Q1 Mean & True Solar Time

Cl.: $\dot{\varphi} = \frac{2\pi}{T}[1+2\varepsilon\cos(\varphi)+O(\varepsilon^2)]$ for the Kepler problem.

Pf.: Note via eq-n (2.2) in the script the (constant) magnitude of the angular momentum is given by

$\ell \equiv \|L\| = mr^2\dot{\varphi}$

$\Rightarrow \dot{\varphi} = \frac{\ell}{mr^2}$

ℓ may be expressed via the geometric shape of the orbit:

Eq-n (2.12) says $d \equiv \frac{\ell^2}{GMm^2}$ and we also know $a = d(1-\varepsilon^2)^{-1}$ (the semi-major axis)

$$\Rightarrow \dot{\varphi} = \frac{\ell}{mr^2} = \frac{\ell}{m}u^2 \underset{\text{eq-n (2.14)}}{=} \frac{\cancel{\ell}}{\cancel{m}} \frac{G^2M^2m^{\cancel{4}\,3}}{\ell^{\cancel{4}\,3}}(1+\varepsilon\cos(\varphi))^2$$

$$= G^2M^2m^3\ell^{-3}[1+\varepsilon\cos(\varphi)]^2$$

$$\ell^{-3} = (GMm^2d)^{-3/2} = (GMm^2a(1-\varepsilon^2))^{-3/2}$$

$$= \frac{(GMm^2(1-\varepsilon^2))^{-3/2}}{a^{3/2}} \underset{\substack{\text{Keplers } 3^{rd} \\ \text{law eq-n (2.15)}}}{=} \frac{(GMm^2(1-\varepsilon^2))^{-3/2}}{\frac{\sqrt{GM}}{2\pi}T} = \frac{2\pi}{T}\frac{(1-\varepsilon^2)^{-3/2}}{m^3G^2M^2}$$

$$= \frac{2\pi}{T}(1-\varepsilon^2)^{-3/2}[1+\varepsilon\cos(\varphi)]^2$$

$$\approx \frac{2\pi}{T}[1+2\varepsilon\cos(\varphi)+O(\varepsilon^2)]$$

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Q.: $\boxed{\varphi(t) = \frac{2\pi}{T}t + 2\varepsilon \sin(\frac{2\pi}{T}t) + O(\varepsilon^2)}$ $\forall t$ in the Kepler problem.

Pf.: We know that when $\varepsilon = 0$, r is const. and

$$\dot{\varphi}_0 = \frac{\ell}{mr_0^2} \Rightarrow \varphi_0(t) = \frac{\ell}{mr_0^2}t + \varphi_0(0)$$

We know that $\boxed{\varphi_0(t+T) - \varphi_0(t) = 2\pi}$

$$\Rightarrow \frac{\ell}{mr_0^2}(\cancel{t}+T) + \cancel{\varphi_0(0)} - \cancel{\frac{\ell}{mr^2}t} - \cancel{\varphi_0(0)} = 2\pi$$

$$\Rightarrow \frac{\ell}{mr_0^2} = \frac{2\pi}{T}$$

$$\Rightarrow \varphi_0(t) = \frac{2\pi}{T}t + \varphi_0(0)$$

Assuming we pick a parametrization s.t. $\varphi_0(0) = 0$, we find $\varphi_0(t) = \frac{2\pi}{T}t$

Since this is the solution when $\varepsilon = 0$, we can write $\varphi(t) = \varphi_0(t) + O(\varepsilon) = \frac{2\pi}{T}t + O(\varepsilon)$

Next note that

$$\varepsilon \cos(f(\varepsilon)) = \varepsilon \cos(f(0)) + O(\varepsilon^2)$$

$$\Rightarrow \varepsilon \cos(\varphi(t)) = \varepsilon \cos(\varphi_0(t)) + O(\varepsilon^2) = \varepsilon \cos(\frac{2\pi}{T}t) + O(\varepsilon^2)$$

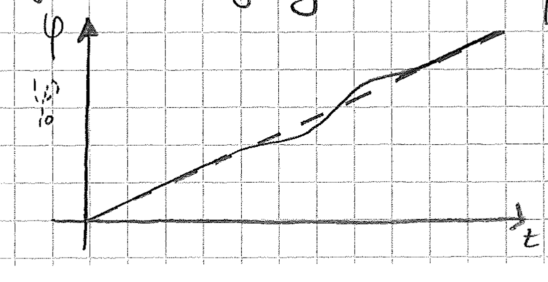
$$\Rightarrow \dot{\varphi}(t) = \frac{2\pi}{T}\left[1 + 2\varepsilon \cos(\frac{2\pi}{T}t)\right] + O(\varepsilon^2)$$

$$\Rightarrow \varphi(t) = \frac{2\pi}{T} + 2\cancel{\frac{2\pi}{T}}\varepsilon\cancel{\frac{T}{2\pi}}\sin(\frac{2\pi}{T}t) + O(\varepsilon^2)$$

Note: $\varphi_0(T) = 2\pi$ as we expect.

$\varphi(T) = 2\pi$ as well. ✓

The difference is that when $\varepsilon \neq 0$, φ as a function of t slightly deviates from a straight line.



Also note that for earth $\overset{\text{and sun}}{}$ $\varepsilon = 0{,}0167$.

We define $(\Delta\varphi)(t) := \varphi(t) - \frac{2\pi}{T} t$

$$\Rightarrow (\Delta\varphi)(t) = 2\varepsilon \sin\left(\frac{2\pi}{T} t\right) + O(\varepsilon^2)$$

Let τ be a given time period.

Then def. $(\Delta t)(t, \tau) := \frac{\Delta\varphi(t)}{2\pi/\tau} = \frac{\varepsilon\tau}{\pi} \sin\left(\frac{2\pi}{T} t\right)$

$\Rightarrow$ $(\Delta t)(t, \tau)$ obtains its max. val. when $t = \frac{1}{4} T$, in which case the max. val. is $\frac{\varepsilon\tau}{\pi}$.

Claim: $(\Delta t)(t, 24 \text{ hours})$ is the difference between the true and mean sun time, if we assume the earth's axial tilt is zero (it is actually $23.44°$), at time (of year) t, over the course of one day.

Proof: Def. $\tau = 24$ hours.

Every day, the earth turns about its axis by angle <u>more</u> than 2π (due to different def. of τ), but after one year, all these surpluses add up to 2π.

Thus, every day the surplus is $2\pi \frac{\tau}{T}$.

After time t, the surplus is $2\pi \frac{t}{T}$.

If ε were equal to zero, this surplus would exactly equal the sun's position and the mean time would equal the true time.

When $\varepsilon \neq 0$, $\varphi(t) - \frac{2\pi}{T} t$ measures how much the sun's true angle (corresponds to $\varphi(t)$) deviates from the mean angle (corresponds to $\frac{2\pi}{T} t$).

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Then this angular deviation, $(\Delta\varphi)(t)$, amounts to temporal deviation over one day via

$$(\Delta t)(t) = \frac{\tau}{2\pi}(\Delta\varphi)(t)$$

Kepler's Law & Newtonian Gravitation

Cl.: If a curve $\gamma : \underset{\text{time}}{\mathbb{R}} \to \underset{\text{space}}{\mathbb{R}^3}$ obeys the first two Kepler laws:

① $\mathrm{im}(\gamma)$ is a planar ellipse in $\mathbb{R}^3$.

② γ sweeps equal area in equal time periods.

Then γ obeys the differential eq-n

$$\boxed{\ddot{\gamma} = -\alpha \frac{\gamma}{\|\gamma\|^3}} \quad \exists \quad \alpha \in (0,\infty)$$

$\Rightarrow$ γ describes the trajectory of a body under the influence of an inverse square attractive force.

Pf.: Via ① and a choice of the coordinate system which does not lead to loss of generality, we may write

$$r(t) = a(1-\varepsilon^2)\left[1+\varepsilon\cos(\varphi(t))\right]^{-1} \quad \forall\, t$$

for some $a>0$, $\varepsilon\in[0,1)$ where $r(t) \equiv \|\gamma(t)\|$

$$\varphi(t) \equiv \operatorname{arctg}\left(+\frac{\langle\gamma(t),e_2\rangle}{\langle\gamma(t),e_1\rangle}\right) \in [0,2\pi)$$

which corresponds to an ellipse in the 1-2 plane w/ eccentricity ε and semi-major axis a.

Cl.: $\langle\dot{\gamma}(t),e_3\rangle = 0 \quad \forall\, t$

Pf.: $\gamma(t) = \langle\gamma(t),e_1\rangle e_1 + \langle\gamma(t),e_2\rangle e_2$ by choice of coord. sys.

$\Rightarrow \dot{\gamma}(t) = \langle\dot{\gamma}(t),e_1\rangle e_1 + \langle\dot{\gamma}(t),e_2\rangle e_2$

Def. $e_r := \cos(\varphi)e_1 + \sin(\varphi)e_2 \Rightarrow \dot{e}_r = -\sin(\varphi)\dot{\varphi}e_1 + \cos(\varphi)\dot{\varphi}e_2 = \dot{\varphi}e_\varphi$

$e_\varphi := -\sin(\varphi)e_1 + \cos(\varphi)e_2 \Rightarrow \dot{e}_\varphi = -\cos(\varphi)\dot{\varphi}e_1 - \sin(\varphi)\dot{\varphi}e_2 = -\dot{\varphi}e_r$

$\Rightarrow$ $\gamma = \|\gamma\|\cos(\varphi)e_1 + \|\gamma\|\sin(\varphi)e_2 = re_r$

$\dot{\gamma} = \dot{r}e_r + r\dot{e}_r = \dot{r}e_r + r\dot{\varphi}e_\varphi$

$\ddot{\gamma} = \ddot{r}e_r + \dot{r}\dot{e}_r + \dot{r}\dot{\varphi}e_\varphi + r\ddot{\varphi}e_\varphi + r\dot{\varphi}\dot{e}_\varphi = (\ddot{r}-r\dot{\varphi}^2)e_r + (r\ddot{\varphi}+2\dot{r}\dot{\varphi})e_\varphi$

$$\Rightarrow \gamma \times \dot{\gamma} = (r e_r) \times (\dot{r} e_r + r \dot{\varphi} e_\varphi)$$
$$= r^2 \dot{\varphi}\, e_r \times e_\varphi$$
$$e_r \times e_\varphi \equiv \left[\cos(\varphi) e_1 + \sin(\varphi) e_2\right] \times \left[-\sin(\varphi) e_1 + \cos(\varphi) e_2\right]$$

$$= \cos(\varphi)^2 e_3 + \sin(\varphi)^2 e_3 = e_3$$

$$\Rightarrow \boxed{\gamma \times \dot{\gamma} = r^2 \dot{\varphi}\, e_3}$$

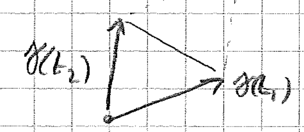
<u>Cl.:</u> (2) $\Rightarrow$ $r^2 \dot{\varphi}$ does not depend on time.

<u>Pf.:</u> Let $\underset{\text{time}}{F : \mathbb{R}} \to \underset{\text{area}}{[0, \infty)}$ be the area swept by γ as a function of time.

(2) $\Leftrightarrow$ $\dot{F}$ does not depend on time.

The area swept from time t_1 to time t_2 if $t_2 \to t_1$ is:

$$\dot{F}(t_1) = \lim_{t_2 \to t_1} \frac{\|\frac{1}{2} \gamma(t_1) \times \gamma(t_2)\|}{t_2 - t_1}$$



(Cross product gives area of parallelogram. If $t_2 \to t_1$, then the area of a triangle converges to the actual area swept)

We write $\gamma(t_2) \approx \gamma(t_1) + \dot{\gamma}(t_1)(t_2 - t_1)$

$$\Rightarrow \dot{F}(t_1) = \frac{1}{2} \| \gamma(t_1) \times \dot{\gamma}(t_1) \| = \frac{1}{2} r^2(t_1)\, \dot{\varphi}(t_1) \qquad \left(\begin{matrix}\text{param. s.t.} \\ \dot{\varphi} \geq 0\end{matrix}\right)$$

$\Rightarrow$ $\gamma \times \dot{\gamma}$ is const.

$\Rightarrow$ $\dot{(\gamma \times \dot{\gamma})} = 0$

But $\dot{(\gamma \times \dot{\gamma})} = \dot{(\varepsilon_{ijk} \gamma_i \dot{\gamma}_j e_k)} = \underbrace{\varepsilon_{ijk} \dot{\gamma}_i \dot{\gamma}_j e_k}_{=0} + \varepsilon_{ijk} \gamma_i \ddot{\gamma}_j e_k$

$$= \gamma \times \ddot{\gamma}$$

$\Rightarrow \gamma \times \ddot{\gamma} = 0 \Leftrightarrow \gamma \parallel \ddot{\gamma} \Leftrightarrow \ddot{\gamma} = \langle \ddot{\gamma}, e_r \rangle e_r$

But $\langle \ddot{\gamma}, e_r \rangle = \ddot{r} - r \dot{\varphi}^2$.

$$\Rightarrow \boxed{\ddot{\gamma} = (\ddot{r} - r \dot{\varphi}^2)\, e_r}$$

Claim: $\ddot{r} - r\dot{\varphi}^2 \propto r^{-2}$

Proof: Take the ellipse eq-n: $r = a(1-\varepsilon^2)[1+\varepsilon\cos(\varphi)]^{-1}$

Take its time derivative to obtain:

$$\dot{r} = a(1-\varepsilon^2)(-1)[1+\varepsilon\cos(\varphi)]^{-2}\varepsilon(-1)\sin(\varphi)\dot{\varphi}$$
$$= a(1-\varepsilon^2)[1+\varepsilon\cos(\varphi)]^{-2}\varepsilon\sin(\varphi)\dot{\varphi}$$
$$= [a(1-\varepsilon^2)]^{-1} r^2\dot{\varphi}\,\varepsilon\sin(\varphi)$$

(recall $r^2\dot{\varphi}$ is const)

Take its ~~derivative~~ derivative again:

$$\ddot{r} = [a(1-\varepsilon^2)]^{-1} r^2\dot{\varphi}\,\varepsilon\cos(\varphi)\dot{\varphi} \qquad (\ast)$$
$$= [a(1-\varepsilon^2)]^{-1}(r^2\dot{\varphi})^2\varepsilon\cos(\varphi)\, r^{-2}$$

$$r\dot{\varphi}^2 = \underbrace{(r^2\dot{\varphi})^2}_{\text{const}} r^{-3}$$

$$\Rightarrow \ddot{r} - r\dot{\varphi} = [a(1-\varepsilon^2)]^{-1}(r^2\dot{\varphi})^2\varepsilon\cos(\varphi)r^{-2} - (r^2\dot{\varphi})^2 r^{-3}$$
$$= (r^2\dot{\varphi})^2 r^{-2}\left\{[a(1-\varepsilon^2)]^{-1}\varepsilon\cos(\varphi) - r^{-1}\right\} =$$
$$= (r^2\dot{\varphi})^2 r^{-2}\left\{[a(1-\varepsilon^2)]^{-1}\varepsilon\cos(\varphi) - [a(1-\varepsilon^2)]^{-1}[1+\varepsilon\cos(\varphi)]\right.$$
$$= -(r^2\dot{\varphi})^2 r^{-2}[a(1-\varepsilon^2)]^{-1} \qquad (r^2\dot{\varphi} \text{ is const})$$

$$\Rightarrow \ddot{\gamma} = -\frac{(r^2\dot{\varphi})^2}{a(1-\varepsilon^2)} r^{-2} e_r = -\underbrace{\frac{(r^2\dot{\varphi})^2}{a(1-\varepsilon^2)}}_{\text{const} =: \alpha > 0 \text{ bcs. } \dot{\varphi} \geq 0 \text{ param}} \frac{\gamma}{\|\gamma\|^3}$$

Claim: $\alpha = 4\pi \frac{a^3}{T^2}$ where T is the period of the ellipse.

Proof: We have found above that $\dot{F}(t) = \frac{1}{2}r^2\dot{\varphi}$ $(= \text{const})$

$$\Rightarrow F(t) = \tfrac{1}{2}r^2\dot{\varphi}t + c_1$$

B.C. is that $F(0) = 0 \Rightarrow c_1 = 0$.

We also know $F(T) = \text{area of ellipse} = \pi a^2\sqrt{1-\varepsilon^2}$

$$\Rightarrow \tfrac{1}{2}r^2\dot{\varphi}T = \pi a^2\sqrt{1-\varepsilon^2} \Rightarrow r^2\dot{\varphi} = \frac{2\pi a^2\sqrt{1-\varepsilon^2}}{T}$$
$$\Rightarrow \alpha \equiv \frac{(r^2\dot{\varphi})^2}{a(1-\varepsilon^2)} = \frac{4\pi^2 a^{\cancel{4}3}\cancel{(1-\varepsilon^2)}}{T^2\cancel{a}\cancel{(1-\varepsilon^2)}} = 4\pi\frac{a^3}{T^2}$$

<u>Corollary</u>: If $\dfrac{a^3}{T^2}$ is independent of the parameters defining the ellipse (a and ε), as in Kepler's 3rd law, then α does not depend on these parameters either.