Analytical Mechanics Recitation Session of Week 3

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1 Epilogue to HW1

1.1 General Notes

- No need to write an equation with (t) all the time if it holds for all t. Simply write a function equation instead of a number equation. That is,

$$x = Ry$$

where x and R are two functions (they depend on t, but that's not a helpful piece of information for that equation) instead of

$$x(t) = R(t)\,y$$

1.2 Question 1

1.2.1 Galilean Invariance of Force Law

Since there has been some confusion about just what it means for the force law to be Galilean invariant, let us clarify that in the most precise way.

Let $\begin{bmatrix} t \\ x \end{bmatrix} \in \mathbb{R}^4$ be any event as measured in some inertial coordinate system S. In that coordinate system, assume that the equations of motion of a system of N particles are of the following form: For any $i \in \{ 1, \ldots, N \}$,

$$m_i \ddot{\gamma}_i = F_i(\gamma_1, \ldots, \gamma_N) \quad (1)$$

where m_i is the mass of the ith particle, $\gamma_i : \mathbb{R} \to \mathbb{R}^3$ is the trajectory of the ith particle as measured in S and $F_i : (\mathbb{R}^3)^N \to \mathbb{R}^3$ is the force on the ith particle (which depends on the positions of all N particles).

Now we perform a Galilean coordinate transformation

$$\begin{bmatrix} t \\ x \end{bmatrix} \mapsto \begin{bmatrix} t' \\ x' \end{bmatrix}$$

into a coordinate system which we label by S'. The equations of motion for the trajectories γ'_i measured in S' can now be obtained by plugging into (1) the expression of γ_i in terms of γ'_i (the reverse transformation). If the resulting equations (for γ'_i) are of the form

$$m_i \ddot{\gamma}'_i = F_i(\gamma'_1, \ldots, \gamma'_N)$$

then we say the force law (1) is Galilean invariant.

Let us be explicit with an example, for the sake of concreteness.

We assume $N = 2$,

$$F_1(\gamma_1, \gamma_2) = -G\frac{m_1 m_2}{\|\gamma_1 - \gamma_2\|^3}(\gamma_1 - \gamma_2)$$

and

$$F_2(\gamma_1, \gamma_2) = -G\frac{m_1 m_2}{\|\gamma_1 - \gamma_2\|^3}(\gamma_2 - \gamma_1)$$

Let $R \in O(3)$ be given and define our transformation as

$$\begin{bmatrix} t \\ x \end{bmatrix} \mapsto \begin{bmatrix} t' \\ x' \end{bmatrix} = \begin{bmatrix} t \\ Rx \end{bmatrix}$$

Then

$$\gamma_i'(t) = R\gamma_i(t)$$

so that

$$\begin{aligned}
m_1\ddot{\gamma}_1'(t) &= m_1 R\ddot{\gamma}_1(t) \\
&= -RG\frac{m_1 m_2}{\|\gamma_1(t) - \gamma_2(t)\|^3}(\gamma_1(t) - \gamma_2(t)) \\
&= -G\frac{m_1 m_2}{\|\gamma_1(t) - \gamma_2(t)\|^3}(R\gamma_1(t) - R\gamma_2(t)) \\
& \quad (R \text{ preserves norms by definition}) \\
&= -G\frac{m_1 m_2}{\|R\gamma_1(t) - R\gamma_2(t)\|^3}(R\gamma_1(t) - R\gamma_2(t)) \\
& \quad (\gamma_i' \equiv R\gamma_i) \\
&= -G\frac{m_1 m_2}{\|\gamma_1'(t) - \gamma_2'(t)\|^3}(\gamma_1'(t) - \gamma_2'(t)) \\
&\equiv F_1(\gamma_1'(t), \gamma_2'(t))
\end{aligned}$$

and similarly for the second mass. Thus, we have found

$$m_i\ddot{\gamma}_i = F_i\left(\{\gamma_j\}_j\right)$$

before the transformation, and

$$m_i\ddot{\gamma}_i' = F_i\left(\{\gamma_j'\}_j\right)$$

after the transformation. This is an invariant force law.

1.3 Question 2

1.3.1 Part one

- R is given as $R : \mathbb{R} \to O(3)$, not anything else! In particular, nothing is given as a map $\mathbb{R}^3 \to \mathbb{R}^3$.

1.3.2 Part Two

- Apparently it's too hard to calculate powers of the matrix

$$\Omega = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

  just by matrix multiplication (although one of you did this). Instead, easier to think of

$$\Omega \equiv \omega \times \cdot$$

  and use properties of cross product!

- It is also a good example of when brute-forcing the solution by diagonalization is not a good idea. Not a single student who attempted this actually succceeded.

- Induction: make sure that $n = 1$ is proven as well.

1.4 Question 3

- Clarify the distinction between vector and pseudo vector. Pseudo-vector is a cross product of two vectors, and under reflection it is invariant.

- Explain what it means for the angular velocity to be additive, and exactly how.

2 Prologue to HW2

2.1 Question 1–Decay of Binary Star

The problem wants us to find a certain criteria that the motion of two bodies (after the second one had exploded) is bounded. For that, we need to understand what "bounded motion" means for the two bodies.

1 Claim. A two-body system which is unbounded has non-negative energy.

Proof. If we denote the displacement (at any given moment) between the two bodies as

$$r := x_1 - x_2$$

Then for the system to be unbounded means that we cannot bound $\|r\|$ as time grows to infinity. That is, for any $R > 0$ there exists some $t_R > 0$ such that $\|r(t_R)\| > R$ (but we allow that $\lim_{t\to\infty}\|r(t)\|$ does not converge necessarily "converge" to ∞).

So let us assume that this is indeed the case for $\|r\|$.

We know that the energy of the system is given by

$$E = T + V$$

where $T \equiv \frac{1}{2}\left(m_1\|\dot{x_1}\|^2 + m_2\|\dot{x_2}\|^2\right)$ is the kinetic energy of the system and

$$V = -G\frac{m_1 m_2}{\|r\|}$$

is its potential energy. We see that $T \geq 0$ and $V \leq 0$ (for all values of parameters). We also see that

$$\lim_{\|r\|\to\infty} V = 0$$

and in addition, via

$$T = E - V$$

and

$$\begin{aligned} \partial_{\|r\|} E &= \frac{dt}{d\|r\|} \underbrace{\partial_t E}_{\text{zero by conservation of energy}} \\ &= 0 \end{aligned}$$

We have

$$\begin{aligned} \lim_{\|r\|\to\infty} T &= \lim_{\|r\|\to\infty} E \\ &= E \end{aligned}$$

so that as $\|r\| \to \infty$, $V \to 0$ and $T \to E$. Because $T \geq 0$ for all parameters and $[0, \infty)$ is closed from the left, that necessarily means that $E \geq 0$. □

2 Corollary. *A two-body system which has negative energy is bounded.*

3 Claim. For a two-body system the kinetic energy of the motion with respect to the center of mass is given by

$$T_s = \frac{1}{2} m \|\dot{r}\|^2$$

with

$$m := \frac{m_1 m_2}{M}$$

being the reduced mass and $M := m_1 + m_2$ the total mass.

Proof. The center of mass is given by

$$X \equiv \frac{1}{M}\left(m_1 x_1 + m_2 x_2\right)$$

Note that

$$\begin{aligned} x_i - X &= \frac{M x_i - m_1 x_1 - m_2 x_2}{M} \\ &= \frac{m_{i'}}{M}\left(x_i - x_{i'}\right) \end{aligned}$$

for all $i \in \{ 1, 2 \}$ where i' is the index in the singleton $\{ 1, 2 \} \setminus \{ i \}$.

Hence we find

$$\begin{aligned} T_S &\equiv \frac{1}{2}\sum_{i=1}^{2} m_i \left\| \dot{x_i} - \dot{X} \right\|^2 \\ &= \frac{1}{2} m \|\dot{r}\|^2 \end{aligned}$$

□

4 Fact. *If a body explodes in an inversion symmetric way (and in particular, spherical symmetric explosion is inversion symmetric), then in its instantaneous rest frame right before the explosion, its velocity before and after the explosion is zero. In equations, we have*

$$\lim_{\varepsilon \to 0^+} \dot{x_2} \left(t_{explosion} - \varepsilon \right) = 0 \tag{2}$$

$$\lim_{\varepsilon \to 0^+} \dot{x_2'} \left(t_{explosion} + \varepsilon \right) = 0 \tag{3}$$

where $t_{explosion} \in \mathbb{R}$ is the time of the explosion, $x_2 : \mathbb{R} \to \mathbb{R}^3$ is the trajectory of the second body measured in the frame in which it is at rest exactly at $t_{explosion}$ (which is why we have (2)) and $x_2' : \mathbb{R} \to \mathbb{R}^3$ is the trajectory of the remainder of the second body after the explosion, in the same frame. The inversion symmetry assumption is exactly (3). Note that x_2's domain is really $(-\infty, t_{explosion})$ and not any time after, as it stops existing after the explosion, and x_2''s domain is really $(t_{explosion}, \infty)$ as it did not exist before the explosion.

5 Fact. *The first body's velocity before the explosion and instantaneously after the explosion is equal, as, according to the hypothesis of the problem, the explosion does not affect it. In equations:*

$$\lim_{\varepsilon \to 0^+} \dot{x_1} \left(t_{explosion} - \varepsilon \right) = \lim_{\varepsilon \to 0^+} \dot{x_1} \left(t_{explosion} + \varepsilon \right) \tag{4}$$

6 Fact. *For the two body problem, one can set up the initial data in such a way that the two bodies move diametrically on two concerntric circles, of possibly different radii. The center of the two circles is the center of mass of the two bodies X. This is what we are told to assume is the situation in the given problem. As a result, we have*

$$\|x_1 - x_2\| = \|x_1\| + \|x_2\| \tag{5}$$

7 Claim. For a two body system moving in a circular fashion (such as the system in our problem),

$$T_S = -\frac{1}{2} V \tag{6}$$

where T_S is the kinetic energy of the two bodies with respect to the center of mass.

Proof. Recall that for circular motion, $\|\dot{x_i}\|$ and $\|x_i\|$ do not depend on time, for all $i \in \{1, 2\}$. We start by computing the kinetic energy of the ith body:

$$\begin{aligned} T_i &\equiv \frac{1}{2}m_i\|\dot{x_i}\|^2 \\ &\left(\|\ddot{x_i}\| = \frac{\|\dot{x_i}\|^2}{\|x_i\|} \text{ for circular motion}\right) \\ &= \frac{1}{2}m_i\|\ddot{x_i}\|\|x_i\| \\ &(m_i\|\ddot{x_i}\| = \|F_i\| \text{ by Newton's second law}) \\ &= \frac{1}{2}\|x_i\|G\frac{m_1m_2}{\|r\|^2} \end{aligned}$$

Thus using (5) we get:

$$\begin{aligned} T_s &\equiv T_1 + T_2 \\ &= \frac{1}{2}G\frac{m_1m_2}{\|r\|^2}\underbrace{(\|x_1\| + \|x_2\|)}_{\|r\|} \\ &= \frac{1}{2}G\frac{m_1m_2}{\|r\|} \\ &\equiv -\frac{1}{2}V \end{aligned}$$

as desired. □

8 Remark. Note that while (6) holds before the explosion (because we were told to assume there is circular motion), no such relation necessarily holds *after* the explosion.

We may now finally solve the problem by showing that $E' < 0$ (note that energy after the explosion is also conserved, so it doesn't matter at which time we evaluate it. We choose to evaluate E' at $t_{\text{explosion}} + \varepsilon$ for some $\varepsilon > 0$):

$$\begin{aligned} E' &\equiv E'(t_{\text{explosion}} + \varepsilon) \\ &(E' \text{ is conserved for all } \varepsilon > 0) \\ &= \lim_{\varepsilon\to 0^+} E'(t_{\text{explosion}} + \varepsilon) \\ &= \lim_{\varepsilon\to 0^+} T_S'(t_{\text{explosion}} + \varepsilon) + V'(t_{\text{explosion}} + \varepsilon) \\ &= \lim_{\varepsilon\to 0^+} \frac{1}{2}m'\left\|\dot{x_1}(t_{\text{explosion}} + \varepsilon) - \dot{x_2'}(t_{\text{explosion}} + \varepsilon)\right\|^2 - G\frac{m_1m_2'}{\|r'(t_{\text{explosion}} + \varepsilon)\|} \end{aligned}$$

We now use (4) and (2) and (3), as well as the fact that for any explosion we must have

$$\lim_{\varepsilon\to 0^+} x_2(t_{\text{explosion}} - \varepsilon) = \lim_{\varepsilon\to 0^+} x_2'(t_{\text{explosion}} + \varepsilon)$$

2.2 One Dimensional Oscillations

Note that even though the text of the question doesn't specify it, it is meant that V is *strictly* monotone on both sides of its minimum (in fact if it's not strictly monotone then the period will diverge!).

So let $V:\mathbb{R}\to\mathbb{R}$ be a function with one minimum, say, at some $m\in\mathbb{R}$ $V(m)=0$, and assume that $V|_{[m,\,\infty)}$ is strictly monotone increasing and $V|_{(-\infty,\,m]}$ is strictly monotone decreasing. Note that a function which is strictly monotone is necessarily injective. In fact V is also surjective onto $[0,\,\infty)$ due to this property. As a result, for a given $E\in\mathbb{R}\backslash\{\,0\,\}$, the sets $V^{-1}(\{\,E\,\})\cap(-\infty,\,m]$ and $V^{-1}(\{\,E\,\})\cap[m,\,\infty)$ are actually singletons. We denote $x_1(E)$ and $x_2(E)$ as the two points inside those singletons respectively (so $x_2(E)>m$ and $x_1(E)<m$).

If a particle of mass $m\equiv 2$ is moving under the influence of the potential V at some energy $E\in\mathbb{R}\backslash\{\,0\,\}$, we denote the time it takes the particle to move from $x_1(E)$ to $x_2(E)$ and then back to $x_1(E)$ by $\tau(E)$.

9 Claim. We have the following expression for $\tau(E)$:

$$\tau(E) = \int_{x_1(E)}^{x_2(E)}\frac{1}{\sqrt{E-V(y)}}dy$$

Proof. Use the fact that

$$\tau(E) \equiv 2\int_0^{\frac{1}{2}\tau(E)}dt$$

Then make a change of variable in the integral

$$y := \gamma(t)$$

where $\gamma:\mathbb{R}\to\mathbb{R}$ is the trajectory of the particle. If we assume that $\gamma(0)=x_1(E)$ and $\gamma\left(\frac{1}{2}\tau(E)\right)=x_2(E)$ we obtain the desired result. □

10 Claim. We have

$$x_2(E)-x_1(E) = \frac{1}{2\pi}\int_0^E\frac{\tau(E')}{\sqrt{E-E'}}dE'$$

Proof. Start of by the observation that

$$2\pi\left(x_2(E)-x_1(E)\right) = 2\int_{x_1(E)}^{x_2(E)}\pi dy$$

The use the fact that

$$\pi = \int_{V(y)}^{E}\frac{\tau(E')}{\sqrt{\left(E'-V(y)\right)\left(E-E'\right)}}dE'$$

independently of y and of E. This is a simple calculation with the change of variables

$$t := \frac{2E' - V(y) - E}{E - V(y)}$$

and then

$$t := \sin(\varphi)$$

Now exchange the order of integration for the two integrals (the inner one which equals π and the outer one from $x_1(E)$ to $x_2(E)$). This is allowed by Fubini's theorem. Note that then the limits of integration change. The (new) inner integral should correspond to $\tau(E')$. □

11 Claim. If V is even (so that $m = 0$) then V may be computed from τ.

Proof. Note that by bijectivity of $V|_{[0,\,\infty)} : [0,\,\infty) \to [0,\,\infty)$ and $V|_{(-\infty,\,0]} : (-\infty,\,0] \to [0,\,\infty)$ we get that

$$x_1 = \left(V|_{(-\infty,\,0]}\right)^{-1}$$

and

$$x_2 = \left(V|_{[0,\,\infty)}\right)^{-1}$$

Because V is even, $x_1 = -x_2$ so that

$$x_2 - x_1 = 2x_2$$

and we need to determine V only on the right, that is, we only need to find $V|_{[0,\,\infty)}$. Thus if we inverted the map

$$\left(V|_{[0,\,\infty)}\right)^{-1} : [0,\,\infty) \to [0,\,\infty)$$

$$E \mapsto \frac{1}{4\pi}\int_0^E \frac{\tau(E')}{\sqrt{E - E'}} dE'$$

we would be finished. □