Additional Notes for 12^{th} Week

7/12/2016

Legendre Transformation

We have defined the L. T. of a map $f: I \to \mathbb{R}$ which is strictly convex and cont. diff. as

$$(\mathcal{L}f)(x) := x((f')^{-1})(x) - f((f')^{-1}(x))$$

where $(f')^{-1}: \mathbb{R} \to I$ is the inverse of f', which $\exists$ bcs:

Claim: f' is injective.

Proof: Since f is strictly convex, $f'' > 0$.

$\Rightarrow$ f' is strictly increasing.

$\Rightarrow$ f' is injective, hence has a left inverse $(f')^{-1}$.

$L(x,v): (\mathbb{R}^f)^2 \to \mathbb{R}$ is the Lagrangian function and its Legendre transform is given by

$H(x,p): (\mathbb{R}^f)^2 \to \mathbb{R}$ $\quad H(x,p) := (\mathcal{L}\, L(x,\cdot))(p)$ (x is fixed)

Explicitly: $H(x,p) \equiv (\mathcal{L}\, L(x,\cdot))(p)$

$$= p(\partial_v L(x,\cdot)^{-1})(p) - L(x, (\partial_v L(x,\cdot)^{-1})(p))$$

$$\partial_v L(x,\cdot) \equiv p(x,\cdot)$$

$$\Rightarrow (\partial_v L(x,\cdot))^{-1} = p(x,\cdot)^{-1} = v \text{ in terms of } p \text{ and } x.$$

Hence we find: $H(x,p) = p\, v(x,p) - L(x, v(x,p))$

which is the usual def. of a Hamiltonian.

Claim: $L(x,\cdot)$ is really strictly convex.

Proof: $L(x,v) \underset{\text{usually}}{=} \frac{1}{2}mv^2 + V(x)$

$\partial_v L(x,v) = mv \qquad \partial_v^2 L(x,v) = m > 0$