Q1 The Discovery of Extrasolar Planets

Star: Body shining via thermonuclear fusion,

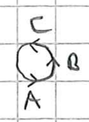
Planet: Body orbiting a star; not massive enough to cause thermonuclear fusion.

Exoplanet ≡ Extrasolar planet: A planet located outside our solar system.

The velocity of a star may be measured using the Doppler effect: The effect of a shift in a wave's frequency due to the movement of its source. We are familiar with this effect from day to day life when we hear a police car with a siren approaching. The sound of the siren have the pitch higher when the source is approaching.

Light behaves in the same way, in particular, by observing the color of stars versus time we can conclude their velocity.

If we assume a star performs circular motion, and that the line between the earth and the star is on the same plane which the circle defines, then we may assume the maximal velocity of the star which we measure is actually its constant speed of circular revolution:



Earth

A: speed max pos. C: speed max. neg.

B: speed min

Cl.: For circular motion $2\pi\|X\| = \|\dot{X}\| T$

where $\|X\|$ is the radius of the circle

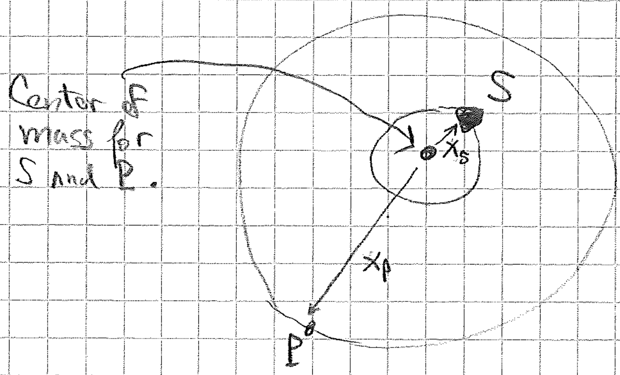
$\|\dot{X}\|$ is the (constant) speed of orbiting

T is the period of the orbit

Pf.: $\|\dot{X}\| = \frac{2\pi\|X\|}{T}$ as in circular motion $\|X\|$ and $\|\dot{X}\|$ are const.

$\Rightarrow$ From observing the star's speed vs. time and assuming circular motion with the geometric setup described above we can obtain the radius of the circular orbit.

Now we assume that $\exists$ a planet (which we cannot see) orbiting near that star, also in circular motion:



With the same geometric setup as above (line from earth to (C.o.M. of S and P) is on the same plane defined by either of the orbits), we want to calculate m_p and $\|x_p\| + \|x_s\|$.

We work in a coordinate system where (C.o.M. of S and P) is fixed at the origin.

$$\Rightarrow \frac{1}{m_p+m_s}(m_p x_p + m_s x_s) = 0$$

$$\Rightarrow \boxed{m_p x_p = -m_s x_s} \quad (*)$$

Note also that for such circular motion,

$r \equiv \|x_s - x_p\| = \|x_s\| + \|x_p\|$

is a constant — the same constant appearing in eq-n (2.15) in

The script as the constant a:

$$a \equiv d(1-\varepsilon^2)^{-1}$$ $\varepsilon=0$ for circular motion

$$= d$$ We have $r(\varphi)^{-1} = d^{-1}[1+\varepsilon\cos(\varphi)]$

$$= r(\varphi)^{-1\,-1}$$

$$= r(\varphi)$$

$$\equiv \|X_S - X_P\|$$ circular diametrical motion

$$= \|X_S\| + \|X_P\|$$

Thus eq-n (2.15) reads:

$$T = \frac{2\pi}{\sqrt{G(m_S+m_P)}} (\|X_S\|+\|X_P\|)^{3/2}$$

with T being the period of the motion.
T is def. as the period of the circle that $X_S - X_P$ makes around zero. Also the period of the orbit of S around P or the period of the orbit of P around S — both are equal to T bcs. we have circular diametrical motion.

$$\Rightarrow \boxed{\|X_S\|+\|X_P\| = \left(\frac{\sqrt{G(m_S+m_P)}\,T}{2\pi}\right)^{2/3} \approx \left(\frac{\sqrt{Gm_S}\,T}{2\pi}\right)^{2/3}}$$

Using the given numbers we find:

$$\|X_S\| + \|X_P\| = \frac{\sqrt[3]{Gm_S}}{(2\pi)^{2/3}}\, T^{2/3} = \frac{\sqrt[3]{G\,0.95M_\odot}\, T^{2/3}}{(2\pi)^{2/3}}$$

$$= \frac{\sqrt[3]{0.95\cdot 1.34\cdot 10^{20}}\, m/\cancel{s^{2/3}}\, (4.23\cdot 86400)^{2/3}\, \cancel{s^{2/3}}}{(2\pi)^{2/3}}$$

$$= 7.55\cdot 10^{9}\ m$$

Next, we obtain m_p from ⊛:

$$m_p x_p = - m_s x_s$$

$$\Rightarrow \quad m_p \|x_p\| = m_s \|x_s\|$$

circular motion

$$\Rightarrow \quad m_p = \frac{m_s \|x_s\|}{\|x_p\|} = \frac{m_s \frac{T\|\dot{x}_s\|}{2\pi}}{\|x_p\| + \|x_s\| - \frac{T\|\dot{x}_s\|}{2\pi}}$$

$$= \frac{1}{2\pi} \frac{m_s T \|\dot{x}_s\|}{\left(\frac{\sqrt{G m_s}\, T}{2\pi}\right)^{2/3} - \frac{T\|\dot{x}_s\|}{2\pi}}$$

Circular motion
$\|\dot{x}_s\| = v_{max}$

$$= \frac{m_s T v_{max}}{2\pi \left(\frac{\sqrt{G m_s}\, T}{2\pi}\right)^{2/3} - T v_{max}}$$

$$= m_s \frac{1}{\sqrt[3]{\frac{2\pi G m_s}{T v_{max}^3}} - 1}$$

Plugging in the numbers we find:

$$\frac{1}{\sqrt[3]{\frac{2\pi G m_s}{T v_{max}^3}} - 1} = \frac{1}{\sqrt[3]{\frac{2\pi\, 0.95\; 1.34\; 10^{20}\; \cancel{m^3}/\cancel{s^2}}{4.23 \cdot 86400\, \cancel{s}\;\; 59^3\; \cancel{m^3}/\cancel{s^3}}} - 1}$$

$$= 0.45 \cdot 10^{-3}$$

$$\Rightarrow \quad m_p = 0.45 \cdot 10^{-3}\, m_s = 0.45 \cdot 10^{-3}\, 0.95 M_\odot$$

$$= 0.43 \cdot 10^{-3} M_\odot$$

≈ half the mass of Jupiter, the biggest planet of our solar system.

Remarks: (1) Official solution makes the approx. $\frac{m_p}{m_s} \ll 1$.

$\overset{(*)}{\Rightarrow} \frac{\|X_s\|}{\|X_p\|} \ll 1$

So actually to be consistent we should have ignored the difference between $\|X_p\|+\|X_s\|$ and $\|X_p\|$ in the calculation of m_p to get:

$$m_p = m_s \frac{\|X_s\|}{\|X_p\|} \approx m_s \frac{\|X_s\|}{\|X_p\|+\|X_s\|}$$

and then plug in the value of $\|X_p\|+\|X_s\|$ found before.

(2) Since in reality we do not know the orientation of the plane of motion w.r.t. line of sight from earth, all we get is a bound (lower, or upper?).

Q2 A Journey to Mars

The Hohmann transfer orbit is an elliptical orbit used to transfer between two circular orbits of different radii in the same plane.

So assume $\exists$ two bodies E and M in circular orbits around a common center S. The motion of E and M is not diametrical as in the previous question.

Assume $R_E < R_M$.

The Hohmann transfer orbit is then the half ellipse whose perihelion is given by R_E,

aphelion R_M

One focus is at S.

Except at beginning and end, only force of gravity from S applies!

6 (a) Cl.: The time of flight on the Hohmann transfer orbit $\frac{1}{2}T$ given by

$$\frac{1}{2}T = \frac{1}{2}\left(\frac{1}{2}(1+q)\right)^{3/2} T_E$$

where $q \equiv R_M / R_E$

Pf.: Recall the ellipse has the trajectory from the focus given by (eq-n below (2.14) in script)

$$u(\varphi) = \ell^{-2} M [1-\cos(\varphi)] + u_e \cos(\varphi)$$

where u_e is the position at $\varphi = 0$

ℓ angular mom of body along Hohmann transf. orbit

$M \equiv m + M_\odot \simeq M_\odot$

mass of body along Hohmann orbit

mass of sun

Recall we def $d := \ell^2 M^{-1}$, $\varepsilon := \sqrt{2E\ell^2 M^{-2} + 1}$

$a := d(1-\varepsilon^2)^{-1}$

and we also found $u_e = d^{-1}(1 \pm \varepsilon)$.

$\Rightarrow R_E \equiv$ Perihelion length $= d(1+\varepsilon)^{-1} = a(1-\varepsilon)$

$R_M \equiv$ Aphelion length $= d(1-\varepsilon)^{-1} = a(1+\varepsilon)$

don't know ε

$\Rightarrow R_E + R_M = 2a$

Now use Kepler's 3rd law (eq-n (2.15) in the script) to find:

$$T = \frac{2\pi}{\sqrt{GM}} a^{3/2} = \frac{2\pi}{\sqrt{GM}} \left[\frac{1}{2}(R_E + R_M)\right]^{3/2}$$

$T_E = \frac{2\pi}{\sqrt{GM}} R_E^{3/2}$

$$\stackrel{\downarrow}{=} T_E R_E^{-3/2} \left[\frac{1}{2}(R_E + R_M)\right]^{3/2}$$

$$= T_E \left[\frac{1}{2}(1+q)\right]^{3/2}$$

Putting in the numbers we find:

$$T = T_E \left[\frac{1}{2}\left(1+\frac{230\cdot 10^6\,\mathrm{km}}{150\cdot 10^6\,\mathrm{km}}\right)\right]^{3/2}$$

$$= 1{,}42\ T_E$$

$$\Rightarrow \boxed{\tfrac{1}{2}T = 0{,}71\ T_E}$$

(b) What is the speed of the body along the Hohmann trajectory:

① Relative to E right after start?

② Relative to M right before end?

E and M are in circular orbits, so that we have

$$\frac{V_\alpha^2}{R_\alpha} = A_\alpha = \frac{F_\alpha}{M_\alpha} = G\frac{M_\alpha M}{R_\alpha^2}\frac{1}{\cancel{M_\alpha}} \qquad \forall\ \alpha \in \{E, M\}$$

$$\Rightarrow V_\alpha = \sqrt{\frac{GM}{R_\alpha}}$$

To get the (absolute) speed of the transferring body, note that at the two positions (perihelion & apehelion) the velocity is perpendicular to the radial direction. Hence:

$$l \equiv \|L\| \equiv \|x \times p\| \equiv \|x \times m\dot{x}\| = m\|x(t_\alpha)\|\,\|\dot{x}(t_\alpha)\|$$

@ peri- or apehelion

where $\alpha \in \{E, M\}$, $t_E := 0$ (start @ E)

$t_M := \frac{1}{2}T$ (end @ M)

$$\text{Also,}\quad \|x(t_\alpha)\| = \begin{cases} R_E & \alpha = E \\ R_M & \alpha = M \end{cases} \equiv R_\alpha$$

$$\Rightarrow \|\dot{x}(t_\alpha)\| = \frac{l}{m R_\alpha}$$

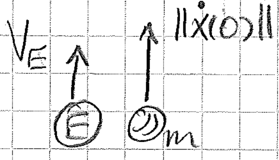
$a \equiv d(1-\varepsilon^2)^{-1}$ $\qquad a(1-\varepsilon^2) = \frac{R_E R_M}{a}$

$d \equiv l^2 M^{-1}$ $\qquad a = \frac{1}{2}(R_E+R_M)$ $\qquad$ Restore units from eqns of HW1

$$l = \sqrt{Md} = \sqrt{Ma(1-\varepsilon^2)} = \sqrt{M\frac{R_E R_M}{a}} = \sqrt{2M\frac{R_E R_M}{R_E+R_M}} = \sqrt{2GMm^2\frac{R_E R_M}{R_E+R_M}}$$

$$\Rightarrow \boxed{\|\dot{x}(t_\alpha)\| = \sqrt{2GM \frac{R_E R_M}{R_\alpha^2} \frac{1}{R_E+R_M}}}$$

(1) For $\alpha = E$,



$\Rightarrow$ The Relative speed is:

$$\|\dot{x}(0)\| - V_E = \sqrt{2GM \frac{R_M}{R_E} \frac{1}{R_M+R_E}} - \sqrt{GM \frac{1}{R_E}}$$

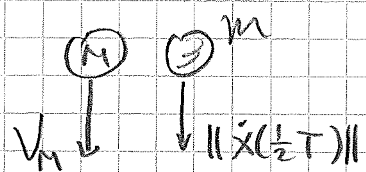
$$= \sqrt{\frac{GM}{R_E}} \left(\sqrt{\frac{2R_M}{R_M+R_E}} - 1 \right)$$

$$= \sqrt{\frac{GM}{R_E}} \left(\sqrt{\frac{2}{1+q^{-1}}} - 1 \right)$$

$$= \sqrt{\frac{GM}{R_E}} \left(\sqrt{\frac{2q}{q+1}} - 1 \right)$$

typo in official solutions?

(2) for $\alpha = M$



Hence the direction is opposite so that the signs must be reversed.

The relative speed is:

$$V_M - \|\dot{x}(\tfrac{1}{2}T)\| = \sqrt{\frac{GM}{R_M}} - \sqrt{\frac{2GMR_E}{R_M} \frac{1}{R_M+R_E}}$$

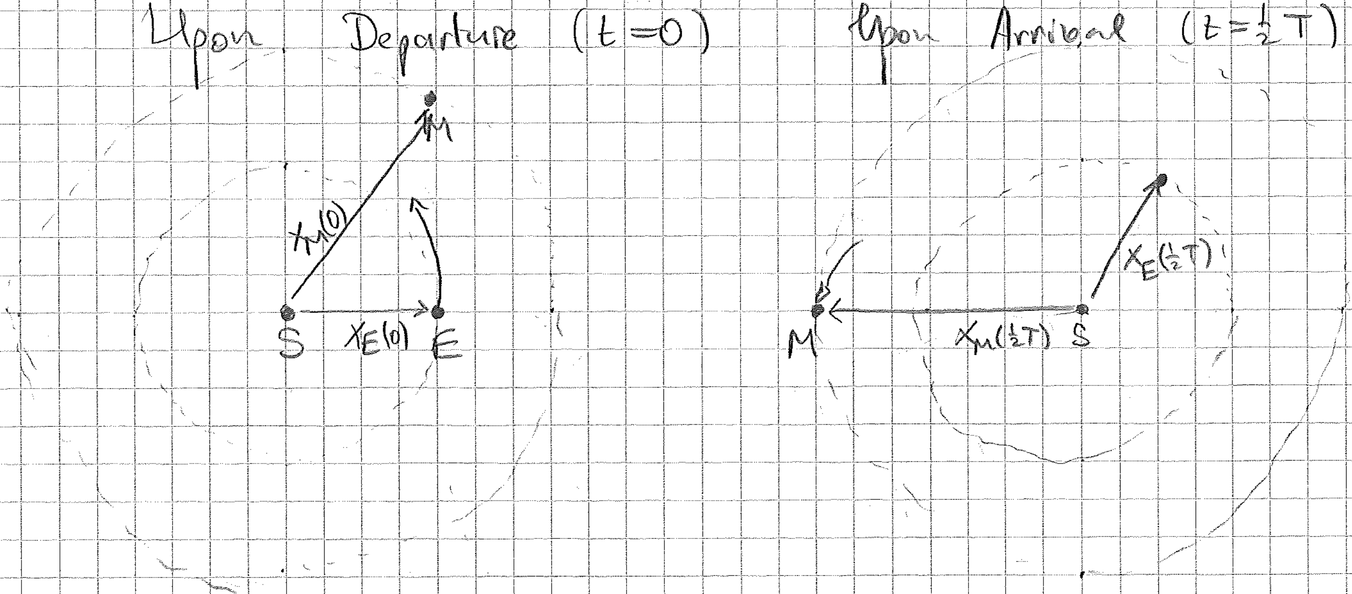
$$= \sqrt{\frac{GM}{R_M}} \left(1 - \sqrt{\frac{2R_E}{R_M+R_E}} \right) = \sqrt{\frac{GM}{R_M}} \left(1 - \sqrt{\frac{2}{q+1}} \right)$$

If we plug in the numbers we get:

$$\|\dot{x}(0)\| - V_E = \sqrt{\frac{1.34 \cdot 10^{20}\, m^3/s^2}{150 \cdot 10^6 \cdot 10^3\, m}} \left(\sqrt{\frac{2 \cdot 1{,}53}{1.53+1}} - 1 \right) = 2{,}98\ km/s$$

$$V_M - \|\dot{x}(\tfrac{1}{2}T)\| = 2{,}67\ km/s$$

(c) How much time has to pass (at least) from the arrival (at $t=\frac{1}{2}T$) to embark on the opposite trajectory?



We denote by X_α the trajectories of $\alpha \in \{E, M\}$ as a function of time, and by φ_α the angle, as measured from the focus (S) as a function of time.

Then: $\varphi_E(0) \equiv 0$ | $\varphi_E(\frac{1}{2}T) = ?$

$\varphi_M(0) = ?$ | $\varphi_M(\frac{1}{2}T) \equiv \pi$ eq-n (2.2) in script

We have $\dot{\varphi}_\alpha = \frac{l_\alpha}{M_\alpha R_\alpha^2}$ = const for circular motion

$$\Rightarrow \varphi_\alpha(t) = \frac{l_\alpha}{M_\alpha R_\alpha^2} t + C_\alpha$$

By $\varphi_E(0)=0$, $C_E=0$.

By $\varphi_E(T_E) \equiv 2\pi$, $\frac{l_E}{M_E R_E^2} T_E = 2\pi \Rightarrow \frac{l_E}{M_E R_E^2} = \frac{2\pi}{T_E}$

$$\Rightarrow \boxed{\varphi_E(t) = \frac{2\pi}{T_E} t} \quad \text{In particular,} \quad \boxed{\varphi_E(\tfrac{1}{2}T) = \frac{\pi T}{T_E}}$$

$$\varphi_M(\tfrac{1}{2}T + T_M) \overset{!}{=} 3\pi$$

$$\Rightarrow \frac{l_M}{M_M R_M^2}(\cancel{\tfrac{1}{2}T} + T_M) + \cancel{C_M} = \cancel{\frac{l_M}{M_M R_M^2}\tfrac{1}{2}T} + \cancel{C_M} + 2\pi \Rightarrow \frac{l_M}{M_M R_M^2} = \frac{2\pi}{T_M}$$

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$$\Rightarrow \quad \varphi_\alpha(t) = \omega_\alpha t + c_\alpha \qquad \text{with} \quad \omega_\alpha := \frac{2\pi}{T_\alpha}$$

$$c_E = 0$$

$$c_M = \varphi_M(\tfrac{1}{2}T) - \omega_M \tfrac{1}{2}T$$
$$= \pi - \omega_M \tfrac{1}{2}T$$

$$\Rightarrow \quad \boxed{\varphi_M(t) = \omega_M(t - \tfrac{1}{2}T) + \pi}$$

Denote by τ the (still unknown) time of departure on the opposite trajectory ($\tau \geq \frac{1}{2}T$).
By departure on the 2^{nd} trajectory, we know:

$$\varphi_M(\tau) = \pi$$

In order to have a successful trip back, we must have

$$\varphi_E(\tau) = -\varphi_E(\tfrac{1}{2}T) + 2\pi n \quad \exists\, n \in \mathbb{Z}$$

(because we know the opposite trajectory has the same period T, as it has the same perihelion & aphelion, so the same a, so by Kepler's 3^{rd}, the same period.
Thus, at time $\tau + \frac{1}{2}T$, $\varphi_E(\tau + \frac{1}{2}T) \stackrel{!}{=} 2\pi n \quad \exists\, n \in \mathbb{Z}$

$$\Rightarrow \quad \varphi_E(\tau) = -\varphi_E(\tfrac{1}{2}T) + 2\pi n \;)$$

φ_E lin. in t

$$\Rightarrow \quad \varphi_E(\tau) - \varphi_M(\tau) = -\varphi_E(\tfrac{1}{2}T) + 2\pi n - \pi$$

But by the linear dependence of φ_α on t we also have

$$\varphi_\alpha(\tau) = \varphi_\alpha(\tfrac{1}{2}T) + \omega_\alpha(\tau - \tfrac{1}{2}T)$$

$$\Rightarrow \quad \varphi_E(\tau) - \varphi_M(\tau) = \varphi_E(\tfrac{1}{2}T) + \omega_E(\tau - \tfrac{1}{2}T) - \varphi_M(\tfrac{1}{2}T) - \omega_M(\tau - \tfrac{1}{2}T)$$

$$= \varphi_E(\tfrac{1}{2}T) + (\omega_E - \omega_M)(\tau - \tfrac{1}{2}T) - \pi$$

$$\Rightarrow \quad 2\varphi_E(\tfrac{1}{2}T) = 2\pi n + (\omega_M - \omega_E)(\tau_n - \tfrac{1}{2}T)$$

$$\Rightarrow \quad \boxed{\tau_n - \tfrac{1}{2}T = \frac{2\varphi_E(\frac{1}{2}T) - 2\pi n}{\omega_M - \omega_E}} \quad \forall\, n \in \mathbb{Z} \text{ s.t. } \tau_n - \tfrac{1}{2}T \geq 0\,!$$

$$\tau_n - \frac{1}{2}T = \frac{2\pi T/T_E - 2\pi n}{\frac{2\pi}{T_M} - \frac{2\pi}{T_E}}$$

$$= \frac{T/T_E - n}{\frac{1}{T_M} - \frac{1}{T_E}}$$

$$= \frac{T - nT_E}{T_E - T_M} T_M$$

$$= \frac{nT_E - T}{T_M - T_E} T_M$$

Cl.: $T_M > T_E$

Pf.: $R_M > R_E$ and using Kepler's law and monotonicity of $(\cdot)^{3/2}$.

We know $T = T_E \left[\frac{1}{2}(1+q)\right]^{3/2}$.

$\frac{T_M}{R_M}$

$$\Rightarrow \boxed{\tau_n - \frac{1}{2}T = \left\{n - \left[\frac{1}{2}(1+q)\right]^{3/2}\right\} \frac{T_M T_E}{T_M - T_E}}$$

This quantity must be positive, so pick smallest $n \in \mathbb{Z}$ s.t. $n - \left[\frac{1}{2}(1+q)\right]^{3/2} \geq 0$

$$\Rightarrow \quad n := \left\lceil \left[\frac{1}{2}(1+q)\right]^{3/2} \right\rceil$$

With $q = 1,53$ we get $n = \left\lceil \left[\frac{1}{2}(1+1,53)\right]^{3/2} \right\rceil = \lceil 1,42 \rceil = 2$.

$$\Rightarrow \tau_2 - \frac{1}{2}T = 0,57 \frac{(687\text{ day})(365\text{ day})}{687\text{ day} - 365\text{ day}} = 449,51\text{ days}$$

Two circular orbits (with different radii and different periods):

$$\varphi_1(t) = \omega_1 t + \alpha_1$$
$$\varphi_2(t) = \omega_2 t + \alpha_2$$

Trip from one orbit to the other takes $\frac{1}{2}T_1$

Start at $t=0$ with $\varphi_1(0) \overset{!}{=} 0 \Rightarrow \alpha_1 = 0$

After $\frac{1}{2}T$, should arrive at other circle. Bcs. the shape of the transf. orbit is half an ellipse, that necessarily means

$$\varphi_2\left(\tfrac{1}{2}T\right) = \pi + 2\pi n_{\alpha_2} \quad \exists\, n_{\alpha_2} \in \mathbb{Z}.$$

Note we allow winding by $2\pi n_{\alpha_2}$ as the most general possibility.

$$\Rightarrow \quad \omega_2 \tfrac{1}{2} T + \alpha_2 = \pi + 2\pi n_{\alpha_2} = \pi(2n_{\alpha_2}+1)$$

$$\Rightarrow \quad \tfrac{1}{2}T = \left[(2n_{\alpha_2}+1)\pi - \alpha_2\right]\omega_2^{-1} \overset{!}{\geq} 0$$

So to have a sol-n, pick n_{α_2} s.t. $(2n_{\alpha_2}+1)\pi - \alpha_2 \geq 0$

$$\Leftrightarrow \quad 2n_{\alpha_2}+1 \geq \frac{\alpha_2}{\pi}$$

$$\Leftrightarrow \quad n_{\alpha_2} \geq \frac{1}{2}\left(\frac{\alpha_2}{\pi} - 1\right)$$

Take the simplest n that works, $n_{\alpha_2} := \left\lceil \frac{1}{2}\left(\frac{\alpha_2}{\pi} - 1\right)\right\rceil$

Note if $\alpha_2 \leq \pi$ then we can simply pick $n_{\alpha_2} = 0$.

$\Rightarrow \exists$ solution, α_2 still unknown.

After arrival at time $\frac{1}{2}T$, want to take the trip back.

The trip back is simply the opposite half ellipse.

Assume the embark on the trip back is at time τ.

$\Rightarrow \quad \varphi_2(\tau)$ is something.

$\varphi_1(\tau)$ is also something.

But since the trip back is also half an ellipse, we must have $\varphi_1(\tau + \frac{1}{2}T) = \varphi_2(\tau) + \pi + 2\pi n \Rightarrow \varphi_1(\tau) = -\varphi_1(\frac{1}{2}T) + \varphi_2(\tau) + (2n+1)\pi$

We seek a solution with $\tau \geq \frac{1}{2}T$ for some $n \in \mathbb{Z}$.

$$\Rightarrow \underbrace{\varphi_1(\tau + \tfrac{1}{2}T)}_{\omega_1(\tau+\frac{1}{2}T)} = \omega_2 \tau + \alpha_2 + \pi + 2\pi n$$

$$\Rightarrow \tau = \left(\alpha_2 + (2n+1)\pi - \omega_1 \tfrac{1}{2}T\right)(\omega_1 - \omega_2)^{-1}$$

$$\Rightarrow \tau - \tfrac{1}{2}T = \left(\alpha_2 + (2n+1)\pi - \omega_1 \tfrac{1}{2}T\right)\underbrace{(\omega_1 - \omega_2)^{-1}}_{\text{positive}} - \tfrac{1}{2}T$$

$$\Rightarrow \varphi_1(\tau) - \varphi_2(\tau) = -\varphi_1(\tfrac{1}{2}T) + (2n+1)\pi$$

Also $\varphi_j(\tau) = \varphi_j(\frac{1}{2}T) + \omega_j(\tau - \frac{1}{2}T)$

$$\begin{aligned} \Rightarrow \varphi_1(\tau) - \varphi_2(\tau) &= \varphi_1(\tfrac{1}{2}T) + \omega_1(\tau - \tfrac{1}{2}T) - \varphi_2(\tfrac{1}{2}T) - \omega_2(\tau - \tfrac{1}{2}T) \\ &= \varphi_1(\tfrac{1}{2}T) + (\omega_1 - \omega_2)(\tau - \tfrac{1}{2}T) - \varphi_2(\tfrac{1}{2}T) \end{aligned}$$

$$\Rightarrow -\varphi_1(\tfrac{1}{2}T) + (2n + \cancel{1})\pi = \varphi_1(\tfrac{1}{2}T) - \underbrace{\varphi_2(\tfrac{1}{2}T)}_{\cancel{\pi +} 2\pi n_{\alpha_2}} + (\omega_1 - \omega_2)(\tau - \tfrac{1}{2}T)$$

$$\Leftrightarrow 2\varphi_1(\tfrac{1}{2}T) = 2(n - n_{\alpha_2})\pi - (\omega_1 - \omega_2)(\tau - \tfrac{1}{2}T)$$