Analytical Mechanics Recitation Session of Week 2

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1 Organizational Information

- Best way to reach me is by email:
  jshapiro@itp.phys.ethz.ch.
- My website, which will contain the summaries of these recitation sessions, as well hints for the homework (in English), and full solutions later (also in English) is at:
  https://people.phys.ethz.ch/∼jshapiro/
- The course website is on Moodle:
  https://moodle-app2.let.ethz.ch/course/view.php?id=2538
- All students are welcome to attend this recitation session. Your homework will be graded only if you are actually registered to my group. If you are a UZH student and are registered to my group you will unfortunately have to change your registration to one of the two UZH assistants and submit your homework to them. Recall that in ETHZ there is no requirement to submit homework in order to attend the test, thus, the main purpose of submitting homework is the privilege to have it graded (UZH has possibly different policies).
- If you want your homework to be graded, you must submit it during the Monday lecture break into the folder bearing my name. That means that the homework to be disucssed today, HW1, shall be submitted in HPV G4 on October 3rd 2016 between 10:30 to 10:45.
- Your graded work shall be returned always two days later in this recitation session. During this session we will spend approximately half of the time discussing the returned homework and the other half trying our best to prepare for the homework due the following Monday.
- During the discussion of the returned homework, students will be asked to come to the blackboard and present their solutions. If I asked you to come to the blackboard and do this, it means I thought your solution was well

written or elegant, or at least acceptable. I very strongly encourage you to accept the invitation to come to the blackboard: the course has more than 230 students so that the recitation sessions are extremely important to make sure one is truly in touch with the material. The best way to guarantee that is to actually present your ideas to others.

- After each Wednesday's session I will post on my website the hints to the homework. You may consult them only after you have honestly attempted to solve the exercise, or at least understand what is being asked.

- You are strongly encouraged to typeset your solution into a computer. If you find LaTeX unnatural to use, you are encouraged to use LyX: http://www.lyx.org/.

- It is imperative that you at least try to solve each homework every week. Not doing so will very quickly make sure you are unable to understand the lectures. If you get stuck trying to solve the homework and really want to find a solution, you may try the following alternatives to get unstuck:

  1. Look at the hints I've posted (but *not* before honestly trying for some time).
  2. Ask me by email.
  3. Ask on the Moodle forums (it may be possible that there will be an anonymous posting option, but even if not, only the assistants read the forums, so you don't have to be concerned that the professor will read your "silly" questions).
  4. Ask on stackexchange: http://physics.stackexchange.com/

- A note about my (perhaps non-standard) notation:

  – The equation $A := B$ means we are now defining A to be equal to B. The equation $A \equiv B$ means that at some point earlier A has been defined to be equal to B. The equation $A = B$ means A turns out to be equal to B.
  – I will probably not bother to put upper arrows or lower bars below vectors in $\mathbb{R}^d$. The type of an object will have to be inferred from the context. Adding a latin subscript to a vector usually denotes the corresponding component.

2 Some Vector Calculus Refreshments

Recall the gradient is defined as the differential operator

$$\nabla : f \quad \mapsto \quad (\partial_i f)\, e_i$$

where $f : \mathbb{R}^d \to \mathbb{R}$ is a differentiable map, ∂_i is the derivative with respect to the ith argument of f, and e_i is the standard basis for $\mathbb{R}^d$, and summation notation has been used (repeating indices are summed over, in this instance, the notation stands for $\sum_{i=1}^{d} (\partial_i f) e_i$. Thus the result of the gradient on the scalar map f is a vector map $\mathbb{R}^d \to \mathbb{R}^d$.

The gradient may be thought of as a generalized derivative, in the sense that its direction is points to the direction of greatest change of f, and its magnitude is the slope of f in that direction. If we consider $f(x) = C$ as the definition of a locus of points in $\mathbb{R}^d$, (a $d-1$-dimensional surface in $\mathbb{R}^d$) then ∇f will give the normal vector to this surface.

2.1 Example. Consider $f(x) := \sum_i \frac{x_i^2}{a_i^2}$ for all x and the equation $f(x) = 1$. This equation defines an ellipsoid. Let us compute its normal vector:

$$\begin{aligned}
(\nabla f)(x) &\equiv \sum_i (\partial_i f) e_i \\
&= \sum_{j,\,i} \partial_i \frac{x_j^2}{a_j^2} e_i \\
&= \sum_{j,\,i} \frac{2x_j \delta_{i,\,j}}{a_j^2} e_i \\
&= 2\sum_i \frac{x_i}{a_i^2} e_i
\end{aligned}$$

The unit vector is then given by

$$\begin{aligned}
\frac{(\nabla f)(x)}{\|(\nabla f)(x)\|} &= \frac{2\sum_i \frac{x_i}{a_i^2} e_i}{\sqrt{\sum_i \left(2\frac{x_i}{a_i^2}\right)^2}} \\
&= \sum_i \frac{1}{\sqrt{\sum_j \frac{x_j^2}{a_j^4}}} x_i e_i
\end{aligned}$$

In particular, evaluating at $x = a_1 e_1$, which is a point on the ellipsoid, we obtain

$$\frac{(\nabla f)(a_1 e_1)}{\|(\nabla f)(a_1 e_1)\|} = e_1$$

which means that the normal vector is point *out* of the ellipsoid.

2.2 Example. We define $r(x) := \|x\|$.

$$\begin{aligned}
(\partial_i r)(x) &= \frac{1}{2} \frac{2x_i}{\sqrt{x_j x_j}} \\
&= \frac{x_i}{\|x\|}
\end{aligned}$$

$$\begin{aligned}
(\nabla r)(x) &= (\partial_i r)(x)\, e_i \\
&= \frac{x_i e_i}{\|x\|} \\
&= \frac{x}{\|x\|}
\end{aligned}$$

Similarly we may also compute

$$\begin{aligned}
\left(\nabla \frac{1}{r}\right)(x) &= \left(\partial_i \frac{1}{r}\right)(x)\, e_i \\
&= \left(-\frac{\partial_i r}{r^2}\right)(x)\, e_i \\
&= -\frac{x}{\|x\|^3}
\end{aligned}$$

and finally, if $f : \mathbb{R} \to \mathbb{R}$ is a given scalar function then

$$\begin{aligned}
(\nabla f \circ r)(x) &= (\partial_i (f \circ r))(x) \\
&= f'(r(x))\,((\partial_i r)(x)) \\
&= f'(\|x\|)\,\frac{x}{\|x\|}
\end{aligned}$$

Recall from the lecture that functions of the form

$$x \mapsto s(\|x\|)\,\frac{x}{\|x\|}$$

where s is some scalar function, are forces which give rise to the same equations of motion under Galilean transformations.

3 Preview of HW1

3.1 Question 1

1. Bear in mind that in order to show some subset of $\mathbb{R}^3$ is contained in a line, a useful criterion to employ is that the set is invariant under rotations about the axis defined by that line. How to show that a subset is contained in a plane?

2. Use the fact (from the theory of ordinary differential equations) that two solutions of the same differential equation with the same initial conditions must be identical.

3. Use the invariance of the equations of motion under Galilean transformations, as well as the invariance of the initial data on the special transformations you pick.

4. For the last part, show that you may assume the second particle starts at rest (why may you do this?). Then, which three vectors define the plane of motion?

3.2 Question 2

- In order to show the existence of $\omega \in \mathbb{R}^3$ such that $\dot{\gamma} = \omega \times \gamma$, use the fact that $\omega \times \cdot$ corresponds to multiplication by some anti-symmetric. Use the fact that $R(t) \in O(3)$ for all t and differentiate its defining equation

$$\mathbb{1} = RR^T$$

to find that some matrix Ω is anti-symmetric. This matrix will then "contain" the right ω.

- Prove a recursion relation for odd and even powers of Ω, plug in the power series expansion of $\exp(\Omega t)$, separate into even and odd powers (why may you do this?), and then identify the power series expansions of cosine and sine after changing the summation indices a few times.

3.3 Question 3

- For the purpose of this exercise, consider only coordinate transformations given by

$$x \mapsto Sx$$

with $S \in O(3)$.

- If $\gamma : \mathbb{R} \to \mathbb{R}^3$ is an orbit, see how $\dot{\gamma}$ transforms under the above mapping. Find how Ω transforms from the relation $\dot{\gamma} = \Omega\gamma$.

- Using $\Omega \equiv \omega \times \cdot$ then see how ω transforms.

- You should find

$$\omega \mapsto \det(S)\, S\omega$$

- Consider two consecutive rotations, R_2 first and then R_1, with their respective angular velocity vectors

$$R_1 \leftrightarrow \omega_1$$

$$R_2 \leftrightarrow \omega_2$$

and find the angular velocity vector ω which corresponds to R_1R_2 (note that R_1R_2 is also a rotation as $O(3)$ is a group).

- You will find it useful to use Ω, and how it transforms. At some point you will also have to use the first part of the exercise.