Q1 Lagrange Functions

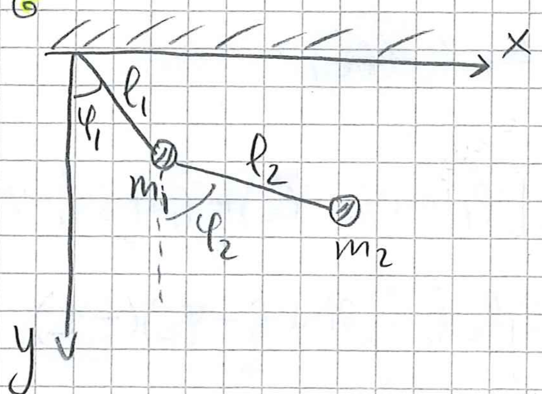
Recall that if the constraints are holonomic (constraints of the form $f(t, q^{i_1}, \ldots, q^{i_j}) = 0 \quad \exists f$) then the Lagrangian (in generalized coordinates $\{q_i\}_i : \mathbb{R} \to \mathbb{R}$) by

$$\boxed{L = T - V}$$

and the equations of motion are given by:

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\alpha} - \frac{\partial L}{\partial q^\alpha} = 0}$$

a) The Double Pendulum (under influence of gravity)



$x_1 = l_1 \sin(\varphi_1)$

$y_1 = l_1 \cos(\varphi_1)$

$x_2 = x_1 + l_2 \sin(\varphi_2) = l_1 \sin(\varphi_1) + l_2 \sin(\varphi_2)$

$y_2 = y_1 + l_2 \cos(\varphi_2) = l_1 \cos(\varphi_1) + l_2 \cos(\varphi_2)$

$$L = T - V = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) + m_1 g y_1 + m_2 g y_2$$

$$= \frac{1}{2}m_1\left(l_1^2 \cos(\varphi_1)^2 \dot{\varphi}_1^2 + l_1^2(-\sin(\varphi_1))^2 \dot{\varphi}_1^2\right) +$$

$$+ \frac{1}{2}m_2\left((l_1 \cos(\varphi_1)\dot{\varphi}_1 + l_2 \cos(\varphi_2)\dot{\varphi}_2)^2 + (-l_1 \sin(\varphi_1)\dot{\varphi}_1 - l_2 \sin(\varphi_2)\dot{\varphi}_2)^2\right)$$

$$+ m_1 g\, l_1 \cos(\varphi_1) + m_2 g\, l_1 \cos(\varphi_1) + m_2 g\, l_2 \cos(\varphi_2)$$

$$= \underline{\frac{1}{2}m_1 l_1^2 \dot{\varphi}_1^2} + \frac{1}{2}m_2\left[\underline{l_1^2 \dot{\varphi}_1^2} + \underline{l_2^2 \dot{\varphi}_2^2} + 2 l_1 l_2 \cos(\varphi_1 - \varphi_2)\dot{\varphi}_1 \dot{\varphi}_2\right]$$

$$\underline{+ (m_1 + m_2) g\, l_1 \cos(\varphi_1)} \underline{+ m_2 g\, l_2 \cos(\varphi_2)}$$

[2] We then obtain the EoM: $(m_1+m_2=:M)$

$$\frac{\partial L}{\partial \dot\varphi_1} = \frac{M}{2} l_1^2 2\dot\varphi_1 + m_2 l_1 l_2 \dot\varphi_2 \cos(\varphi_1-\varphi_2)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\varphi_1} = M l_1^2 \ddot\varphi_1 + m_2 l_1 l_2 \left[\ddot\varphi_2 \cos(\varphi_1-\varphi_2) - \dot\varphi_2 \sin(\varphi_1-\varphi_2)(\dot\varphi_1-\dot\varphi_2)\right]$$

$$\frac{\partial L}{\partial \varphi_1} = -Mgl_1 \sin(\varphi_1) - m_2 l_1 l_2 \dot\varphi_1 \dot\varphi_2 \sin(\varphi_1-\varphi_2)$$

So the first EoM is:

$$M l_1^2 \ddot\varphi_1 + m_2 l_1 l_2 \ddot\varphi_2 \cos(\varphi_1-\varphi_2) + m_2 l_1 l_2 \dot\varphi_2^2 \sin(\varphi_1-\varphi_2)$$
$$\cancel{-m_2 l_1 l_2 \dot\varphi_2 \dot\varphi_1 \sin(\varphi_1-\varphi_2)} + Mgl_1 \sin(\varphi_1) \cancel{+ m_2 l_1 l_2 \dot\varphi_1 \dot\varphi_2 \sin(\varphi_1-\varphi_2)} = 0$$

$$\boxed{\ddot\varphi_1 + \frac{g}{l_1}\sin(\varphi_1) + \frac{m_2 l_2}{M l_1}\left[\cos(\varphi_1-\varphi_2)\ddot\varphi_2 + \sin(\varphi_1-\varphi_2)\dot\varphi_2^2\right] = 0}$$

$$\frac{\partial L}{\partial \dot\varphi_2} = m_2 l_2^2 \dot\varphi_2 + m_2 l_1 l_2 \dot\varphi_1 \cos(\varphi_1-\varphi_2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\varphi_2}\right) = m_2 l_2^2 \ddot\varphi_2 + m_2 l_1 l_2 \left[\ddot\varphi_1 \cos(\varphi_1-\varphi_2) + \dot\varphi_1 \sin(\varphi_1-\varphi_2)(\dot\varphi_1-\dot\varphi_2)\right]$$

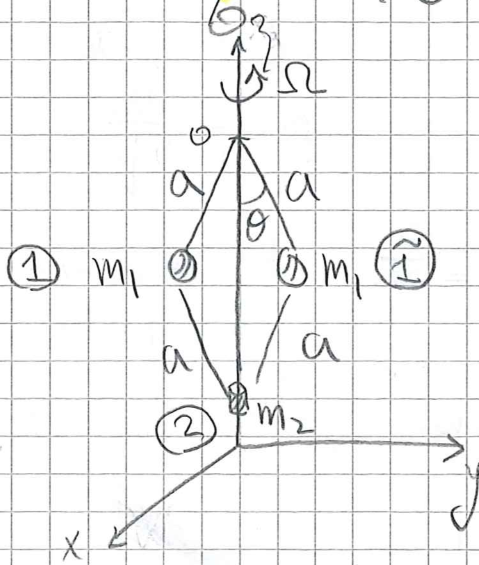
$$\frac{\partial L}{\partial \varphi_2} = -m_2 g l_2 \sin(\varphi_2) - m_2 l_1 l_2 \dot\varphi_1 \dot\varphi_2 \sin(\varphi_1-\varphi_2)(-1)$$

So the second EoM is:

$$\cancel{m_2} l_2^2 \ddot\varphi_2 + \cancel{m_2} l_1 l_2 \left[\ddot\varphi_1 \cos(\varphi_1-\varphi_2) - \dot\varphi_1 \sin(\varphi_1-\varphi_2)(\dot\varphi_1 \cancel{-\dot\varphi_2})\right]$$
$$+\cancel{m_2} g l_2 \sin(\varphi_2) \cancel{- m_2 l_1 l_2 \dot\varphi_1 \dot\varphi_2 \sin(\varphi_1-\varphi_2)} = 0$$

$$\boxed{\ddot\varphi_2 + \frac{g}{l_2}\sin(\varphi_2) + \frac{l_1}{l_2}\left[\cos(\varphi_1-\varphi_2)\ddot\varphi_1 - \sin(\varphi_1-\varphi_2)\dot\varphi_1^2\right]}$$

b) Centrifugal Governor ⑥



Recall we'll have stability if the oscillations are of the form $\ddot{\theta} + \underbrace{\quad}_{\text{positive}} \theta = 0$.
(definition (4.13))

$$x_2 = 0,\ y_2 = 0,\ z_2 = -2a\cos(\theta)$$

$$x_1 = a\sin(\theta)\cos(\Omega t)$$

$$y_1 = a\sin(\theta)\sin(\Omega t)$$

$$z_1 = -a\cos(\theta)$$

$$x_{\tilde{1}} = a\sin(\theta)\cos(\Omega t + \pi) = -a\sin(\theta)\cos(\Omega t)$$

$$y_{\tilde{1}} = a\sin(\theta)\sin(\Omega t + \pi) = -a\sin(\theta)\sin(\Omega t)$$

$$z_{\tilde{1}} = -a\cos(\theta)$$

$$L = T - V = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2) + \frac{1}{2}m_1(\dot{x}_{\tilde{1}}^2 + \dot{y}_{\tilde{1}}^2 + \dot{z}_{\tilde{1}}^2) +$$

$$+ \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2) - m_1 g z_1 - m_1 g z_{\tilde{1}} - m_2 g z_2$$

$$= \frac{1}{2}m_1 a^2 \Big[\big(\cos(\theta)\dot{\theta}\cos(\Omega t) - \sin(\theta)\sin(\Omega t)\Omega\big)^2 +$$

$$+ \big(\cos(\theta)\dot{\theta}\sin(\Omega t) + \sin(\theta)\cos(\Omega t)\Omega\big)^2 + \big(\sin(\theta)\dot{\theta}\big)^2 +$$

$$+ \big(-\cos(\theta)\dot{\theta}\cos(\Omega t) + \sin(\theta)\sin(\Omega t)\Omega\big)^2 +$$

$$+ \big(-\cos(\theta)\dot{\theta}\sin(\Omega t) - \sin(\theta)\cos(\Omega t)\Omega\big)^2 + \big(\sin(\theta)\dot{\theta}\big)^2 \Big]$$

$$+ \frac{1}{2}m_2\big(+2a\sin(\theta)\dot{\theta}\big)^2 + m_1 g a\cos(\theta) + m_1 g a\cos(\theta) + m_2 g 2a\cos(\theta)$$

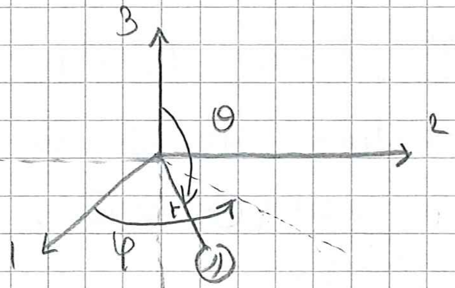
$$= m_1 a^2\left[\dot{\theta}^2 + \Omega^2\sin(\theta)^2\right] + 2m_2 a^2\sin(\theta)^2\dot{\theta}^2 + 2(m_1 + m_2) g a\cos(\theta)$$

$$\frac{\partial L}{\partial \dot{\theta}} = 2m_1 a^2\dot{\theta} + 2m_2 a^2\sin(\theta)^2 2\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 2m_1 a^2\ddot{\theta} + 4m_2 a^2\sin(\theta)^2\ddot{\theta} + 8m_2 a^2\sin(\theta)\cos(\theta)\dot{\theta}^2$$

$$\frac{\partial L}{\partial \theta} = m_1 a^2\Omega^2 2\sin(\theta)\cos(\theta) + 2m_2 a^2 2\sin(\theta)\cos(\theta)\dot{\theta}^2 - 2(m_1 + m_2) g a\sin(\theta)$$

4 (c) The Spherical Pendulum



Recall the conjugate momenta are given by $p_i = \frac{\partial L}{\partial \dot{q}_i}$.

Hence $\dot{p}_i = \frac{\partial L}{\partial q_i}$ by Euler-Lagrange equation.

$\Rightarrow$ If L is indep. of q_i then p_i is constant.

Here we'll find p_φ is const.
Also the energy is const.
We'll write the energy in terms of θ and p_φ alone, and get the relation $E = T_\theta + U_\theta$. Bcs. $T_\theta > 0$, $U_\theta < E$. This will constrain us away from certain configuration points.

$$x = r\sin(\theta)\cos(\varphi) \qquad \dot{x} = r[\cos(\theta)\dot{\theta}\cos(\varphi) - \sin(\theta)\sin(\varphi)\dot{\varphi}]$$

$$y = r\sin(\theta)\sin(\varphi) \qquad \dot{y} = r[\cos(\theta)\dot{\theta}\sin(\varphi) + \sin(\theta)\cos(\varphi)\dot{\varphi}]$$

$$z = r\cos(\theta) \qquad \dot{z} = -r\sin(\theta)\dot{\theta}$$

$$L = T - V = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$

$$= \frac{1}{2}mr^2\left[\sin(\theta)^2\dot{\varphi}^2 + \dot{\theta}^2\right] - mgr\cos(\theta)$$

Note difference to official sol-ns. Because their θ is our $\pi - \theta$.

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = mr^2\ddot{\theta} \qquad \frac{\partial L}{\partial \theta} = mr^2\sin(\theta)\cos(\theta)\dot{\varphi}^2 + mgr\sin(\theta)$$

E.o.M. for θ: $\cancel{mr^2}\ddot{\theta} - \cancel{mr^2}\sin(\theta)\cos(\theta)\dot{\varphi}^2 - \cancel{m}g\overset{r^{-1}}{\cancel{r}}\sin(\theta) = 0$

$$\boxed{\ddot{\theta} - \sin(\theta)\cos(\theta)\dot{\varphi}^2 - \frac{g}{r}\sin(\theta) = 0}$$

Hence the E.o.M. is:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\cancel{2}m_1 \cancel{a^2}\ddot\theta + \overset{2}{\cancel{4}} m_2 \cancel{a^2} \sin(\theta)^2 \ddot\theta + \overset{4}{\cancel{8}} m_2 \cancel{a^2} \sin(\theta)\cos(\theta)\dot\theta^2 -$$

$$- m_1 \cancel{a^2} \Omega^2 \cancel{2} \sin(\theta)\cos(\theta) - \overset{2}{\cancel{4}} m_2 \cancel{a^2} \sin(\theta)\cos(\theta)\dot\theta^2 + \cancel{2}(m_1+m_2) g \overset{a^{-1}}{\cancel{a}} \sin(\theta) = 0$$

$$\boxed{\left[m_1 + 2m_2\sin(\theta)^2\right]\ddot\theta + 2m_2\sin(\theta)\cos(\theta)\dot\theta^2 + \sin(\theta)\left[\frac{m_1+m_2}{a}g - m_1\Omega^2\cos(\theta)\right] = 0}$$

Equilibrium $\Leftrightarrow$ $\dot\theta = 0$

Implies the condition

$$\sin(\theta)\left[\frac{m_1+m_2}{a}g - m_1\Omega^2\cos(\theta)\right] = 0$$

$\sin(\theta) = 0$

$\downarrow$

$\theta_n = n\pi \; ; n\in\mathbb{Z}$

By construction of the device, only $\theta = 0$ is possible.

$\frac{m_1+m_2}{a}g - m_1\Omega^2\cos(\theta) = 0$

$\downarrow$

$\cos(\theta) = \frac{m_1+m_2}{a m_1 \Omega^2} g$

$\boxed{\text{Hence if } \frac{m_1+m_2}{a m_1\Omega^2} g > 1 \;(*) \quad \nexists \text{ solution here, as } |\cos(\theta)| \le 1.}$

If $\frac{m_1+m_2}{a m_1 \Omega^2} g \le 1$,

$\theta_n = \arccos\left(\frac{m_1+m_2}{a m_1\Omega^2} g\right) + 2\pi n \; ; n\in\mathbb{Z}$

Only $n=0$ is possible as $\theta < \pi$.

Study oscillations near equilibrium to decide about stability:

Case 1: θ near 0. (Everything to linear order in θ)

Get $m_1\ddot\theta + \theta\left[\frac{m_1+m_2}{a}g - m_1\Omega^2\right] = \mathcal{O}(\theta^2)$

3.6

<u>Note</u>: here we assumed $\dot{\theta}$ is of order θ.
$\ddot{\theta}$ is of order θ.

This works by an iterative solution of the E.o.M.

When ⊛ holds, the bracket in front of θ is positive, so that by def. (4.13) the oscillating system is stable.

When ⊛ fails to hold, $\theta = 0$ is not a stable point.

<u>Case 2</u>: $\theta = \overbrace{\arccos\left(\frac{m_1+m_2}{a m_1 \Omega^2} g\right)}^{=:\alpha} + \varphi$, $\varphi \ll 1$

and ⊛ fails to hold.
(Work to linear order in φ)

$$\sin(\theta) = \sin(\alpha+\varphi) \approx \sin(\alpha) + \cos(\alpha)\varphi + O(\varphi^2)$$
$$\cos(\theta) = \cos(\alpha+\varphi) \approx \cos(\alpha) - \sin(\alpha)\varphi + O(\varphi^2)$$
$$\sin(\alpha) = \sin\left(\arccos\left(\frac{\cdots}{\cdots}\right)\right) = \sqrt{1-\left(\frac{\cdots}{\cdots}\right)^2}$$

Thus the E.o.M. becomes:

$$\left[m_1 + 2m_2 \sin(\alpha)^2\right]\ddot{\varphi} + \left[\sin(\alpha) + \cos(\alpha)\varphi\right] \times$$
$$\times\left[\cancel{\frac{m_1+m_2}{a} g} - m_1\Omega^2\left(\cancel{\frac{m_1+m_2}{a m_1 \Omega^2} g} - \sin(\alpha)\varphi\right)\right] = 0$$

$$\left[m_1 + 2m_2 \sin(\alpha)^2\right]\ddot{\varphi} + \underbrace{\sin(\alpha)^2 m_1 \Omega^2}_{\text{again positive}} \varphi = 0$$

$\Rightarrow$ Oscillations are stable.

$$\frac{\partial L}{\partial \dot{\varphi}} = mr^2 \sin(\theta)^2 \dot{\varphi} \qquad \frac{\partial L}{\partial \varphi} = 0$$

$$\Rightarrow \boxed{\frac{d}{dt}\big(\underbrace{mr^2 \sin(\theta)^2 \dot{\varphi}}_{=: p_\varphi}\big) = 0} \quad \text{EOM for } \varphi$$

Hence the conjugate momentum to φ is constant.

The total energy $E = T + V$ is also constant.

$$E = \frac{1}{2} mr^2 \left[\sin(\theta)^2 \dot{\varphi}^2 + \dot{\theta}^2\right] + mgr\cos(\theta)$$

$$\dot{\varphi} = \frac{p_\varphi}{mr^2 \sin(\theta)^2}$$

$$\stackrel{\triangle}{=} \frac{1}{2} mr^2 \left[\cancel{\sin(\theta)^2} \frac{p_\varphi^2}{m^2 r^4 \sin(\theta)^{\cancel{4}}} + \dot{\theta}^2\right] + mgr\cos(\theta)$$

$$= \underbrace{\frac{1}{2} mr^2 \dot{\theta}^2}_{\tilde{T}} + \underbrace{\frac{1}{2}\frac{p_\varphi^2}{mr^2} \sin(\theta)^{-2} + mgr\cos(\theta)}_{=: U(\theta)}$$

E is constant, so for a fixed value of E and p_φ (which is non-zero), θ may only take values s.t. $U(\theta) \leq E$ (bcs. $\tilde{T} \geq 0$).

That means that those values of θ s.t. $U(\theta) > E$ are not allowed. So we need to solve the inequality

$$U(\theta(E, p_\varphi)) > E$$

$$\underbrace{\frac{1}{2}\frac{p_\varphi^2}{mr^2}}_{=: A} \sin(\theta(E, p_\varphi))^{-2} + \underbrace{mgr}_{=: B} \cos(\theta(E, p_\varphi)) - E > 0$$

Define $x := \sin(\theta(E, p_\varphi))$. $\Rightarrow \cos(\theta(E, p_\varphi)) = \sqrt{1 - x^2}$

$$\Leftrightarrow \quad Ax^{-2} + B\sqrt{1 - x^2} - E > 0$$

$$\Leftrightarrow \quad Ex^2 - A < B\sqrt{1 - x^2} \qquad \Leftrightarrow \quad E^2x^4 - 2EAx^2 + A^2 < B^2 - B^2x^2$$

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$$\Leftrightarrow \quad E^2 x^4 + (B^2 - 2EA)x^2 + A^2 - B^2 < 0$$

$$x^2 = \frac{-(B^2-2EA) \pm \sqrt{(B^2-2EA)^2 - 4E^2(A^2-B^2)}}{2E^2}$$

$$= \frac{2EA - B^2 \pm \sqrt{B^4 - 4EAB^2 + 4E^2B^2}}{2E^2}$$

$$= \frac{2EA - B^2 \pm B^2\sqrt{B^2 - 4EA + 4E^2}}{2E^2}$$

$$\Leftrightarrow \quad \frac{2EA - B^2 - B^2\sqrt{B^2-4EA+4E^2}}{2E^2} < x < \frac{2EA - B^2 - B^2\sqrt{B^2-4EA+4E^2}}{2E^2}$$

...

Can check that this happens, for example, for values of θ suff. close to 0 or π.

$\boxed{Q2}$ Quadrupole Ion Trap — The Return

Recall from part 1 (HW8) we had the EoM:
$$m\ddot{\gamma} = -\frac{e\varphi(t)}{r_0^2}\left[\gamma_1 e_1 + \gamma_2 e_2 - 2\gamma_3 e_3\right]$$

We choose
$$\varphi(t) := \frac{m r_0^2}{e}\begin{cases} a_+ & t \in (0, T/2) \\ a_- & t \in (T/2, T) \end{cases} =: \frac{m r_0^2}{e} a(t)$$

and assume $|a_\pm| T^2 \ll 1$.

Define $\varepsilon_\pm := \sqrt{a_\pm}\, T/2$, $\quad \omega_\pm := \sqrt{a_\pm}$

$$\ddot{\gamma}_i = -a(t)\gamma_i \qquad \forall\, i \in \{1,2\}$$

$$\gamma_3 = 2a(t)\gamma_3$$

$\lambda_i := \begin{bmatrix} \gamma_i \\ \dot{\gamma}_i \end{bmatrix} \quad \forall i \in \{1,2,3\}$

$A(t) = \begin{bmatrix} 0 & 1 \\ -a(t) & 0 \end{bmatrix}$

$\boxed{\dot{\lambda}_i = A\lambda_i}$

Deal first with the case $i \in \{1,2\}$.

The propagator for the system is given by:
$$P(T) = p(T/2; \omega_-)\, p(T/2; \omega_+)$$

where
$$p(t, \omega) \equiv \begin{bmatrix} \cos(\omega t) & \omega^{-1}\sin(\omega t) \\ -\omega\sin(\omega t) & \cos(\omega t) \end{bmatrix}$$

Recall from pp. 38 that the system is stable if $|\mathrm{tr}(P(T))| \leq 2$.

This is because $|\mathrm{tr}(P(T))| \leq 2$ implies the eigenvalues of $P(T)$ are within the unit circle on $\mathbb{C}$.

But the eigenvalues of $P(T)$ are important because:

$\lambda_i(t) = P(t)\lambda_i(0)$

Since A is T-periodic, so is P, and

$P(nT) = P(T)^n$

$\Rightarrow$ If e-vals inside S^1, $n \to \infty$ gives stable behavior!

Recall also that system is stable iff it is stable $\forall\, i \in \{1,2,3\}$.

Thus we compute the trace and find:

$$P(T) = \begin{bmatrix} \cos(\frac{T}{2}\omega_-)\cos(\frac{T}{2}\omega_+) - \frac{\omega_+}{\omega_-}\sin(\frac{T}{2}\omega_-)\sin(\frac{T}{2}\omega_+) & \cdots \\ \cdots & \bigcirc \end{bmatrix}$$

$$\cos(\tfrac{T}{2}\omega_-)\cos(\tfrac{T}{2}\omega_+) - \frac{\omega_-}{\omega_+}\sin(\tfrac{T}{2}\omega_-)\sin(\tfrac{T}{2}\omega_+)$$

$$\Rightarrow tr(P(T)) = 2\cos(\tfrac{T}{2}\omega_-)\cos(\tfrac{T}{2}\omega_+) - \left(\frac{\omega_+}{\omega_-}+\frac{\omega_-}{\omega_+}\right)\sin(\tfrac{T}{2}\omega_-)\sin(\tfrac{T}{2}\omega_+)$$

$\varepsilon_\pm \equiv \omega_\pm \frac{T}{2}$

$$= 2\cos(\varepsilon_-)\cos(\varepsilon_+) - \underbrace{\left(\frac{\omega_+^2+\omega_-^2}{\omega_-\omega_+}\right)}_{\frac{\varepsilon_+^2+\varepsilon_-^2}{\varepsilon_+\varepsilon_-} = \frac{\omega_+^2(\cancel{T/2})^2+\omega_-^2(\cancel{T/2})^2}{\omega_+(\cancel{T/2})\,\omega_-(\cancel{T/2})} = \frac{\omega_+^2+\omega_-^2}{\omega_+\omega_-}}\sin(\varepsilon_-)\sin(\varepsilon_+)$$

$$= 2\cos(\varepsilon_-)\cos(\varepsilon_+) - (\varepsilon_+^2+\varepsilon_-^2)\frac{\sin(\varepsilon_-)}{\varepsilon_-}\frac{\sin(\varepsilon_+)}{\varepsilon_+}$$

Note that if $a_\pm < 0$, $\omega_\pm \in i\mathbb{R} \Rightarrow \varepsilon_\pm \in i\mathbb{R}$, $\varepsilon_\pm^2 \in \mathbb{R}$.

Even in that case, $\cos(i|\alpha|) = \cosh(|\alpha|) \in \mathbb{R}$

$$\frac{\sin(i|\alpha|)}{i|\alpha|} = \frac{i\sinh(|\alpha|)}{i|\alpha|} = \frac{\sinh(|\alpha|)}{|\alpha|} \in \mathbb{R}$$

The assumption that $|a_\pm|T^2 \ll 1$ implies $|\varepsilon_\pm| \ll 1$.

Hence we make a Taylor expansion of cosine & sine up to <u>sixth</u> order in $|\varepsilon_\pm|$ to get:

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)$$

$$\sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120} + O(x^6)$$

$$\Rightarrow tr(P(T)) = 2 - 2(\varepsilon_-^2+\varepsilon_+^2) + \frac{1}{12}\left[3(\varepsilon_-^4+\varepsilon_+^4) + 10\,\varepsilon_-^2\varepsilon_+^2\right] + O(|\varepsilon_\pm|^6)$$

$$\in \mathbb{R}$$

Recall $|\alpha| \leq 2 \Leftrightarrow -2 \leq \alpha \leq 2$

Hence the condition for stability is now:

$$-2 \leq 2 - 2(\varepsilon_-^2 + \varepsilon_+^2) + \frac{1}{12}\left[3(\varepsilon_-^4 + \varepsilon_+^4) + 10\varepsilon_-^2\varepsilon_+^2\right] \leq 2$$

$$\Rightarrow \quad -2(\varepsilon_-^2 + \varepsilon_+^2) + \frac{1}{12}\left[3(\varepsilon_-^4 + \varepsilon_+^4) + 10\varepsilon_-^2\varepsilon_+^2\right] \leq 0$$

$$\Leftrightarrow \quad -\cancel{2}(a_- + a_+)\frac{T^2}{\cancel{4}_{\,2}} + \frac{1}{12}\left[3(a_-^2 + a_+^2)\frac{T^4}{16} + 10 a_- a_+ \frac{T^4}{16}\right] \leq 0$$

Define $\quad b_\pm := \mp\frac{1}{2}(a_+ \pm a_-)T^2.$

Note $\quad 3(a_-^2 + a_+^2) + 10 a_- a_+ = 4(a_- + a_+)^2 - (a_- - a_+)^2$

$$= 4\left(b_+ \frac{2}{T^2}\right)^2 - \left(b_- \frac{2}{T^2}\right)^2$$

$$= \frac{4}{T^4}\left(4b_+^2 - b_-^2\right)$$

Hence we find the condition:

$$b_+ + \frac{1}{12}\frac{\cancel{T^4}}{\cancel{16}_{\,4}}\frac{4}{\cancel{T^4}}(4b_+^2 - b_-^2) \leq 0$$

$$\Leftrightarrow \quad \boxed{b_+ + \frac{1}{48}(4b_+^2 - b_-^2) \leq 0}$$

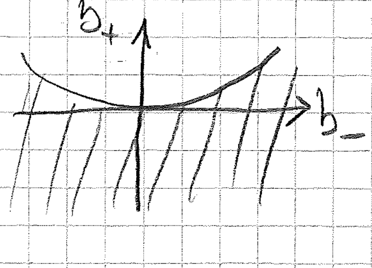
$$\Leftrightarrow \quad b_+^2 + 12 b_+ - \frac{1}{4}b_-^2 \leq 0$$

$$\Leftrightarrow \quad (b_+ + 6)^2 - 36 - \frac{1}{4}b_-^2 \leq 0 \qquad \text{(hyperbola)}$$

$$\Leftrightarrow \quad -6 - \frac{1}{2}\sqrt{144 + b_-^2} \leq b_+ \leq -6 + \frac{1}{2}\sqrt{144 + b_-^2}$$

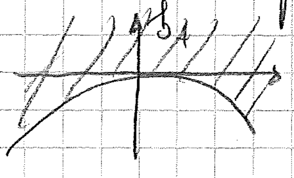
up to $O(b_-^2)$

$$\Leftrightarrow \quad -12 - \frac{b_-^2}{48} \leq \boxed{b_+ \leq \frac{b_-^2}{48}}$$



For the third axis we need the replacement $a_\pm \mapsto -2a_\pm$.

$$\Rightarrow \quad b_+ \geq -\frac{b_-^2}{24}$$



$\xRightarrow{\text{inter-section}}$