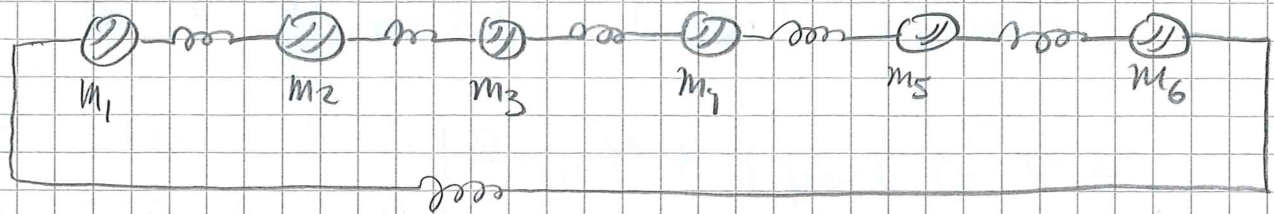
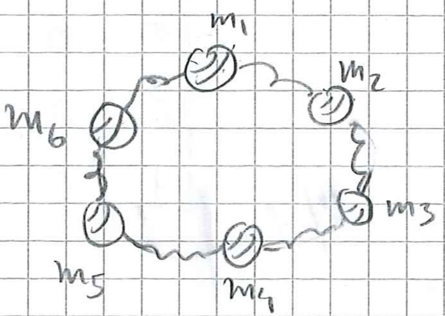
Q1 An Oscillating Chain

Six masses are placed in one dimension with springs in between them: (movement only in 1D)



Or six masses on a circle: (movement only on circle; springs are also constrained to lie on the circle)



The springs are all identical w/ spring const. $f \in \mathbb{R}_{\geq 0}$.

We have, for some $m \in \mathbb{R}_{\geq 0}$, $m_1 = m_4 = 2m$

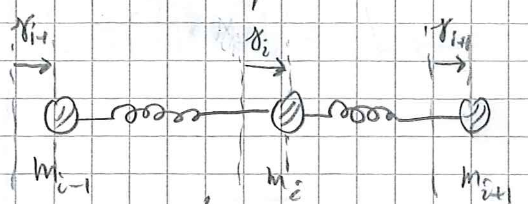
$m_2 = m_3 = m_5 = m_6 = m$

Note the symmetry of the picture under reflections about the vertical and horizontal axes.

Let $\gamma_i : \mathbb{R} \to \mathbb{R}$ be the displacement (along the circle) of the i^{th} mass from equilibrium position with the springs.

arclength displacement

Then by Hooke's law we have:



$$m_i \ddot{\gamma}_i = \underbrace{-f(\gamma_i - \gamma_{i+1})}_{\overleftarrow{\text{force}}} \underbrace{-f(\gamma_i - \gamma_{i-1})}_{\overleftarrow{\text{force}}} \quad \forall\, i \in \mathbb{Z}/6\mathbb{Z} + 1$$

$\boxed{2}$ The kinetic energy is

$$T(\dot{\gamma}) = \frac{1}{2}2m(\dot{\gamma}_1^2 + \dot{\gamma}_4^2) + \frac{1}{2}m(\dot{\gamma}_2^2 + \dot{\gamma}_3^2 + \dot{\gamma}_5^2 + \dot{\gamma}_6^2) = \dot{\gamma}^T m \begin{bmatrix} 1 & & & & & \\ & 1/2 & & & & \\ & & 1/2 & & & \\ & & & 1 & & \\ & & & & 1/2 & \\ & & & & & 1/2 \end{bmatrix} \dot{\gamma}$$

The potential energy is

$$V(\gamma) \underset{\gamma_7 := \gamma_1}{=} \frac{1}{2} f \sum_{i=1}^{6} (\gamma_i - \gamma_{i+1})^2 = f \sum_{i=1}^{6} \left[\frac{1}{2}\gamma_i^2 + \frac{1}{2}\gamma_{i+1}^2 - \gamma_i \gamma_{i+1}\right]$$

$$= f \sum_{i=1}^{6} \sum_{j=1}^{6} \gamma_i \delta_{ij} \gamma_j - f \sum_{i=1}^{6} \sum_{j=1}^{6} \gamma_i \delta_{i,j-1} \gamma_j$$

$$V = \gamma^T (f \mathbb{1}_{6\times 6}) \gamma + \gamma^T \left(-f \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\right) \gamma$$

Hence, as matrices,

$$\tilde{T} = m \operatorname{diag}(1, 1/2, 1/2, 1, 1/2, 1/2)$$

$$\tilde{V} = f \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \tilde{V}_S := \frac{1}{2}(\tilde{V} + \tilde{V}^T) = f \begin{bmatrix} 1 & -1/2 & 0 & 0 & 0 & -1/2 \\ & 1 & -1/2 & 0 & 0 & 0 \\ & & 1 & -1/2 & 0 & 0 \\ & & & 1 & -1/2 & 0 \\ & & & & 1 & -1/2 \\ & & & & & 1 \end{bmatrix}$$

Because $\tilde{T} = \tilde{T}^T$ and $\tilde{T} > 0$, $\exists\, L : \tilde{T} = L^T L$ and L is invertible.

Here it's easy to compute

$$L = \sqrt{m}\, \mathrm{diag}\,(1, 1/\sqrt{2}, 1/\sqrt{2}, 1, 1/\sqrt{2}, 1/\sqrt{2})$$

Hence

$$\tilde{V} := (L^{-1})^T \tfrac{1}{2}(V+V^T) L^{-1} = \frac{f}{m}\begin{bmatrix} 1 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} \\ & 2 & -1 & 0 & 0 & 0 \\ & & 2 & -\frac{1}{\sqrt{2}} & 0 & 0 \\ & & & 1 & -\frac{1}{\sqrt{2}} & 0 \\ & & & & 2 & -1 \\ & & & & & 2 \end{bmatrix}$$

Using Mathematica we find that the eigenvalues of $\tilde{V}$ are given by:

$$\boxed{\{0, 1, 2, 3, 2 \pm \sqrt{2}\}\, f/m}$$

The story would be finished now, except, you don't always have Mathematica, or sometimes the problem is too complicated even for Mathematica to solve, so we need to be able to solve this ourselves. But it's hard to diagonalize a 6×6 matrix, so use symmetries to reduce it to smaller blocks.

As noted above, the picture is symmetric about reflections about the horizontal & vertical axes.

In terms of transf. $\mathbb{R}^6 \to \mathbb{R}^6$ these correspond to:

$$R_v : (x_1, x_2, x_3, x_4, x_5, x_6) \longmapsto (x_1, x_6, x_5, x_4, x_3, x_2)$$

$$R_h : (x_1, x_2, x_3, x_4, x_5, x_6) \longmapsto (x_4, x_3, x_2, x_1, x_6, x_5)$$

We verify that they're really symmetries in the sense of ~~the~~ the def. on pp. 36 in the script:

4

As matrices, $\tilde{R_v} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$

$$\tilde{R_h} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$T(R_\alpha x) = T(x) \quad \forall \ \alpha \in \{v, h\}$ (simple verification)

$V(R_\alpha x) = V(x) \quad \forall \ \alpha \in \{v, h\}$ " "

Furthermore: $R_v R_h = R_h R_v$

$$R_\alpha{}^2 = \mathbb{1}$$

$\Rightarrow$ If $R_\alpha v = \lambda v$, $R_\alpha^2 v = R_\alpha \lambda v = \lambda^2 v \Rightarrow \lambda^2 = 1$

But $R_\alpha \in O(6)$ (bcs. $T \circ R_\alpha = T$) so that $\det(R_\alpha) = \pm 1$.

$\Rightarrow |\lambda| = 1 \ \forall$ e-val $\lambda \Rightarrow \boxed{\lambda = \pm 1}$.

(This could've been computed by hand for our particular R_v, R_h).

Because $[R_v, R_h] = 0$, they may be diagonalized simultaneously and we find:

$$\mathbb{R}^6 = W_{1,1} \oplus W_{1,-1} \oplus W_{-1,1} \oplus W_{-1,-1}$$

where $W_{i,j} := \{ v \in \mathbb{R}^6 \mid R_v\, v = i v,\ R_h\, v = j v \}$

(*) $W_{1,1} = \{ v \in \mathbb{R}^6 \mid v_1 = v_4 \wedge v_2 = v_3 = v_5 = v_6 \} \cong \mathbb{R}^2$

Then $\tilde{V}\big|_{W_{1,1}}$ should be a 2×2 matrix which we obtain via:

$$\tilde{V} \begin{bmatrix} \alpha \\ \beta \\ \beta \\ \alpha \\ \beta \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha - \sqrt{2}\beta \\ -\frac{\alpha}{\sqrt{2}} + \beta \\ \vdots \\ \vdots \end{bmatrix} \in W_{1,1}$$

$$\Rightarrow \tilde{V}\big|_{W_{1,1}} \cong \begin{bmatrix} 1 & -\sqrt{2} \\ -1/\sqrt{2} & 1 \end{bmatrix} \quad \text{w/ eigenvalues } \boxed{2, 0}. \checkmark$$

(*) $W_{1,-1} = \{ v \in \mathbb{R}^6 \mid v_1 = -v_4 \wedge v_2 = -v_3 = v_6 = -v_5 \} \cong \mathbb{R}^2$

$$\tilde{V} \begin{bmatrix} \alpha \\ \beta \\ -\beta \\ -\alpha \\ -\beta \\ +\beta \end{bmatrix} = \begin{bmatrix} \alpha - \sqrt{2}\beta \\ -\frac{\alpha}{\sqrt{2}} + 3\beta \\ \vdots \\ \vdots \end{bmatrix} \in W_{1,-1} \Rightarrow \tilde{V}\big|_{W_{1,-1}} = \begin{bmatrix} 1 & -\sqrt{2} \\ -\frac{1}{\sqrt{2}} & 3 \end{bmatrix}$$

w/ eigenvalues $\boxed{2 \pm \sqrt{2}}$. $\checkmark$

(*) $W_{-1,1} = \{ v \in \mathbb{R}^6 \mid v_1 = v_4 = 0 \wedge v_2 = -v_6 = v_3 = -v_5 \} \cong \mathbb{R}$

$$\tilde{V} \begin{bmatrix} 0 \\ \alpha \\ \alpha \\ 0 \\ -\alpha \\ -\alpha \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha \\ \alpha \\ 0 \\ -\alpha \\ -\alpha \end{bmatrix} \Rightarrow \tilde{V}\big|_{W_{-1,1}} = \mathbb{1}_{2\times 2} \text{ w/ eigenvalue } \boxed{1}. \checkmark$$

$\boxed{6}$ (*) $\mathbb{W}_{-1,-1} = \{ v \in \mathbb{R}^6 \mid v_1 = v_4 = 0 \wedge v_2 = -v_3 = v_5 = -v_6 \} \overset{?}{\subseteq} \mathbb{R}^6$.

$$\tilde{V} \begin{bmatrix} 0 \\ \alpha \\ -\alpha \\ 0 \\ \alpha \\ -\alpha \end{bmatrix} = \begin{bmatrix} 0 \\ 3\alpha \\ -3\alpha \\ 0 \\ 3\alpha \\ -3\alpha \end{bmatrix} \Rightarrow \tilde{V}\Big|_{\mathbb{W}_{-1,-1}} = 3$$

w/ eigenvalue $\boxed{3}$. ✓

Paul's Trap – Quadrupole Ion Trap

a) **Cl.:** The equilibrium of a charged particle in an external time-indep el. field in vaccuum is unstable.

Pf.: The EoM for a charged particle is given by $m\ddot{\gamma} = e\,E(\gamma)$
where $m \equiv$ mass of particle
$e \equiv$ charge of particle
$\gamma : \mathbb{R} \to \mathbb{R}^3 \equiv$ traj. of particle
$E : \mathbb{R}^3 \to \mathbb{R}^3 \equiv$ el. field.

Equilibrium is reached when $\ddot{\gamma} = 0$, so when
$$E(x_0) = 0 \qquad \text{(alg. eq-n for } x_0 \in \mathbb{R}^3\text{)}$$
By the hint, recall $E = -\nabla\phi$ where $\phi : \mathbb{R}^3 \to \mathbb{R}$ is the el. pot.

$$\Rightarrow \quad (-\nabla\phi)(x_0) = 0$$

$\Rightarrow$ x_0 is an extremal pt. of ϕ.

Assume $\phi(x_0) = 0$ (can always shift el. pot. by const.) so:
$$\phi(x) \approx (\partial_i\partial_j\phi)(x_0)\, x_i x_j + O(|x_\ell|^3)$$

$$\Rightarrow (-\nabla\phi)(x) = -(e_\ell\,\partial_\ell\phi)(x) = -e_\ell(\partial_\ell\partial_j\phi)(x_0)\,x_j$$

$\Rightarrow$ EoM near x_0 is:

$$m\ddot{\gamma} = -e\,H(x_0)\,\gamma$$

where $H(x_0)$ is the matrix w/ entries $\{(\partial_i\partial_j\phi)(x_0)\}_{i,j}$.
Bcs. $H(x_0)$ is symmetric, it may be diagonalized w/ e-vals h_1, h_2, h_3 w/ eigenvectors e_1, e_2, e_3.
Solve the EoM w/ the Ansatz $\gamma(t) = a\,e^{i\lambda t}$

get $\dot{\gamma} = a e^{i\lambda t} i\lambda$

$\ddot{\gamma} = a e^{i\lambda t}(-\lambda^2)$

$$\Rightarrow \quad m a \cancel{e^{i\lambda t}}(-\lambda^2) = -e H(x_0) a \cancel{e^{i\lambda t}}$$

$$\Rightarrow \quad H(x_0) a = \frac{m}{e}\lambda^2 a$$

$$\Rightarrow \quad \frac{m}{e}\lambda_i^2 = h_i \quad \text{and} \quad a = e_i \quad \forall\, i \in \{1,2,3\}$$

are the possible solutions.

By def., the equil. is stable if $\lambda^2 > 0$.

Note $tr(H(x_0)) = (\Delta \phi)(x_0)$ (easy verification).

But by Poisson's eq-n, $(\Delta\phi)(x_0) = 0$ (no charge at that pt.)

$\Rightarrow \quad tr(H(x_0)) = 0 \quad \Leftrightarrow \quad h_1 + h_2 + h_3 = 0$

By the hint we're assuming $H(x_0) \neq 0$.

$\Rightarrow \quad \exists\, i : h_i < 0$.

$\Rightarrow$ For that i, $\lambda^2 < 0 \Rightarrow$ System is <u>unstable</u>.

(b) <u>Cl.:</u> The EoM of a charged particle under the influence of an el. pot. $\phi(x,t) = \varphi(t)\dfrac{x_1^2 + x_2^2 - 2x_3^2}{2r_0^2}$

where $r_0 \in \mathbb{R}_{\geq 0}$ and $\varphi : \mathbb{R} \to \mathbb{R}$ is some map

w/ $\varphi(t+T) = \varphi(t)\ \forall t,\ \exists\, T \in \mathbb{R}_{>0}$.

Pf.: We have $m\ddot{\gamma} = e E(\gamma) = -e(\nabla\phi)(\gamma)$

$(\nabla\phi) = e_i \partial_i \phi$

$(\partial_i \phi)(x) = \dfrac{\varphi(t)}{2r_0^2} \cdot 2x_i(\delta_{i,1} + \delta_{i,2} - 2\delta_{i,3})$

$$\Rightarrow \quad \boxed{m\ddot{\gamma} = -\frac{e\varphi(t)}{r_0^2}\left[\gamma_1 e_1 + \gamma_2 e_2 - 2\gamma_3 e_3\right]}$$