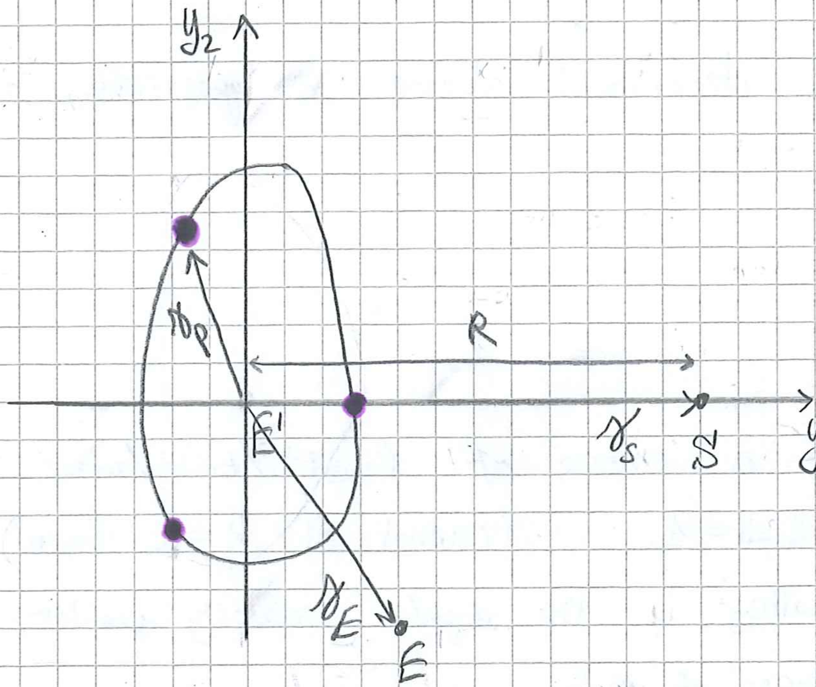
Q1 The Lisa Constellation



● – A space probe.

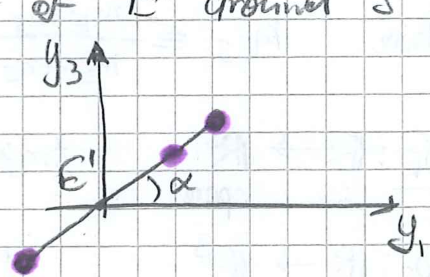
R – earth's orbital radius = E' orbital radius

Three space probes orbit some fictitious) pt. E', which itself orbits the sun S on the same orbit as the earth E but with a delay of approx. 20° (i.e. if E were at a certain pt. at time t, E' will reach that pt. in

$$20^\circ \frac{2\pi}{360^\circ} \frac{T}{2\pi} = \frac{\beta T}{2\pi}$$

time duration) and with the same period (by Kepler's 3rd).

The plane of orbit of the probes around E' is inclined w.r.t. the plane of orbit of E' around S ($\equiv$ the ecliptic) with an angle α:



The orbiting of the probes around E' is performed in such a way so that at any given moment the

[2] three probes form an equilateral triangle whose side length is d ($\approx 3 \cdot 10^{-2} R$).
No other forces (no engines etc.) apply on the probes.

Cl.: The situation described above is possible, in the approx. where

Pf.: We start w/ a choice of length/time/mass units s.t. $G = m_E = \|\omega\| = 1$ (cannot set $R=1$ here)
where $\omega \equiv \|\omega\| e_3$ is the angular velocity vector,
$m_E \equiv$ mass of earth
$\|\omega\| \equiv \frac{2\pi}{1 \text{ year}}$ in ordinary units

Since we are in the restricted problem, we assume the probes do not affect earth & the sun's motion at all. We thus follow the description of the Sun-Jupiter-Asteroid system as on page 20 in the script:
By eq-n (1.25) in the script we have for the relative motion of the earth & the sun:

$$m_{ES}\ddot{\gamma}_{ES} = K_{ES} - 2m_{ES}(\omega \times \dot{\gamma}_{ES}) - m_{ES}(\dot{\omega} \times \gamma) - m_{ES}\omega \times (\omega \times \gamma_{ES}) - m a_{ES}$$

where $m_{ES} \equiv \frac{m_E m_S}{m_E + m_S}$ $\quad \gamma_{ES} := \gamma_E - \gamma_S$

origin @ E-S CoM

$\gamma_E : \underset{\text{time}}{\mathbb{R}} \to \underset{\text{space}}{\mathbb{R}^3}$ trajectory of E in co-rotating fixed origin frame.

$\gamma_S : \mathbb{R} \to \mathbb{R}^3$ " " S " " " "

K_{ES} force on m_{ES} fictitious particle $= -G\frac{m_S m_E}{\|\gamma_{ES}\|^3}\gamma_{ES}$.

a_{ES} acceleration of the origin of our frame $= 0$ (origin fixed).
also $\dot{\omega} = 0$

$\|\gamma_{ES}\| \equiv R$

Bcs. we're in the co-rotating frame, we actually have $\dot{\gamma}_E = \dot{\gamma}_S = 0$. Also, we have $\gamma_S = Re_1$, $\langle \gamma_E, e_3 \rangle = 0$.

Hence eq-n (1.25) now reduces to:

$$0 = -G\frac{m_S m_E}{R^3}\gamma_{ES} - m_{ES}\,\omega\times(\omega\times\gamma_{ES})$$

$$\omega\times(\omega\times\gamma_{ES}) = \underbrace{(\omega\cdot\gamma_{ES})\omega}_{0} - \|\omega\|^2\gamma_{ES}$$

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$$= -\|\omega\|^2\gamma_{ES}$$

$$\Rightarrow\ G\frac{1}{R^3}\gamma_{ES} = \frac{1}{m_E+m_S}\|\omega\|^2\gamma_{ES} \Rightarrow \frac{G}{R^3} = \frac{\|\omega\|^2}{m_E+m_S}$$

Thus with our choice of units ($G = m_E = \|\omega\| = 1$) we get:

$$\boxed{1 + m_S = R^3} \xrightarrow{m_S \gg 1} \boxed{m_S = R^3}$$

Now that we have $m_S = R^3$, go to the co-rotating frame whose origin is fixed at E' (before origin was fixed at C.o.M. of E–S) so that γ_E, γ_S are now the trajectories in this new frame.

Our goal is to get $\gamma_p : \underset{\text{time}}{\mathbb{R}} \to \underset{\text{space}}{\mathbb{R}^3}$, the trajectory of one of the probes (doesn't matter which) in this frame.

We may now write eq-n (1.25) for the probe:

$$m_p\ddot{\gamma}_p = K_p - 2m_p(\omega\times\dot{\gamma}_p) - m_p\underbrace{(\cancel{\dot{\omega}\times\gamma_p})}_{0} - m_p\,\omega\times(\omega\times\gamma_p) - m_p a_{E'}$$

where $\gamma_p : \underset{\text{time}}{\mathbb{R}} \to \underset{\text{space}}{\mathbb{R}^3}$ is the traj. of the probe (one of the three, doesn't matter which) in the co-rotating y-frame.

$$K_p = -G\frac{m_p m_E}{\|\gamma_p - \gamma_E\|^3}(\gamma_p - \gamma_E) - G\frac{m_p m_S}{\|\gamma_p - \gamma_S\|}(\gamma_p - \gamma_S)$$

is the force on the probe in the co-rotating frame.

$a_{E'} = \|\omega\|^2 R e_1$ ($\equiv$ accel. of origin of y-frame as measured in inertial frame, always directed towards sun! for circular motion of E')

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$$\omega \times (\omega \times \gamma_p) = (\omega \cdot \gamma_p)\omega - \|\omega\|^2 \gamma_p$$
$$= \|\omega\|^2 \langle e_3, \gamma_p \rangle e_3 - \|\omega\|^2 \gamma_p$$
$$= -\|\omega\|^2 \underbrace{(\gamma_p \text{ projected onto } 1-2 \text{ plane})}_{\langle e_1, \gamma_p\rangle e_1 + \langle e_2, \gamma_p \rangle e_2}$$

$$\omega \times \dot{\gamma}_p = e_i\, \varepsilon_{ijk}\, \omega_j (\dot{\gamma}_p)_k = e_i\, \varepsilon_{i3k} \|\omega\| (\dot{\gamma}_p)_k$$

$$= e_1 \underbrace{\varepsilon_{132}}_{=-1} \|\omega\| (\dot{\gamma}_p)_2 + e_2 \underbrace{e_{231}}_{=1} \|\omega\| (\dot{\gamma}_p)_1$$
$$= \|\omega\| \left[(\dot{\gamma}_p)_1 e_2 - (\dot{\gamma}_p)_2 e_1 \right]$$

(Note that m_p drops out of the EoM as all forces are proportional to it — we didn't really need to set $m=1$)
The EoM becomes: (in $G = \|\omega\| = m_p = 1$ units & $m_s \approx R^3$)

$$\ddot{\gamma}_p = \underbrace{-\frac{(\gamma_p - \gamma_E)}{\|\gamma_p - \gamma_E\|^3}}_{\substack{\text{grav. of } E \\ F_{G_E}}} \underbrace{- \frac{R^3 (\gamma_p - \gamma_s)}{\|\gamma_p - \gamma_s\|^3}}_{\substack{\text{grav. of } S' \\ F_{G_s}}} \underbrace{-2\left[(\dot{\gamma}_p)_1 e_2 - (\dot{\gamma}_p)_2 e_1\right]}_{\substack{\text{Coriolis} \\ C}} + \underbrace{\left[(\gamma_p)_1 e_1 + (\gamma_p)_2 e_2\right]}_{\substack{\text{centrifugal} \\ Z}}$$

$$\underbrace{-Re_1}_{\text{d'Alembert force}}$$

Sun fixed at $\gamma_s = Re_1$

Note $\gamma_p - \gamma_s = \gamma_p - Re_1 = R\,(R^{-1}\gamma_p - e_1)$

$$F_{G_s} = -\frac{R^3 R (R^{-1}\gamma_p - e_1)}{\|R(R^{-1}\gamma_p - e_1)\|^3} = -R \frac{(R^{-1}\gamma_p - e_1)}{\|R^{-1}\gamma_p - e_1\|^3}$$

Note γ_p is the displacement between the probe and E', which is $\frac{d}{\sqrt{3}}$ (equilateral centered at E' w/ side length d), and we have $d \approx 3 \cdot 10^{-2} R$.
Hence we have $\|R^{-1}\gamma_p\| \ll 1$. So we want to approx. F_{G_s} in that small parameter.

We need to use the multi-variable Taylor expansion, which takes the form: for any $f: \mathbb{R}^3 \to \mathbb{R}^3$

$$f(x) \approx f(0) + \underbrace{e_i (\partial_j (f)_i)(0)\, x_j}_{e_i (\nabla f_i)(0) \cdot x} + O(\|x\|^2)$$

If $f(x) = G \frac{x - v}{\|x-v\|^3} \implies f_i(x) = G \frac{x_i - v_i}{\|x-v\|^3}$

$$\implies (\partial_j f_i)(0) = G \left. \frac{\delta_{ij} \|x-v\|^3 - (x_i - v_i)\, 3\|x-v\|^2 \frac{x_j - v_j}{\|x-v\|}}{\|x-v\|^6} \right|_{x=0}$$

$$= G \frac{\delta_{ij}}{\|v\|^3} - 3G \frac{v_i v_j}{\|v\|^5}$$

$$\implies f(x) \approx -G \frac{v}{\|v\|^3} + e_i \, G \frac{\delta_{ij}}{\|v\|^3} x_j - 3G \frac{v(v \cdot x)}{\|v\|^5} + O(\|x\|^2)$$

$$= -G \frac{v}{\|v\|^3} + G \frac{x}{\|v\|^3} - 3G \frac{v(v \cdot x)}{\|v\|^5} + O(\|x\|^2)$$

$$= G \frac{x - v - 3v(v\cdot x)\|v\|^{-2}}{\|v\|^3} + O(\|x\|^2)$$

$$\implies F_{G_S} \approx -R\left(R^{-1}\gamma_p - e_1 - 3e_1(e_1 \cdot R^{-1}\gamma_p)\right) + O(\|R^{-1}\gamma_p\|^2)$$

$$= \underset{\uparrow}{Re_1} + \underbrace{3e_1(e_1 \cdot \gamma_p) - e_3(e_3 \cdot \gamma_p)} \underbrace{- e_2(e_2 \cdot \gamma_p) - e_1(e_1 \cdot \gamma_p)}$$

cancels with d'Alembert force $m_p a_{E'}$

cancels w/ Centrifugal force Z for

The left over which is not cancelled by fictitious forces is called the tidal forces, T.

$$\boxed{T \equiv 3e_1(e_1 \cdot \gamma_p) - e_3(e_3 \cdot \gamma_p)}$$

Cl.: $\frac{\|F_{G_E}\|}{\|T\|} \ll 1$ and hence we may neglect F_{G_E}.

Pf.: We have $\|F_{G_E}\| = \|\gamma_p - \gamma_E\|^{-2}$.

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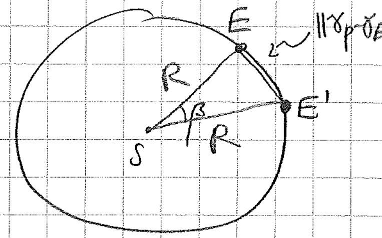
On the other hand,

$$\|T\| = \sqrt{\left(3(\gamma_p)_1\right)^2 + \left(-(\gamma_p)_3\right)^2}$$



Note $\|\gamma_p - \gamma_E\| \approx \beta R$

use arclength instead of sidelength of triangle bcs. they are almost the same

Otherwise we'd have to use the law of cosines:

$$\|\gamma_p - \gamma_E\|^2 = 2R^2 - 2R^2\cos(\beta)$$

$$\|\gamma_p - \gamma_E\| = R\sqrt{2}\underbrace{\sqrt{1-\cos(\beta)}}_{\sqrt{2}\sin(\beta/2)} \approx R\beta$$

Cl.: $R \approx 70$ in our units.

Pf.: Recall, $G = \|\omega\| = m_E = 1$

Let τ and λ be the new units of time & length.

Then: $G \overset{!}{=} \dfrac{\lambda^3}{m_E \tau^2}$ $\qquad \|\omega\| = \dfrac{2\pi}{T}$

$$\|\omega\| \overset{!}{=} \tau^{-1}$$

$$\Rightarrow \lambda = \sqrt[3]{\frac{m_E G}{\|\omega\|^2}}$$

$$\Rightarrow \tilde{R} = \frac{R}{\lambda} = \frac{R}{\sqrt[3]{\frac{m_E G}{\|\omega\|^2}}} = \frac{147 \cdot 10^9 \,\cancel{\text{meters}}}{\sqrt[3]{\dfrac{5.97 \cdot 10^{24}\,\cancel{kg}\; 6.67 \cdot 10^{-11}\,\cancel{\frac{\text{meter}^3}{kg\,sec^2}}}{(31557600\,\cancel{sec})^{-2}(2\pi)^2}}}$$

$$\hat{=} 69.1823$$

$$\Rightarrow \frac{\|F_{G_E}\|}{\|T\|} \sim \frac{(\beta R)^{-2}}{d} \overset{\beta \approx 0.35,\, d \approx 3\cdot 10^{-2} R,\, R \approx 70}{=} \frac{(0.35 \cdot 70)^{-2}}{3 \cdot 10^{-2} \cdot 70} \hat{=} 8 \cdot 10^{-4} \sim 10^{-3} \ll 1.$$

As a result the EoM becomes:

$$\ddot{\gamma}_p = 3e_1(\gamma_p)_1 - e_3(\gamma_p)_3 - 2\left[e_2(\dot{\gamma}_p)_1 - e_1(\dot{\gamma}_p)_2\right]$$

Or in components:

$$\boxed{\begin{cases} (\ddot{\gamma}_p)_1 = 3(\gamma_p)_1 + 2(\dot{\gamma}_p)_2 \\ (\ddot{\gamma}_p)_2 = -2(\dot{\gamma}_p)_1 \\ (\ddot{\gamma}_p)_3 = -(\gamma_p)_3 \end{cases}}$$

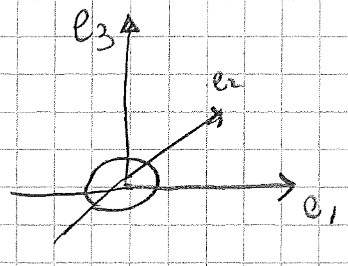
[redacted]

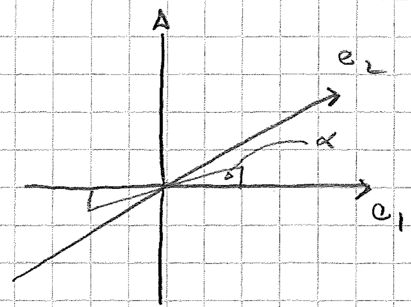
$\boxed{7\ Alt}$

A circle in the 1-2 plane w/ radius r is given by: $[0,2\pi)\ni t \xmapsto{f} r\left[\cos(a(t-t_0))e_1 + \sin(a(t-t_0))e_2\right]\in \mathbb{R}^2$

radius angular velocity initial orientation

We then tilt the plane of the circle by α:





The axis of rotation is e_2. The rotation matrix is:

$$R_\alpha = \begin{bmatrix} \cos(\alpha) & 0 & \sin(\alpha) \\ 0 & 1 & 0 \\ -\sin(\alpha) & 0 & \cos(\alpha) \end{bmatrix}$$

Then the rotated circle is f': $f' \equiv Rf$

$$f'(t) = Rf(t) = \begin{bmatrix} \cos(\alpha) & 0 & \sin(\alpha) \\ 0 & 1 & 0 \\ -\sin(\alpha) & 0 & \cos(\alpha) \end{bmatrix}\begin{bmatrix} r\cos(a(t-t_0)) \\ r\sin(a(t-t_0)) \\ 0 \end{bmatrix} =$$

$$= \begin{bmatrix} r\cos(a(t-t_0))\cos(\alpha) \\ r\sin(a(t-t_0)) \\ -r\cos(a(t-t_0))\sin(\alpha) \end{bmatrix}$$

Now check that f' obeys EoM:

$$\dot{f}'(t) = \begin{bmatrix} -r\cos(\alpha)\sin(a(t-t_0))a \\ r\cos(a(t-t_0))a \\ r\sin(\alpha)\sin(a(t-t_0))a \end{bmatrix} \quad \ddot{f}'(t) = \begin{bmatrix} -r\cos(\alpha)\cos(a(t-t_0))a^2 \\ -r\sin(a(t-t_0))a^2 \\ r\sin(\alpha)\cos(a(t-t_0))a^2 \end{bmatrix}$$

$\boxed{8\ \text{Alt}}$

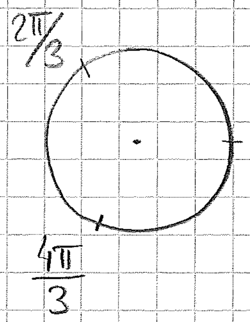
$$\begin{cases} -\cancel{r}\cos(\alpha)\, a^2 \cancel{\cos(a(t-t_0))} \overset{!}{=} 3\cancel{r}\cos(\alpha)\cancel{\cos(a(t-t_0))} + 2\cancel{r}a\cancel{\cos(a(t-t_0))} & (1) \\ -\cancel{r}a^{\cancel{2}}\cancel{\sin(a(t-t_0))} \overset{!}{=} +2\cancel{r}\cos(\alpha)\cancel{a}\cancel{\sin(a(t-t_0))} & (2) \\ r\cancel{\sin(\alpha)}\, a^2\cancel{\cos(a(t-t_0))} \overset{!}{=} +r\cancel{\sin(\alpha)}\cancel{\cos(a(t-t_0))} & (3) \end{cases}$$

(3) $\Rightarrow a^2 = 1 \Rightarrow \boxed{a = \pm 1}$

(2) $\Rightarrow -a = 2\cos(\alpha) \Rightarrow \boxed{\cos(\alpha) = \frac{-a}{2}} = \pm 1/2$

(1) $\Rightarrow -\underbrace{\underbrace{\cos(\alpha)}_{-\frac{a}{2}}\underbrace{a^2}_{=1}}_{\frac{1}{2}a} = \underbrace{3\underbrace{\cos(\alpha)}_{-\frac{a}{2}}+2a}_{\frac{1}{2}a} \checkmark \Rightarrow$ $\boxed{\text{No new information,}}$
(But eq-n is satisfied!)

Because the three probes form an equilateral on a circle, we have:



$0,\ \frac{2\pi}{3},\ \frac{4\pi}{3}$

as the initial positions (t_0) for each of the three probes.

Note $a = 1$ means the orbiting of the probes about E' has the same period as the earth around the sun (bcs. $|\omega| = 1$).

Q2 Phase Portraits of Dampened Oscillations

The Dampened Oscillator is:

$$m\ddot{x} = -f x - r\dot{x}$$

where $m \in (0,\infty)$, $f \in (0,\infty)$, $r \in (0,\infty)$.

We define $\alpha := \sqrt{\frac{f}{m}}$, $\beta := \frac{r}{2m}$.

Then $\ddot{x} = -\alpha^2 x - 2\beta\dot{x}$

Define $z : \mathbb{R} \to \mathbb{R}^2$ by $t \longmapsto \begin{bmatrix} x(t) \\ \alpha^{-1}\dot{x}(t) \end{bmatrix}$.

In terms of z, the EoM becomes:

$$\dot{z} = \begin{bmatrix} \dot{x} \\ \alpha^{-1}\ddot{x} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ -\alpha x - 2\beta\alpha^{-1}\dot{x} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \alpha \\ -\alpha & -2\beta \end{bmatrix}}_{A} \begin{bmatrix} x \\ \alpha^{-1}\dot{x} \end{bmatrix}$$

$$\boxed{\dot{z} = Az}$$

This defines a vector field $\mathbb{R}^2 \ni z \longmapsto Az \in \mathbb{R}^2$

Anyway, our goal is to draw solutions to the EoM in the $\dot{x}, x$ plane (instead of $t, \dot{x}, x$ space).

<u>Example</u>: If $\boxed{\beta = 0}$ we get a simple Harmonic oscillator $\ddot{x} = -\alpha^2 x$

whose sol-n is $\boxed{x(t) = A\cos(\omega t) + B\sin(\omega t)}$

$$\Rightarrow \dot{x}(t) = -\omega A\sin(\omega t) + \omega B\cos(\omega t)$$

$$\ddot{x}(t) = -\omega^2 A\cos(\omega t) - \omega^2 B\sin(\omega t)$$

$$\Rightarrow -\omega^2 = -\alpha^2 \Rightarrow \boxed{\omega = \pm\alpha}$$

$$\boxed{x(0) = A} \quad \boxed{\dot{x}(0) = \omega B}$$

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$$\begin{bmatrix} \alpha^{-1}\dot{x}(t) \\ x(t) \end{bmatrix} = \begin{bmatrix} -A\sin(\omega t) + B\cos(\omega t) \\ A\cos(\omega t) + B\sin(\omega t) \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{bmatrix}}_{\text{Counter clockwise rotation of } \omega t \text{ degrees}} \begin{bmatrix} B \\ A \end{bmatrix}$$

initial position in phase space

$$\Rightarrow \boxed{\text{Circles w/ angular velocity } \pm\alpha \text{ and radius } \sqrt{B^2+A^2} = \sqrt{(\alpha^{-1}\dot{x}(0))^2 + (x(0))^2}}$$

General solution: (pp. 31 in script)

Define $\omega_0 := \sqrt{\alpha^2 - \beta^2}$

We have $\dot{z}(t) = A z(t)$

$$\Rightarrow z(t) = \exp(At)\, z(0)$$

How to compute $\exp(At)$?

① Mathematica's MatrixExp[tA] . $\rightsquigarrow$ FullSimplify[ ].

② $e^{At} = e^{-\beta t} e^{(A+\beta \mathbb{1})t}$

$$A + \beta\mathbb{1} = \begin{bmatrix} 0 & \alpha \\ -\alpha & -2\beta \end{bmatrix} + \begin{bmatrix} \beta & 0 \\ 0 & \beta \end{bmatrix} = \begin{bmatrix} \beta & \alpha \\ -\alpha & -\beta \end{bmatrix}$$

$$(A+\beta\mathbb{1})^2 = \begin{bmatrix} \beta & \alpha \\ -\alpha & -\beta \end{bmatrix}^2 = \begin{bmatrix} \beta & \alpha \\ -\alpha & -\beta \end{bmatrix}\begin{bmatrix} \beta & \alpha \\ -\alpha & -\beta \end{bmatrix} = \begin{bmatrix} \beta^2-\alpha^2 & 0 \\ 0 & -\alpha^2+\beta^2 \end{bmatrix} =$$

$$= -\omega_0^2\, \mathbb{1}_{2\times 2}$$

$$\Rightarrow \exp((A+\beta\mathbb{1})t) = \sum_{n=0}^{\infty} \frac{1}{n!}\left((A+\beta\mathbb{1})t\right)^n =$$

$$= \sum_{\substack{n=0 \\ n\in 2\mathbb{N}_{\geq 0}}}^{\infty} \frac{1}{n!}((A+\beta\mathbb{1})t)^n + \sum_{\substack{n=1 \\ n\in 2\mathbb{N}_{\geq 0}+1}}^{\infty} \frac{1}{n!}((A+\beta\mathbb{1})t)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{(2n)!} ((A+\beta \mathbb{1})t)^{2n} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} ((A+\beta \mathbb{1})t)^{2n+1}$$

$$= \underbrace{\sum_{n=0}^{\infty} \frac{1}{(2n)!} (-\omega_0^2)^n t^{2n}}_{\cos(\omega_0 t)} + \underbrace{\sum_{n=0}^{\infty} \frac{1}{(2n+1)!} (-\omega_0^2)^n (A+\beta \mathbb{1}) t^{2n+1}}_{\sin(\omega_0 t)\, \omega_0^{-1} (A+\beta \mathbb{1})}$$

This was assuming $\boxed{\omega_0 \neq 0}$.

Hence

$$\boxed{\begin{aligned} \zeta(t) &= \exp(At)\,\zeta(0) \\ &= e^{-\beta t} \left[\cos(\omega_0 t) + \frac{1}{\omega_0} \sin(\omega_0 t)(A+\beta \mathbb{1}) \right] \zeta(0) \end{aligned}}$$

If $\omega_0 \in \mathbb{C}$ $(\beta > \alpha)$ then:

$$\cos(\omega_0 t) = \cosh(|\omega_0| t)$$

$$\omega_0^{-1} \sin(\omega_0 t) = \frac{1}{|\omega_0|} \sinh(|\omega_0| t)$$

If $\omega_0 = 0$ $(\beta = \alpha)$ then:

$$(A+\beta \mathbb{1})^2 = 0 \Rightarrow \exp((A+\beta \mathbb{1})t) = \mathbb{1} + (A+\beta \mathbb{1})t$$

$$\Rightarrow \boxed{\zeta(t) = e^{-\beta t} \left[\mathbb{1} + (A+\beta \mathbb{1}) t \right] \zeta(0)}$$