Analytical Mechanics Recitation Session of Week 14

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1 The Hamilton-Jacobi Equation

1.1 Generators of Canonical Transformations

Let $g:\mathbb{R}^{2f}\times\mathbb{R}\to\mathbb{R}$ be a canonical transformation of coordinates: If the old trajectory in phase space $\mathbb{R}^{2f}$ is $\gamma:\mathbb{R}\to\mathbb{R}^{2f}$ then the new one $\tilde{\gamma}:\mathbb{R}\to\mathbb{R}^{2f}$ is related to γ via:

$$\gamma(t) = g(\tilde{\gamma}(t),\,t) \quad \forall t\in\mathbb{R}$$

And we also have

$$H(q,\,p,\,t) = \tilde{H}(\tilde{q},\,\tilde{p},\,t)$$

which means

$$H(g(\tilde{q},\,\tilde{p},\,t),\,t) = \tilde{H}(\tilde{q},\,\tilde{p},\,t) \qquad \forall(\tilde{q},\,\tilde{p},\,t)\in\mathbb{R}^{2f}\times\mathbb{R}$$

The question is if we can come up with ways to "generate" g in easy ways, ensuring that it will automatically be canonical.

One way to make sure of that is to ensure that the two Lagrangians corresponding to H and $\tilde{H}$ are equivalent, that is, equal up to a function's derivative with respect to time:

$$\sum_{i=1}^{f}\gamma_{p_i}(t)\,\dot{\gamma}_{q_i}(t)-H(\gamma_{q_i}(t),\,\gamma_{p_i}(t),\,t)\overset{!}{=}\sum_{i=1}^{f}\tilde{\gamma}_{p_i}(t)\,\dot{\tilde{\gamma}}_{q_i}(t)-\tilde{H}(\tilde{\gamma}_{q_i}(t),\,\tilde{\gamma}_{p_i}(t),\,t)+\dot{F}(t) \tag{1}$$

where F is *any* function of time.

It *turns out* that it's most useful to assume that F has the following particular dependence on time:

$$F(t) = G_1(\gamma_q(t),\,\tilde{\gamma}_q(t),\,t)$$

or

$$F(t) = G_2(\gamma_q(t),\,\tilde{\gamma}_p(t),\,t)$$

or other combinations which mix the positions and momenta of the transformed and untransformed coordinates. Consider the first case. Then

$$\begin{aligned}\dot{F}(t) &= \sum_{i=1}^{f} \left(\partial_{q_i} G_1\right)\left(\gamma_q(t), \tilde{\gamma}_q(t), t\right) \dot{\gamma_{q_i}}(t) + \\ &\quad + \left(\partial_{\tilde{q}_i} G_1\right)\left(\gamma_q(t), \tilde{\gamma}_q(t), t\right) \dot{\tilde{\gamma}}_{q_i}(t) + \\ &\quad + \left(\partial_t G_1\right)\left(\gamma_q(t), \tilde{\gamma}_q(t), t\right)\end{aligned}$$

and we find (1) becomes

$$\begin{aligned}\sum_{i=1}^{f} \gamma_{p_i}(t) \dot{\gamma}_{q_i}(t) - H\left(\gamma_{q_i}(t), \gamma_{p_i}(t), t\right) &\stackrel{!}{=} \sum_{i=1}^{f} \tilde{\gamma}_{p_i}(t) \dot{\tilde{\gamma}}_{q_i}(t) - \tilde{H}\left(\tilde{\gamma}_{q_i}(t), \tilde{\gamma}_{p_i}(t), t\right) + \\ &\quad \left(\partial_{q_i} G_1\right)\left(\gamma_q(t), \tilde{\gamma}_q(t), t\right) \dot{\gamma_{q_i}}(t) + \\ &\quad + \left(\partial_{\tilde{q}_i} G_1\right)\left(\gamma_q(t), \tilde{\gamma}_q(t), t\right) \dot{\tilde{\gamma}}_{q_i}(t) + \\ &\quad + \left(\partial_t G_1\right)\left(\gamma_q(t), \tilde{\gamma}_q(t), t\right)\end{aligned}$$

This equation can only hold for all t if we have the following three equations:

$$\begin{aligned}\gamma_{p_i}(t) &= \left(\partial_{q_i} G_1\right)\left(\gamma_q(t), \tilde{\gamma}_q(t), t\right) \\ \tilde{\gamma}_{p_i}(t) &= -\left(\partial_{\tilde{q}_i} G_1\right)\left(\gamma_q(t), \tilde{\gamma}_q(t), t\right) \\ -H\left(\gamma_{q_i}(t), \gamma_{p_i}(t), t\right) &= -\tilde{H}\left(\tilde{\gamma}_{q_i}(t), \tilde{\gamma}_{p_i}(t), t\right) + \left(\partial_t G_1\right)\left(\gamma_q(t), \tilde{\gamma}_q(t), t\right)\end{aligned}$$

We assume we can invert the first (set of f) equations to get $\tilde{\gamma}_q(t)$ in terms of $\gamma_{p_i}(t)$ and $\gamma_q(t)$ and t. Once we found $\tilde{\gamma}_q(t)$, we place it in the second (set of f) equations to get $\tilde{\gamma}_{p_i}(t)$ in terms of $\gamma_{p_i}(t)$ and $\gamma_q(t)$ and t. This completes the exercise, since this data is exactly g^{-1}.

1 Example. If we pick

$$G_1\left(\gamma_q(t), \tilde{\gamma}_q(t), t\right) := \sum_{i=1}^{f} \gamma_{q_i}(t) \tilde{\gamma}_{q_i}(t)$$

Then the transformation we get exchanges the positions and momenta:

$$\gamma_{p_i}(t) = \tilde{\gamma}_{q_i}(t)$$

$$\tilde{\gamma}_{p_i}(t) = -\gamma_{q_i}(t)$$

So that $g : \mathbb{R}^{2f} \to \mathbb{R}^{2f}$ is given by the matrix

$$g = \begin{bmatrix} 0 & -\mathbb{1}_f \\ \mathbb{1}_f & 0 \end{bmatrix}$$

which is incidentally the standard symplectic form (so of course it is a canonical transformation).

2 Example. (The 1D Harmonic Oscillator) We begin with the one dimensional Harmonic oscillator:

$$H(q,\, p) \;=\; \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right) \tag{2}$$

$$G_1\left(\gamma_q(t),\, \tilde{\gamma}_q(t),\, t\right) \;:=\; \frac{1}{2} m\omega \left(\gamma_q(t)\right)^2 \cot\left(\tilde{\gamma}_q(t)\right)$$

We then find

$$\begin{aligned}
\gamma_p(t) &= \left(\partial_q G_1\right)\left(\gamma_q(t),\, \tilde{\gamma}_q(t),\, t\right) = m\omega\gamma_q(t)\cot\left(\tilde{\gamma}_q(t)\right) \\
\tilde{\gamma}_p(t) &= -\left(\partial_{\tilde{q}} G_1\right)\left(\gamma_q(t),\, \tilde{\gamma}_q(t),\, t\right) = \frac{1}{2} m\omega\left(\gamma_q(t)\right)^2 \frac{1}{\sin\left(\tilde{\gamma}_q(t)\right)^2}
\end{aligned}$$

Solve for $\gamma_p(t)$ and $\gamma_q(t)$ to find

$$\gamma_q(t) \;=\; \sqrt{\frac{2}{m\omega}\tilde{\gamma}_p(t)}\sin\left(\tilde{\gamma}_q(t)\right) \tag{3}$$

and

$$\gamma_p(t) \;=\; \sqrt{2m\omega\tilde{\gamma}_p(t)}\cos\left(\tilde{\gamma}_q(t)\right) \tag{4}$$

We find then the new Hamiltonian is:

$$\begin{aligned}
\tilde{H}(\tilde{q},\, \tilde{p}) &= H(q,\, p) \\
&= H\left(\sqrt{\frac{2}{m\omega}\tilde{p}}\sin(\tilde{q}),\, \sqrt{2m\omega\tilde{p}}\cos(\tilde{q})\right) \\
&= \frac{1}{2m}\left\{\left[\sqrt{2m\omega\tilde{p}}\cos(\tilde{q})\right]^2 + m^2\omega^2\left[\sqrt{\frac{2}{m\omega}\tilde{p}}\sin(\tilde{q})\right]^2\right\} \\
&= \omega\tilde{p}
\end{aligned}$$

Hence the new canonical equations of motion give

$$\begin{aligned}
\dot{\tilde{\gamma}}_p &= -\frac{\partial\tilde{H}}{\partial\tilde{q}} = 0 \\
\dot{\tilde{\gamma}}_q(t) &= -\frac{\partial\tilde{H}}{\partial\tilde{p}} = \omega
\end{aligned}$$

so that

$$\tilde{\gamma}_p(t) \;=\; \frac{E}{\omega}$$

and

$$\tilde{\gamma}_q(t) \;=\; \omega t + const$$

Placing these back in (3) and (4) gives us the solution.

1.2 The Hamilton-Jacobi Equation

(page 430 of Goldstein)

We look for a generating functional S of canonical transformations

$$(\tilde{q},\, \tilde{p},\, t) \quad \overset{g}{\mapsto} \quad (q,\, p)$$

such that after the transformation $g : \mathbb{R}^{2f} \times \mathbb{R} \to \mathbb{R}^{2f}$, the new Hamiltonian will be zero:

$$H\left(q,\, p,\, t\right) \quad = \quad \tilde{H}\left(\tilde{q},\, \tilde{p},\, t\right) \overset{!}{=} 0$$

The consequence of $\tilde{H} = 0$ means that the new canonical equations of motion imply the new trajectories $\tilde{\gamma} : \mathbb{R} \to \mathbb{R}^{2f}$ are constant:

$$\partial_t \gamma_{\tilde{q}_i} \quad = \quad \left(\partial_{\tilde{p}_i} \tilde{H}\right)\left(\gamma_{\tilde{q}},\, \gamma_{\tilde{p}}\right) = 0$$

$$\partial_t \gamma_{\tilde{p}_i} \quad = \quad -\left(\partial_{\tilde{q}_i} \tilde{H}\right)\left(\gamma_{\tilde{q}},\, \gamma_{\tilde{p}}\right) = 0$$

So that

$$\gamma_{\tilde{q}_i}\left(t\right) \quad = \quad \gamma_{\tilde{q}_i}\left(0\right) \quad \forall t \in \mathbb{R}$$

$$\gamma_{\tilde{p}_i}\left(t\right) \quad = \quad \gamma_{\tilde{p}_i}\left(0\right) \quad \forall t \in \mathbb{R}$$

Then if we could figure out what g is, then we will have solved the problem (that is, obtained the trajectories γ_q and γ_p) because we have

$$\begin{aligned} \gamma\left(t\right) &= g\left(\tilde{\gamma}\left(t\right),\, t\right) \\ &= g\left(\tilde{\gamma}\left(0\right),\, 0\right) \end{aligned}$$

so that really all the data we need is encoded inside of g. As we know, the generating functional generates g, so finding the generating functional S corresponding to g itself encodes all the data we need. The Hamilton-Jacobi equation is a partial differential equation to find that special S.

Relating to the generator above, we pick S of the form

$$S\left(t\right) \quad = \quad G_2\left(\gamma_q\left(t\right),\, \tilde{\gamma}_p\left(t\right),\, t\right) - \sum_{i=1}^{f} \gamma_{q_i}\left(t\right) \tilde{\gamma}_{p_i}\left(t\right)$$

for some G_2, which leads to transformation equations of the form

$$\begin{aligned} \gamma_{p_i}\left(t\right) &= \frac{\partial G_2}{\partial q_i}\left(\gamma_q\left(t\right),\, \tilde{\gamma}_p\left(t\right),\, t\right) \\ \gamma_{\tilde{q}_i}\left(t\right) &= \frac{\partial G_2}{\partial \tilde{p}_i}\left(\gamma_q\left(t\right),\, \tilde{\gamma}_p\left(t\right),\, t\right) \\ -H\left(\gamma_q\left(t\right),\, \gamma_p\left(t\right),\, t\right) &= \left(\partial_t G_1\right)\left(\gamma_q\left(t\right),\, \tilde{\gamma}_p\left(t\right),\, t\right) \end{aligned}$$

Hence we find

$$H\left(\gamma_q\left(t\right),\, \frac{\partial G_2}{\partial q_i}\left(\gamma_q\left(t\right),\, \tilde{\gamma}_p\left(t\right),\, t\right),\, t\right) + \left(\partial_t G_2\right)\left(\gamma_q\left(t\right),\, \tilde{\gamma}_p\left(t\right),\, t\right) \quad = \quad 0 \tag{5}$$

which is the Hamilton-Jacobi equation, a partial-differential equation in $f+1$ variables $(\gamma_q(t), \tilde{\gamma}_p(t), t)$.

If H does not explicitly depend on time, we must have H a constant in time, the energy E, so that $(\partial_t G_2)(\gamma_q(t), \tilde{\gamma}_p(t), t) \overset{!}{=} -E$. Thus we could write G_2's explicit dependene on time as

$$\frac{\partial G_2}{\partial t} = -E$$

for *some* constant E, which we call the energy.

3 Example. (The 1D Harmonic Oscillator) We begin again with (2) and we use this H for the Hamilton-Jacobi equation:

$$\begin{aligned} H\left(\gamma_q(t), \frac{\partial G_2}{\partial q}(\gamma_q(t), \tilde{\gamma}_p(t), t), t\right) &= E \\ &\updownarrow \\ \frac{1}{2m}\left\{\left[\frac{\partial G_2}{\partial q}(\gamma_q(t), \tilde{\gamma}_p(t), t)\right]^2 + m^2\omega^2\gamma_q(t)^2\right\} &= E \end{aligned}$$

and the unknown to be found is the function G_2. We can actually integrate this last equation to find, for some function G_3,

$$G_2(\gamma_q(t), \tilde{\gamma}_p(t), t) = \int^{\gamma_q(t)} \sqrt{2mE - m^2\omega^2 q'^2}\mathrm{d}q' + G_3(\tilde{\gamma}_p(t)) - Et$$

Recall that by definition $\tilde{\gamma}_p(t) = \tilde{\gamma}_p(0)$ so that $G_3(\tilde{\gamma}_p(t)) = G_3(\tilde{\gamma}_p(0))$ and is thus merely a constant. Since G_2 enters into our equations only derivatives, we discard this constant. The explicit dependence on t enters as in our earlier discussion:

$$G_2(\gamma_q(t), \tilde{\gamma}_p(t), t) = \int^{\gamma_q(t)} \sqrt{2mE - m^2\omega^2 q'^2}\mathrm{d}q' - Et$$

Note that since $\tilde{\gamma}_p(t) = \tilde{\gamma}_p(0)$, we are actually free to choose this constant as we please (it will merely lead to a change in the function G_2, but it will not change is functional dependence on $\gamma_q(t)$), and we can just as well choose it to be E. Thus we find

$$\begin{aligned} \gamma_{\tilde{q}}(0) &\equiv \gamma_{\tilde{q}}(t) \\ &= \frac{\partial G_2}{\partial \tilde{p}_i}(\gamma_q(t), \tilde{\gamma}_p(t), t) \\ &= \frac{\partial G_2}{\partial E}(\gamma_q(t), E, t) \\ &= \int^{\gamma_q(t)} \frac{m}{\sqrt{2mE - m^2\omega^2 q'^2}}\mathrm{d}q' - t \\ &= \sqrt{\frac{m}{2E}}\int^{\gamma_q(t)} \frac{1}{\sqrt{1 - \frac{1}{2E}m\omega^2 q'^2}}\mathrm{d}q' - t \end{aligned}$$

This can be integrated and gives:

$$\begin{aligned}\gamma_{\tilde{q}}(0) &= \sqrt{\frac{m}{2E}}\sqrt{2E\frac{1}{m\omega^2}}\arcsin\left(\sqrt{\frac{1}{2E}m\omega^2}\gamma_q(t)\right)-t\\ &= \frac{1}{\omega}\arcsin\left(\sqrt{\frac{m\omega^2}{2E}}\gamma_q(t)\right)-t\end{aligned}$$

Which implies

$$\gamma_q(t) = \sqrt{\frac{2E}{m\omega^2}}\sin\left(\omega t-\gamma_{\tilde{q}}(0)\right)$$

and we can also find the momentum $\gamma_p(t)$ via $\gamma_{p_i}(t)=\frac{\partial G_2}{\partial q_i}\left(\gamma_q(t),\,\tilde{\gamma}_p(t),\,t\right)$. Hence we find the usual solution for the harmonic oscillator.

Thus, the canonical transformation we found moves us to the (constant) coordinates which are the total energy E and the initial phase $\gamma_{\tilde{q}}(0)$.

2 Poincare Transformations

Recall that we started classical mechanics with the observation that for any two events $(x,\,y)\in\left(\mathbb{R}^4\right)^2$ (where $\mathbb{R}^4$ represents time in the zeroth component and space in the remaining three components) measured in any coordinate system, the two quantities:

1. $|x_0-y_0|$ the time-separation.
2. $\sqrt{\sum_{i=1}^{3}(x_i-y_i)^2}$ the spatial-separation.

carry absolute meaning (that is, beyond that which corresponds to a particular choice of a coordinate system) and that any transformation of coordinate systems must preserve these two quantities. The set of all such possible transformations g are given by

$$g\left(\begin{bmatrix}x_0\\x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}\lambda x_0+a\\ R(x_0)\,x_s+b(x_s)\end{bmatrix}\qquad\forall x\in\mathbb{R}^4$$

where $x_s\equiv\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\in\mathbb{R}^3$ is the spatial component of x, and the transformation is parametrized by $\lambda\in\{\pm1\}$, $a\in\mathbb{R}$, $R:\mathbb{R}\to O(3)$, $b:\mathbb{R}\to\mathbb{R}^3$.

In special relativity, the new concept is that the speed of light must be constant no matter in which frame we observe the light. It turns out that in order for that to hold, we must insist that yet *another* quantity is conserved when performing coordinate transformations $\mathbb{R}^4\to\mathbb{R}^4$.

4 Definition. We define *the Minkowski metric* η as a bilinear form $\mathbb{R}^4\times\mathbb{R}^4\to\mathbb{R}$ given by the following:

$$\eta(x,\,y) := x_0y_0-\sum_{i=1}^{3}x_iy_i$$

Note that as a matrix, we may write $g = \mathrm{diag}(-1,\,1,\,1,\,1)$ and then

$$\eta\,(x,\,y) \quad\equiv\quad \langle x,\,gy\rangle$$

5 Definition. A *Poincaré transformation* is an isometry of η, that is, a transformation Λ such that for all $(x,\,y) \in \left(\mathbb{R}^4\right)^2$ we have

$$\boxed{\eta\,(\Lambda\,(x-y)\,,\,\Lambda\,(x-y)) = \eta\,(x-y,\,x-y)}$$

6 Claim. The Poincare group (The group of Poincare transformations) is a subgroup of the affine group of $\mathbb{R}^4$.

Proof. Following Steven Weinberg:

We have that the Jacobian of a Poincare transformation Λ must also be a Poincare transformation as a linear map (make a Taylor expansion of Λ and employ the constraint order by order in x):

$$\langle (D\Lambda)\,x,\,gD\Lambda x\rangle \quad=\quad \langle x,\,gx\rangle \qquad \forall x\in\mathbb{R}^4$$

(This is essentially because the differential spacetime intervial is preserved) where

$$(D\Lambda)_{i,\,j} \quad\equiv\quad \partial_j\Lambda_i$$

In matrix notation the first equation is thus

$$\begin{aligned}(D\Lambda)^T g\,(D\Lambda) &= g\\ \left((D\Lambda)^T\right)_{ij} g_{jk}\,(D\Lambda)_{kl} &= g_{il}\\ (\partial_i\Lambda_j)\,g_{jk}\,(\partial_l\Lambda_k) &= g_{il}\end{aligned}$$

Differentiate with respect to the mth coordinate to get

$$(\partial_m\partial_i\Lambda_j)\,g_{jk}\,(\partial_l\Lambda_k) + (\partial_i\Lambda_j)\,g_{jk}\,(\partial_m\partial_l\Lambda_k) \quad=\quad 0$$

Now we add to this the same equation with $m\leftrightarrow i$ interchanged, and subtract the same with $m\leftrightarrow l$ interchanged to obtain

$$\begin{aligned}0 \quad&=\quad (\partial_m\partial_i\Lambda_j)\,g_{jk}\,(\partial_l\Lambda_k) + (\partial_i\Lambda_j)\,g_{jk}\,(\partial_m\partial_l\Lambda_k) +\\ &\quad\; + (\partial_i\partial_m\Lambda_j)\,g_{jk}\,(\partial_l\Lambda_k) + (\partial_m\Lambda_j)\,g_{jk}\,(\partial_i\partial_l\Lambda_k) -\\ &\quad\; - (\partial_l\partial_i\Lambda_j)\,g_{jk}\,(\partial_m\Lambda_k) + (\partial_i\Lambda_j)\,g_{jk}\,(\partial_l\partial_m\Lambda_k)\\ &\quad\; \text{(Use the fact that } \partial_\alpha\partial_\beta\Lambda_\gamma = \partial_\beta\partial_\alpha\Lambda_\gamma \text{ and } g_{jk}=g_{kj})\\ &=\quad 2g_{jk}\,(\partial_m\partial_i\Lambda_j)\,(\partial_l\Lambda_k)\end{aligned}$$

But g is invertible and so is $\partial_l \Lambda_k$, so that

$$\boxed{\partial_m \partial_i \Lambda_j = 0}$$

which is equivalent to Λ being affine. $\square$

7 Remark. Since Λ is affine it may be written as a matrix and a vector $\Lambda(x) = Mx + v$ in which case the requirement becomes

$$M^T g M = g$$

The set of all such matrices M is called the Lorentz group L.

8 Remark. All in all, the Poincare transformations include:

1. Translations (four degrees of freedom).
2. Reflection (three degrees of freedom, the orientation of the plane of reflection). Note that a rotation in space is obtained via a composition of an even number of rotations.
3. Boosts (three degrees of freedom, the velocity of the boost).