Analytical Mechanics Recitation Session of Week 13

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December 14, 2016

1 Go with the Flow

Let $f \in \mathbb{N}_{\geq 1}$ (the number of degrees of freedom in the system) and define $\Gamma := \mathbb{R}^{2f}$ (the phase space manifold) which has a differentiable manifold structure.

1 Remark. In this document (perhaps contrary to before) Γ is the phase space (position and momentum), $\gamma : \mathbb{R} \to \Gamma$ is a typical trajectory (time parametrized) in phase space. As usual, if a function depends on both $\mathbb{R}$ (time) and Γ (phase space), then the symbol ∂ alone means time derivative, ∂_i means derivative with respect to the ith coordinate in Γ (so if $i \in \{ 1, \dots, f \}$ then ∂_i is derivative with respect to the position part of phase space and ∂_{i+f} is derivative with respect to the momentum part of the phase space.

2 Definition. A *flow* is a group morphism $\varphi : \mathbb{R} \to Aut(\Gamma)$ where $\mathbb{R}$ is considered as the additive group, and Γ has the structure of a differentiable manifold (and it is in to that structure that automorphisms of Γ refer to, not to the structure of a vector space! In this regard the term automorphism is perhaps confusing).

3 Definition. Given a flow $\varphi : \mathbb{R} \to Aut(\Gamma)$, its *orbits*, *trajectories*, or *integral curves* is the following set of trajectories

$$\mathcal{O}(\varphi) \quad := \quad \{ \gamma : \mathbb{R} \to \Gamma \mid \gamma(t) := (\varphi(t))(x) \text{ for all } t \in \mathbb{R} \text{ for some } x \in \Gamma \}$$

Since we know that $\varphi(0) = \mathbb{1}_\Gamma$ (φ is a group morphism), this means that the trajectories of φ are obtained by varying over all possible starting points.

4 Claim. For all $(x, y) \in \Gamma^2$, define $x \sim y$ iff $\exists (\gamma, t_x, t_y) \in \mathcal{O}(\varphi) \times \mathbb{R}^2 : (\gamma(t_z) = z \forall z \in \{ x, y \})$ iff there is an orbit connecting x and y. Then $\sim$ is an equivalence relation on Γ. Hence $\mathcal{O}(\varphi)$ partitions Γ into the images of disjoint orbits.

5 Definition. Given a flow $\varphi : \mathbb{R} \to Aut(\Gamma)$, the vector field induced by φ, $v_\varphi : \Gamma \to \Gamma$, is defined as the vector field giving the veloctiy vector of an orbit which passes through the given point (through every point there passes an orbit by the previous claim).

Given a point $x \in \Gamma$, we have an orbit $\gamma_x \in \mathcal{O}(\varphi)$ that passes through x at time zero given by: $\gamma_x(t) \equiv (\varphi(t))(x)$ for all $t \in \mathbb{R}$. Indeed,

$$\begin{aligned}\gamma_x(0) &= (\varphi(0))(x) \\ & \quad (\varphi \text{ is a group morphism}) \\ &\equiv (\mathbb{1}_\Gamma)(x) \\ &= x\end{aligned}$$

We thus define

$$\begin{aligned}v_\varphi(x) &:= (\partial\gamma_x)(0) \\ &\equiv \left.(\mathbb{R} \ni t \mapsto (\varphi(t))(x) \in \Gamma)'\right|_{t=0}\end{aligned}$$

6 Definition. Given a vector field $v : \Gamma \to \Gamma$, a flow $\varphi_v : \mathbb{R} \to Aut(\Gamma)$ is (sometimes) defined as follows. Let $(t, x) \in \mathbb{R} \times \Gamma$ be given. Then we seek a solution $\gamma : \mathbb{R} \to \Gamma$ to the first order differential equation

$$v \circ \gamma = \partial\gamma$$

with the initial condition $\gamma(0) = x$. By the Picard–Lindelöf theorem, if v is Lipschitz continuous there exists a unique solution γ at least locally (that is, there is some $\varepsilon > 0$ such that the equation is solved by γ at least on $(-\varepsilon, \varepsilon)$ (instead of $\mathbb{R}$)). If that unique solution may actually be extended from $(-\varepsilon, \varepsilon)$ to $\mathbb{R}$ then v is called *complete*. For complete vector fields we define the induced flow

$$(\varphi_v(t))(x) := \gamma(t)$$

7 Claim. Not every vector field is complete.

Proof. Consider $v : \mathbb{R} \to \mathbb{R}$ given by $v(x) := x^2 + 1$. Then the differential equation to solve to get its integral curves is

$$\partial\gamma = \gamma^2 + 1$$

which is solved by $\gamma = \tan(\cdot - C)$ for some $C \in \mathbb{R}$. If our initial condition is $\gamma(0) = x$ then $x = \tan(-C)$ so that $C = -\arctan(x)$ and we find

$$\gamma(t) = \tan(t + \arctan(x))$$

Of course this solution cannot work globally: $\tan \equiv \frac{\sin}{\cos}$ is undefined on $\frac{\pi}{2}\mathbb{Z}$. □

8 Claim. If v is compactly supported then it is complete.

9 Definition. Given a flow $\varphi : \mathbb{R} \to Aut(\Gamma)$, we define its *associated Jacobian matrix* $A_\varphi : \mathbb{R} \times \Gamma \to Mat_{2f\times 2f}(\mathbb{R})$ via the entries $(i,\, j) \in \{\, 1,\, \dots,\, 2f \,\}^2$

$$(A_\varphi(t,\, x))_{i,\, j} \quad := \quad (\partial_j (\varphi(t))_i)(x) \qquad \forall (t,\, x) \in \mathbb{R} \times \Gamma$$

10 Definition. A flow $\varphi : \mathbb{R} \to Aut(\Gamma)$ is *canonical* iff its associated Jacobian matrix $A_\varphi(t,\, x)$ is symplectic for all $(t,\, x) \in \mathbb{R} \times \Gamma$.

11 Claim. Let $\varphi : \mathbb{R} \to Aut(\Gamma)$ be a flow. Then φ is canonical iff there is some map $F_\varphi : \Gamma \to \mathbb{R}$ such that for any $x \in \Gamma$, the following differential equation for the unknown path $\mathbb{R} \ni t \mapsto (\varphi(t))(x) \in \Gamma$ is obeyed

$$\boxed{\Omega \partial (\varphi(t))(x) = \nabla F_\varphi \circ (\varphi(t))(x)}$$

with the boundary condition

$$(\varphi(0))(x) \quad = \quad x$$

F_φ is called *the generating function of the canonical flow* φ. In particular we find that H is the generating function of the canonical flow given by all physical trajectories.

Proof. Let v_φ be the vector field defined from φ. Then we have by definition for any $x \in \Gamma$ the differential equation for the unknown path $\mathbb{R} \ni t \mapsto \gamma_x(t) \in \Gamma$ (where we have defined $\gamma_x(t) := (\varphi(t))(x)$ for brevity):

$$\partial \gamma_x \quad = \quad v_\varphi \circ \gamma_x \tag{1}$$

Then

$$\begin{aligned}
((\partial A_\varphi)(t,\, x))_{i,\, j} \quad &\equiv \quad (\partial \partial_j (\varphi(t))_i)(x) \\
&\quad\;\; \text{(We may exchange the order of differentiation)} \\
&= \quad (\partial_j \partial (\varphi(t))_i)(x) \\
&\quad\;\; \text{(Use the equation above)} \\
&= \quad \partial_j (v_\varphi)_i \circ ((\varphi(t))(x)) \\
&\quad\;\; \text{(Use the chain rule)} \\
&= \quad \sum_l \left(\left(\partial_l (v_\varphi)_i\right) \circ ((\varphi(t))(x)) \right) \partial_j ((\varphi(t))(x))_l \\
&\quad\;\; \text{(Use the definition of } A_{\varphi_v}\text{)} \\
&= \quad \sum_l \left(\left(\partial_l (v_\varphi)_i\right) \circ ((\varphi(t))(x)) \right) (A_\varphi(t,\, x))_{l,\, j}
\end{aligned}$$

If we define a new matrix V_φ by components $V_{il} := \partial_l (v_\varphi)_i$ then we find

$$\partial A_\varphi \quad = \quad (V_\varphi \circ \gamma_x) A_\varphi$$

Note that $\varphi(0) = \mathbb{1}_\Gamma$ (group morphism) so that $A_\varphi(0,\, x) = \mathbb{1}_{2f\times 2f}$ for any

$x \in \Gamma$. But the identity matrix is symplectic. So we find

$$\left(A_\varphi\left(0,\, x\right)\right)^T \Omega A_\varphi\left(0,\, x\right) \quad = \quad \Omega$$

If φ is to be canonical, we need to have that $A_\varphi\left(t,\, x\right)$ is symplectic for *any* t. That is,

$$\begin{aligned}\left(A_\varphi\left(t,\, x\right)\right)^T \Omega A_\varphi\left(t,\, x\right) &\overset{!}{=} \Omega \\ &= \left(A_\varphi\left(0,\, x\right)\right)^T \Omega A_\varphi\left(0,\, x\right)\end{aligned}$$

so that means we need the matrix-valued function of t

$$t \quad \mapsto \quad \left(A_\varphi\left(t,\, x\right)\right)^T \Omega A_\varphi\left(t,\, x\right)$$

to be constant in time:

$$\partial\left[\left(A_\varphi\left(t,\, x\right)\right)^T \Omega A_\varphi\left(t,\, x\right)\right] \quad \overset{!}{=} \quad 0 \tag{2}$$

But

$$\begin{aligned}\partial\left[\left(A_\varphi\left(t,\, x\right)\right)^T \Omega A_\varphi\left(t,\, x\right)\right] &= \left[\partial\left(A_\varphi\left(t,\, x\right)\right)^T\right] \Omega A_\varphi\left(t,\, x\right) + \left(A_\varphi\left(t,\, x\right)\right)^T \Omega \partial A_\varphi\left(t,\, x\right) \\ &\quad \text{(Use } \partial A_\varphi = \left(V_\varphi \circ \gamma_x\right) A_\varphi\text{)} \\ &= A_\varphi\left(t,\, x\right)^T \left(V_\varphi \circ \gamma_x\left(t\right)\right)^T \Omega A_\varphi\left(t,\, x\right) + \\ &\quad + \left(A_\varphi\left(t,\, x\right)\right)^T \Omega\left(V_\varphi \circ \gamma_x\left(t\right)\right) A_\varphi\left(t,\, x\right) \\ &\quad \text{(Factorize)} \\ &= A_\varphi\left(t,\, x\right)^T \left[\left(V_\varphi \circ \gamma_x\left(t\right)\right)^T \Omega + \Omega\left(V_\varphi \circ \gamma_x\left(t\right)\right)\right] A_\varphi\left(t,\, x\right)\end{aligned}$$

Now recall $\varphi\left(t\right)$ is an automorphism for any t, so that A_φ must be invertible as a matrix, hence (2) implies

$$\begin{aligned}\left(V_\varphi \circ \gamma_x\left(t\right)\right)^T \Omega + \Omega\left(V_\varphi \circ \gamma_x\left(t\right)\right) &= 0 \\ &\updownarrow \\ \left(V_\varphi \circ \gamma_x\left(t\right)\right)^T \Omega &= -\Omega\left(V_\varphi \circ \gamma_x\left(t\right)\right) \\ &\updownarrow \quad \left(\Omega^T = -\Omega\right) \\ \left(\Omega V_\varphi \circ \gamma_x\left(t\right)\right)^T &= \Omega V_\varphi \circ \gamma_x\left(t\right)\end{aligned}$$

and hence by hint in the last question in homework 11 (symmetric matrix can be diagonalized) we find that there must be some $F_\varphi : \Gamma \to \mathbb{R}$ whose gradient is Ωv_φ:

$$\Omega v_\varphi \quad = \quad \nabla F_\varphi$$

Plugging in the value of v_φ from (1) by evaluating at γ_x we find

$$\begin{array}{rcl}\Omega v_\varphi \circ \gamma_x & = & \nabla F_\varphi \circ \gamma_x \\ & \updownarrow & \\ \Omega \partial \gamma_x & = & \nabla F_\varphi \circ \gamma_x\end{array}$$

And then placing back the definition of γ_x we find the result. □

2 Time Dependence of Generators

Recall that the canonical equations of motion for a trajectory in phase space $\gamma : \mathbb{R} \to \Gamma$ are given by

$$\dot{\gamma} = \Omega^T (\nabla H) \circ \gamma$$

where H is the Hamiltonian. Then we have for any scalar quantity $F : \Gamma \to \mathbb{R}$, the time derivative of it evaluated on a trajectory which is a solution of the equations of motion is given by:

$$\begin{array}{rcl}\partial (F \circ \gamma) & = & \sum_{i=1}^{2f} [(\partial_i F) \circ \gamma] \, \partial \gamma_i \\ & \equiv & \langle \nabla F \circ \gamma, \, \partial \gamma \rangle_\Gamma \\ & \equiv & \langle \nabla F \circ \gamma, \, \dot{\gamma} \rangle_\Gamma \\ & & \text{(Above E.o.M.)} \\ & = & \left\langle \nabla F \circ \gamma, \, \Omega^T (\nabla H) \circ \gamma \right\rangle_\Gamma \\ & = & \left\langle \nabla F, \, \Omega^T \nabla H \right\rangle_\Gamma \circ \gamma \\ & = & \langle \Omega \nabla F, \, \nabla H \rangle_\Gamma \circ \gamma \\ & = & \langle \nabla H, \, \Omega \nabla F \rangle_\Gamma \circ \gamma\end{array}$$

We thus define the Poisson bracket of two scalars $(A, \, B) \in \left(\mathbb{R}^\Gamma\right)^2$ as

$$\{ \, A, \, B \, \} \; := \; \langle \nabla A, \, \Omega \nabla B \rangle_\Gamma$$

and find

$$\partial (F \circ \gamma) = \{H, \, F\} \circ \gamma$$

3 Canonical Transformations

In this section we consider the phase space $\Gamma \equiv \mathbb{R}^{2f}$ as a differentiable manifold and not so much as a vector space.

12 Definition. A bijection $b : \Gamma \to \Gamma$ is called a canonical transformation iff it leaves the canonical equations of motion for the trajectory $\gamma : \mathbb{R} \to \Gamma$ invariant:

$$\begin{array}{rcl}\Omega \partial \gamma & = & (\nabla H) \circ \gamma \\ & \updownarrow & \\ \Omega \partial \left(\underbrace{b^{-1} \circ \gamma}_{\tilde{\gamma}} \right) & = & \left(\nabla \left(\underbrace{H \circ b}_{\tilde{H}} \right) \right) \circ \left(\underbrace{b^{-1} \circ \gamma}_{\tilde{\gamma}} \right)\end{array}$$

This means that

$$\begin{aligned}
\Omega\partial\gamma &= \Omega\partial\left(b\circ\tilde{\gamma}\right)\\
&= \Omega\sum_{i=1}^{2f}\left(\partial_i b\circ\tilde{\gamma}\right)\partial\tilde{\gamma}_i\\
&\overset{!}{=} e_i\left(\partial_i H\right)\circ\gamma\\
&= e_i\left(\partial_i\left(\tilde{H}\circ b^{-1}\right)\right)\circ b\circ\tilde{\gamma}\\
&= e_i\left(\left(\left(\partial_j\tilde{H}\right)\circ b^{-1}\right)\underbrace{\partial_i\left(b^{-1}\right)_j}_{\left(B^{-1}\right)_{ji}}\right)\circ b\circ\tilde{\gamma}
\end{aligned}$$

We find that

$$\left(B^T\Omega B\right)\partial\tilde{\gamma} \overset{!}{=} \nabla\tilde{H}\circ\tilde{\gamma}$$

which implies that

$$B^T\Omega B \overset{!}{=} \Omega$$

that is, that B is symplectic (evaluated anywhere).

Thus we conclude: $b:\Gamma\to\Gamma$ is a canonical transformation iff the matrix-valued function whose elements are the functions $\partial_i b_j:\Gamma\to\mathbb{R}$ is symplectic.

3.1 Generating Canonical Transformations

As in HW11Q3, we have a coordinate canonical transformation which possibly depends on time $b:\Gamma\times\mathbb{R}\to\Gamma$. It induces a new trajectory as

$$\gamma = b\circ\left(\tilde{\gamma}\times\mathbb{1}_{\mathbb{R}}\right)$$

If b is to be canonical, then the Lagrangians of the two systems must be equivalent, that is, the same up to a total time derivative:

$$\sum_{i=1}^{f}\gamma_{i+f}\left(t\right)\dot{\gamma}_i\left(t\right)-H\left(\gamma\left(t\right),\,t\right) \overset{!}{=} \sum_{i=1}^{f}\tilde{\gamma}_{i+f}\left(t\right)\dot{\tilde{\gamma}}_i\left(t\right)-\tilde{H}\left(\tilde{\gamma}\left(t\right),\,t\right)+\dot{S}_0\left(\tilde{\gamma}\left(t\right),\,t\right)$$

for some scalar field $S_0:\Gamma\times\mathbb{R}\to\mathbb{R}$.

This implies the identity of differential forms

$$\sum_{i=1}^{f}\gamma_{i+f}\left(t\right)\mathrm{d}\gamma_i\left(t\right)-H\left(\gamma\left(t\right),\,t\right)\mathrm{d}t \overset{!}{=} \sum_{i=1}^{f}\tilde{\gamma}_{i+f}\left(t\right)\mathrm{d}\tilde{\gamma}_i\left(t\right)-\tilde{H}\left(\tilde{\gamma}\left(t\right),\,t\right)\mathrm{d}t+\mathrm{d}S_0\left(\tilde{\gamma}\left(t\right),\,t\right)$$

Define

$$S := S_0+\sum_{i=1}^{f}\tilde{\gamma}_{i+f}\left(t\right)\tilde{\gamma}_i\left(t\right)$$

Then

$$
\begin{aligned}
\mathrm{d}S &= \mathrm{d}S_0 + \sum_{i=1}^{f} \mathrm{d}\tilde{\gamma}_{i+f}(t)\,\tilde{\gamma}_i(t) \\
&= \mathrm{d}S_0 + \sum_{i=1}^{f} \left[\mathrm{d}\tilde{\gamma}_{i+f}(t)\tilde{\gamma}_i(t) + \tilde{\gamma}_{i+f}(t)\,\mathrm{d}\tilde{\gamma}_i(t)\right] \\
&= \sum_{i=1}^{f} \gamma_{i+f}(t)\,\mathrm{d}\gamma_i(t) - H(\gamma(t), t)\,\mathrm{d}t + \tilde{H}(\tilde{\gamma}(t), t)\,\mathrm{d}t + \sum_{i=1}^{f} \mathrm{d}\tilde{\gamma}_{i+f}(t)\tilde{\gamma}_i(t) \\
&= \sum_{i=1}^{f} \left[\gamma_{i+f}(t)\,\mathrm{d}\gamma_i(t) + \tilde{\gamma}_i(t)\,\mathrm{d}\tilde{\gamma}_{i+f}(t)\right] + \left[\tilde{H}(\tilde{\gamma}(t), t) - H(\gamma(t), t)\right]\mathrm{d}t
\end{aligned}
$$

Assume we could express S as a function of $\{\gamma_i\}_{i=1}^{f}$ and $\{\tilde{\gamma}_{i+f}\}_{i=1}^{f}$. Then the above equation implies

$$
\begin{aligned}
\partial_{\gamma_i} S &= \gamma_{i+f} \\
\partial_{\tilde{\gamma}_{i+f}} S &= \tilde{\gamma}_i \\
\partial_t S &= \tilde{H} - H
\end{aligned}
$$

We use the second equation to find $\{\gamma_i\}$ in terms of $\{\tilde{\gamma}_i\}_{i=1}^{2f}$ and t, and place this in the first and third equations to find $\{\gamma_i\}_{i=1}^{2f}$ in terms of $\{\tilde{\gamma}_i\}_{i=1}^{2f}$ and t. This is why S is called a generator of canonical transformations.

Different choices of S lead to (necessarily) various canonical transformations, which is why this is useful.