\sqrt{GM}}\, a^{3/2} = \frac{2\pi \; (58\cdot 10^{9}\, m)^{3/2}}{\sqrt{6.67\cdot 10^{-11} m^3\, kg^{-1} s^{-2} \; 1.98\cdot 10^{30} kg}} =$$

$$\approx 7.63\cdot 10^{6}\ \text{sec} = 0.24\ \text{year} = 88\ \text{days}$$

$$\Rightarrow \Delta\varphi^{G.R.}_{Mercury} \text{ per 100 years} = 5.01\cdot 10^{-7}\cdot 0.24^{-1}\cdot 100$$
$$= 2.08\cdot 10^{-4}\ \text{rad}$$
$$= 2.08\cdot 10^{-4}\ 4.84^{-1}\cdot 10^{6}\ \text{arcsec}$$
$$= \boxed{43\ \text{arcsec}}$$

Note: This is within 1% of observations.
Was one of the earliest verifications of G.R.

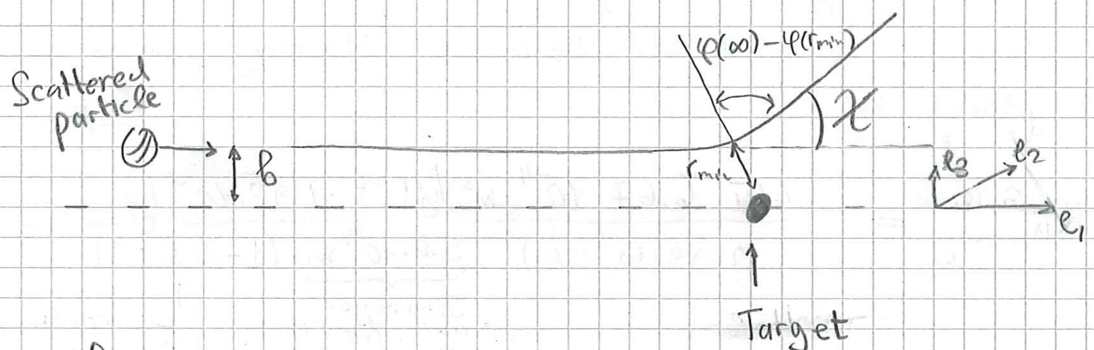
Calculation of Scattering Cross Section ⑥

⑥

A particle of energy E is subject to the repulsive central force of the form $F(x) = k\|x\|^{-4}x$

Note that $F(r) = -(\nabla V)(r)$, so that $V(r) = \frac{1}{2}kr^{-2}$ works!

$$-(\nabla V)(r) = -\frac{1}{2}k(-2)r^{-3}\frac{x}{r} = k\|x\|^{-4}x \checkmark$$



b is called the impact parameter.

χ is called the scattering angle.

$r_{min} \equiv$ closest distance to target along trajectory

Note that as in the script we have $V(r) \to 0$ as $r \to \infty$.
As a result, assuming $\|x(-\infty)\| \to \infty$ (particle comes from far away) we find:

$$E = \frac{1}{2}m\|\dot{x}(t)\|^2 + V(\|x(t)\|)$$
$$= \lim_{t\to-\infty} \frac{1}{2}m\|\dot{x}(t)\|^2$$
$$\Rightarrow \quad m\|\dot{x}(t)\| \xrightarrow{t\to-\infty} \sqrt{2mE}$$

At $t=-\infty$, $x = be_3$, $\dot{x} \parallel e_1$

$$\ell \equiv \|L\| \equiv \|x \times m\dot{x}\| = b\sqrt{2mE}$$

(a) Cl.: $$\chi(b) = \pi\left(1 - \frac{\sqrt{2E}\,b}{\sqrt{2Eb^2+k}}\right)$$

Pf.: Recall from eq-n (2.5) in the script that

$$\varphi(r) = +\varphi(r_0) \pm \int_{r_0}^{r} \frac{\ell\, dx}{x^2\sqrt{2m(E-\mathcal{U}(x))}}$$

$$\Rightarrow \quad \underbrace{\varphi(\infty) - \varphi(r_{min})}_{\text{deflection of angle from } r_{min} \text{ till } \infty.} = \int_{r_{min}}^{\infty} \frac{\ell\, dx}{x^2\sqrt{2m(E-\mathcal{U}(x))}}$$

$\equiv \pi/2 - \frac{1}{2}\chi$

Thus we find:

$$\chi = \pi - 2\int_{r_{min}}^{\infty} \frac{\ell\, dx}{x^2\sqrt{2m(E-U(x))}}$$

$$\ell^2 = b^2 2mE$$

$$\frac{\ell^2}{2mr^2} = \frac{b^2 \cancel{2m}E}{\cancel{2m}r^2}$$

$$= \pi - 2\int_{r_{min}}^{\infty} \frac{b\sqrt{2mE}\, dx}{x^2\sqrt{2m(E - b^2Ex^{-2} - V(x))}}$$

$$= \pi - 2\int_{r_{min}}^{\infty} \frac{b\, dx}{x^2\sqrt{1 - b^2x^{-2} - E^{-1}V(x)}}$$

$$V(x) = \frac{1}{2}kx^{-2}$$

$$= \pi - 2\int_{r_{min}}^{\infty} \frac{b\, dx}{x^2\sqrt{1 - b^2x^{-2} - E^{-1}\frac{1}{2}kx^{-2}}}$$

$$= \pi - 2b\int_{r_{min}}^{\infty} \frac{dx}{\sqrt{x^4 - (b^2 + \frac{1}{2}E^{-1}k)x^2}}$$

Note that the extremal pt. r_{min} can be obtained via:

$$\|\dot{x}(t_{ext.})\| \equiv 0, \quad \|x(t_{ext.})\| \equiv r_{min}$$

$$\Rightarrow \quad E = \frac{1}{2}m\underbrace{\|\dot{x}(t_{ext.})\|^2}_{=0} + U(r_{min})$$

$$\Rightarrow \quad E = \frac{b^2E}{r_{min}^2} + \frac{1}{2}k\, r_{min}^{-2} \Rightarrow r_{min} = \sqrt{\frac{b^2E + \frac{1}{2}k}{E}} = \sqrt{b^2 + \frac{1}{2}E^{-1}k}$$

$$= \pi - 2b\int_{r_{min}}^{\infty} \frac{dx}{\sqrt{x^4 - r_{min}^2x^2}}$$

$$t := r_{min}^{-1}x$$

$$= \pi - 2b\int_{1}^{\infty} \frac{r_{min}\, dt}{\sqrt{r_{min}^4t^4 - r_{min}^4t^2}}$$

$$= \pi - \frac{2b}{r_{min}}\underbrace{\int_{1}^{\infty} \frac{dt}{\sqrt{t^4 - t^2}}}_{I = \pi/2 \text{ (by Mathematica)}}$$

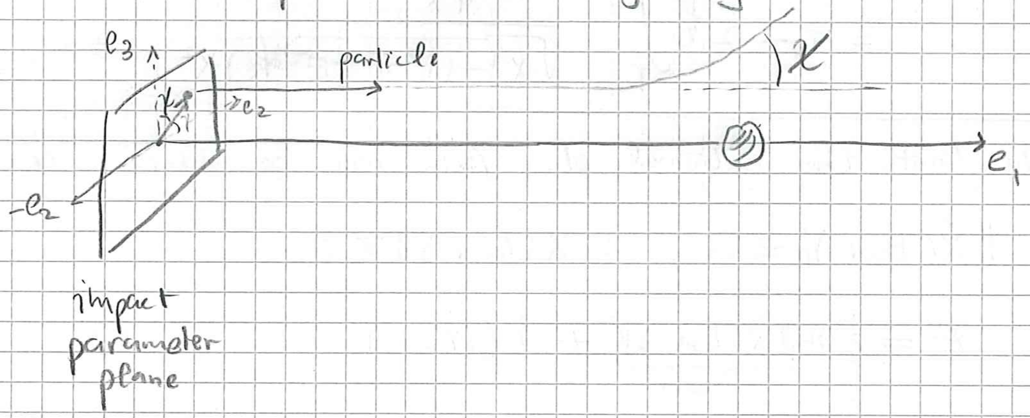
Note we also know for $k=0$ (no target), $\chi \equiv 0$.

$$\Rightarrow \chi\big|_{k=0} = \pi - \frac{2b}{r_{min}}\bigg|_{k=0} I \overset{!}{=} 0 \qquad \left(r_{min}\big|_{k=0} = b \right)$$

So without calculating the integral we find $I = \pi/2$ as well.

$$\Rightarrow \chi = \pi - \pi b\, r_{min}^{-1}$$
$$= \pi\left(1 - \frac{b}{r_{min}}\right)$$
$$= \pi\left(1 - \frac{b}{\sqrt{b^2 + \frac{1}{2}kE^{-1}}}\right) \qquad r_{min} = \sqrt{b^2 + \frac{1}{2}kE^{-1}}$$
$$= \pi\left(1 - \frac{b\sqrt{2E}}{\sqrt{2Eb^2 + k}}\right)$$

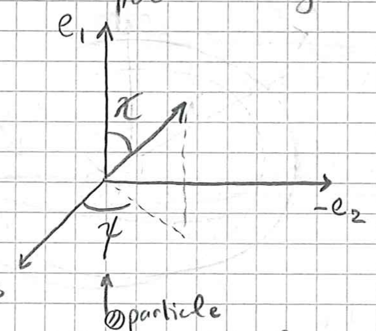
(b) The differential cross section is defined as the ratio between the differential area element in the plane of the impact parameter and the solid angle element in the direction of the particle's exit trajectory



Note that ψ, χ define a point on the 2-sphere S^2:

ψ is the azimuthal angle

χ is the polar angle



$\Rightarrow$ Differential area element on impact param. plane:

$$d\sigma = b\, db\, d\psi$$

Different solid angle element on S^2 around target:

$$d\Omega = \sin(\chi)\, d\chi\, d\psi$$

$$\Rightarrow \left|\frac{d\sigma}{d\Omega}\right| = \left|\frac{b\, db\, \cancel{d\psi}}{\sin(\chi)\, d\chi\, \cancel{d\psi}}\right| = \left|\frac{b}{\sin(\chi)}\frac{db}{d\chi}\right| = \frac{b}{\sin(\chi)}\left|\left(\frac{d\chi}{db}\right)^{-1}\right|$$

Cl.: In our problem,

$$\frac{d\sigma}{d\Omega} = \frac{k}{2\pi E} \frac{1-x}{x^2(2-x)^2 \sin(\pi x)}$$

where $x \equiv \chi \pi^{-1}$

Pf.: We have from (a) $\chi(b) = \pi\left(1 - \frac{b\sqrt{2E}}{\sqrt{2Eb^2+k}}\right)$

$$\Rightarrow \frac{d\chi}{db} = \pi\left(-\sqrt{2E}\,\frac{\sqrt{2Eb^2+k} - b\,\frac{1}{2}(2Eb^2+k)^{-1/2}\,2E\,2b}{2Eb^2+k}\right)$$

$$= -\sqrt{2E}\,\pi\left(\frac{1}{\sqrt{2Eb^2+k}} - \frac{1}{2}\,\frac{4Eb^2}{(2Eb^2+k)^{3/2}}\right)$$

$$= \frac{-\sqrt{2E}\,\pi}{(2Eb^2+k)^{3/2}}\left(\cancel{2Eb^2}+k-\cancel{2Eb^2}\right) = -\frac{\pi k\sqrt{2E}}{(2Eb^2+k)^{3/2}}$$

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin(\chi)}\left|\left(\frac{d\chi}{db}\right)^{-1}\right| = \frac{b}{\sin(\pi x)}\,\frac{(2Eb^2+k)^{3/2}}{\pi k\sqrt{2E}}$$

Next note

$$1-x = 1-\frac{\chi}{\pi} = \frac{b\sqrt{2E}}{\sqrt{2Eb^2+k}}$$

$$x(2-x) = 1-(1-x)^2 = 1-\frac{b^2\,2E}{2Eb^2+k} = \frac{k}{2Eb^2+k}$$

$$\Rightarrow \frac{1-x}{x^2(2-x)^2} = \frac{b\sqrt{2E}}{\sqrt{2Eb^2+k}}\,\frac{(2Eb^2+k)^2}{k^2} = \frac{b\,(2Eb^2+k)^{3/2}}{k\sqrt{2E}}\,\frac{2E}{k}$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{1}{\pi\sin(\pi x)}\,\frac{k}{2E}\,\frac{1-x}{x^2(2-x)^2}$$