Analytical Mechanics Recitation Session of Week 4

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October 12, 2016

1 Prologue to HW3

1.1 Question 1

In this question, there is a rocket which expels mass at a certain constant velocity v_0 with respect to its rest frame. The rocket's initial mass is m_0. You are to ignore air friction and gravity. Your goal is to write down an equation relating the rocket's remaining mass m to its velocity v.

To obtain the equation, you will have to solve a differential equation for v as a function of m.

To get the differential equation, do the following:

1. Write down the momentum of the rocket at a given instance of time (disregarding any of the already expelled mass), in which it had velocity v and mass m.
2. After a short period of time, the momentum of the rocket is composed of two terms: the just expelled mass with its velocity, and what remains of the rocket. If we label the mass that has just been expelled as dm (with the convention that $dm < 0$) and the addition to the rocket's velocity due to the exhaust by dv (with the convention that $dv > 0$) then we get for the momentum

$$(m + dm)(v + dv) + (-dm)(v - v_0)$$

3. Now apply momentum conservation.

For the second part of the exercise, we no longer ignore gravity. Thus, there is a homogeneous gravitational field g, and we again would like to find a relation between the velocity of the rocket and its mass. This time the velocity will also contain explicit dependence in time t.

1. Work in an (accelerating) reference frame in which the gravitational field is zero (you've seen in the lecture that this is possible).
2. In that frame, apply now the first part of the question.
3. Now transform back the expression for the velocity you found into the non-accelerating reference frame.

1.2 Question 2

- Note that there is a mistake on the exercise sheet, you should rather find that $\lambda = \frac{g}{l}$.
- The main point about the first part of this exercise is to understand just what to neglect, and why.
- Note that the earth's rotation is given by angular veloctiy

$$\begin{aligned}\omega &= \frac{2\pi}{\text{day}} \\ &= \frac{2\pi}{86400\text{sec}} \\ &\approx 10^{-5}\frac{1}{\text{sec}}\end{aligned}$$

  which is a very small number. Argue then why the centrifugal terms can be neglected, compared with the other terms.
- Once you neglect the centrifugal terms, there is a further approximation that is made, namely, that the oscillations along the 1 and 2 axis are very small compared to the length of the thread: $\frac{y_1}{l}$ amd $\frac{y_2}{l}$ are very small numbers. To apply this approximation, recall that the pendulum is constrained by the relation

$$\|y\| = l$$

  so that

$$y_3 = -l\sqrt{1 - \left(\frac{y_1}{l}\right)^2 - \left(\frac{y_2}{l}\right)^2}$$

  and since the two terms in the square root are assumed to be very small, we have

$$y_3 \approx -l$$

  When we place this in the third component for the equation of motion we find the value of λ (assuming that the higher corrections to λ will exactly cancel the other non-constant terms).
- Once we find λ the other two remaining equations (together with $\dot{y_3} \approx 0$) give the result.

2 Epilogue to HW2

2.1 Question 1

- Note that we need $E' < 0$, not merely $E' \leq 0$! $E' = 0$ might still be unbounded.
- $\|x_1 - x_2\| = \|x_1\| + \|x_2\|$ for diametric points! (no minus signs between norms).
- Precise meaning of inversion symmetry.

2.2 Question 2

- x_1 and x_2 are functions of energy, not time! They are *not* trajectories. Do not mix up the notation $x(t)$ with $x_1(E)$. In particular, $\dot{x_1}(t)$ or $\dot{x_2}(t)$ do not make any sense.

- Why are you allowed to make the change of variables $y := \gamma(t)$? (If you don't like dividing by differentials)

- Formula (39) in Rudin's PMA Chapter 6 (Theorem 6.19) is:

$$\int_a^b f(y)\,\mathrm{d}y = \int_A^B f(\varphi(t))\,\varphi'(t)\,\mathrm{d}y$$

  where $\varphi : [A, B] \to [a, b]$ is continuous and strictly increasing and surjective, such that φ' is Riemann integrable on $[A, B]$ and f is Riemann integrable on $[a, b]$.

  - We apply this result with $(a, b) := (x_1(E), x_2(E))$, $f := x \mapsto \frac{1}{\sqrt{E-V(x)}}$, and $(A, B) := \left(0, \frac{1}{2}\tau(E)\right)$. We then choose $\varphi := \gamma$, and φ is: continuous (by physical assumption), strictly increasing on this interval, and surjective (by definition). Also, note f is Riemann integrable on $(x_1(E), x_2(E))$ (due to the square root). So we may apply the theorem in Rudin to obtain the desired change of variables:

$$\begin{aligned}
\int_{x_1(E)}^{x_2(E)} \frac{1}{\sqrt{E-V(y)}}\mathrm{d}y &= \int_0^{\frac{1}{2}\tau(E)} \frac{1}{\sqrt{E-V(\varphi(t))}}\varphi'(t)\,\mathrm{d}t \\
&= \int_0^{\frac{1}{2}\tau(E)} \frac{1}{\sqrt{E-V(\gamma(t))}}\dot{\gamma}(t)\,\mathrm{d}t \\
&= \int_0^{\frac{1}{2}\tau(E)} \frac{1}{\dot{\gamma}(t)}\dot{\gamma}(t)\,\mathrm{d}t \\
&= \int_0^{\frac{1}{2}\tau(E)} \mathrm{d}t \\
&= \frac{1}{2}\tau(E)
\end{aligned}$$

- Explain again how to exchange the order of the limits, and *why* it is necessary at all:

$$\int_0^E \int_{x_1(E)}^{x_2(E)} \cdot\mathrm{d}y\mathrm{d}E' = \int_{x_1(E)}^{x_2(E)} \int_{V(y)}^E \cdot\mathrm{d}E'\mathrm{d}y$$

- We know

$$V^{-1}(E) = \frac{1}{4\pi}\int_0^E \frac{\tau(E')}{\sqrt{E-E'}}\mathrm{d}E'$$

  so that we need to solve the equation

$$x = \frac{1}{4\pi}\int_0^{V(x)} \frac{\tau(E')}{\sqrt{V(x)-E'}}\mathrm{d}E'$$

  for $V(x)$ (need to have the explicit form of τ in order to do that).