Analytical Mechanics Recitation Session of Week 5

Jacob Shapiro

October 19, 2016

1 Prologue to HW4

Use the redacted solutions, they contain some background information about this exercise.

1.1 Question 1–Perihelion Precession

This question uses pages 11 till 16 from the script heavily, so be sure to read them prior to attempting this question.

1.1.1 Part i)

You simply need to figure out what $\tilde{U}(u)$ is for our case, as defined in equation (2.3) and above (2.6) in the script. Then simply plug this into (2.7).

1.1.2 Part ii)

The concept is as follows: we choose initial conditions such that if α were equal to zero, we would get an elliptic orbit. Now we "turn on" α and assume that for sufficiently small α, we would get orbits which may deviate from ellipses, but want to quantify precisely how. One way is to to compute the perihelion precession per orbit, which is what this exercise is all about. We shall do all calculations up to linear order in α.

- The perihelion is defined as the minimum value r obtains along the trajectory. Thus, φ_0 is a perihelion angle iff $r(\varphi_0)$ is minimal. This implies that $u'(\varphi_0) = 0$ (recall $u \equiv r^{-1}$ so that r minimal means u maximal).
- The first occurrence of the perihelion is, by choice of parametrization, at $\varphi = 0$. Let us denote the next occurrence of the perihelion, after a full orbit, by φ_0. Thus, for the $\alpha = 0$ case $\varphi_0 = 2\pi$. The perihelion precession (what we have to actually compute here) is denoted by $\Delta\varphi$ so that by definition

$$\varphi_0 = 2\pi + \Delta\varphi$$

- Now we make a Taylor expansion of u' around 2π (recall $u'(\varphi_0) = 0$ because the perihelion is an extremal point of the trajectory) in the small parameter $\Delta\varphi$ (which is going to be linear in α in our approximation):

$$\begin{aligned} \underbrace{u'(\varphi_0)}_{\equiv 0} &= u'(2\pi + \Delta\varphi) \\ &\approx u'(2\pi) + u''(2\pi)\Delta\varphi + O(\alpha^2) \end{aligned}$$

hence

$$\Delta\varphi \approx -\frac{u'(2\pi)}{u''(2\pi)} + O(\alpha^2) \tag{1}$$

- We start by analyzing the $\alpha = 0$ case (see the redacted solutions for full details).

- We denote by u_0 the solution to the $\alpha = 0$ equations of motion. That is, u_0 satisfies

$$u_0'' + u_0 = l^{-2}M$$

This equation is solved with the initial conditions $u_0'(0) = 0$ (we want to parametrize u_0 such that at $\varphi = 0$ it starts precisely at the perihelion) so that $u_0(0)$ should be the perihelion. $u_0(0)$ may be obtained from the energy equation $E = T + V$ evaluated at $\varphi = 0$. We find:

$$u_0(0) = d^{-1}(1 + \varepsilon)$$

where $d \equiv l^2 M^{-1}$ and $\varepsilon \equiv \sqrt{2El^2M^{-2} + 1}$. Then the trajectory is given by

$$u_0(\varphi) = d^{-1}[1 + \varepsilon\cos(\varphi)] \tag{2}$$

- Next we turn to consider the full equation $u'' + u = l^{-2}(M - 3\alpha u^2)$. The initial values are the same: $u'(0) = 0$ and $u(0) = d^{-1}(1 + \varepsilon)$.

- Define $v := u - u_0$. Since u depends on α, v depends on α as well. Furthermore, v must be at least linear in α because for $\alpha = 0$ we have $u = u_0$ so that $v = 0$.

- v obeys the initial conditions $\begin{cases} v(0) &= 0 \\ v'(0) &= 0 \end{cases}$ and the differential equation $v'' + v = -3l^{-2}\alpha(v - u_0)^2$. Because we are interested only in linear orders in α, and v is at least of linear order in α, we may just as well write

$$v'' + v = -3l^{-2}\alpha u_0^2$$

into which we may readily plug in (2):

$$v'' + v = -3l^{-2}\alpha d^{-2} - 6l^{-2}\alpha d^{-2}\varepsilon\cos - 3l^{-2}\alpha d^{-2}\varepsilon^2\cos^2 \tag{3}$$

(this is a function equation)

- In terms of v, the perihelion precession $\Delta\varphi$ is given by

$$\begin{aligned}
\Delta\varphi &\approx -\frac{u'(2\pi)}{u''(2\pi)} \\
&= -\frac{v'(2\pi)+u_0'(2\pi)}{v''(2\pi)+u_0''(2\pi)} \\
& \quad \left(u_0'(2\pi)\equiv 0 \wedge u_0''(2\pi)=-\varepsilon d^{-1}\right) \\
&= -\frac{v'(2\pi)}{v''(2\pi)-\varepsilon d^{-1}} \\
& \quad \left(v''(2\pi) \text{ is at least linear in } \alpha\right) \\
&\approx \frac{d}{\varepsilon}v'(2\pi)+O\left(\alpha^2\right)
\end{aligned}$$

- We write $v' = v'_{per} + v'_{n.p.}$ where v'_{per} is the periodic part of v' (that is, $v'_{per}(\varphi) = v'_{per}(\varphi+2\pi)$ for all φ) and $v'_{n.p.} \equiv v' - v'_{per}$. Then

$$\begin{aligned}
v'(2\pi) &= v'_{per}(2\pi)+v'_{n.p.}(2\pi) \\
&= v'_{per}(0)+v'_{n.p.}(2\pi) \qquad (4)
\end{aligned}$$

- Now we inspect (3) to understand which part of its solution is periodic and which isn't. The full solution is a sum of the solution to the homogeneous equation and the particular solutions. The homogeneous solution is periodic (being a linear combination of a sine and a cosine) and so its derivative is also periodic. We use the hint on the exercise which gives us the particular solutions. Of these, the first and third terms produce periodic particular solutions and so do their derivatives. Only the second term produces a non-periodic solution with non-periodic derivative. It is proportional to

$$v_{par,2}(\varphi) \quad \propto \quad \varphi\sin(\varphi)$$

so that

$$v'_{par,2}(\varphi) \quad \propto \quad \sin(\varphi)+\varphi\cos(\varphi)$$

As a result, $v'_{n.p.}(0)=0$ so that continuing (4) we have

$$\begin{aligned}
v'(2\pi) &= v'_{per}(0)+0+v'_{n.p.}(2\pi) \\
&= v'_{per}(0)+v'_{n.p.}(0)+v'_{n.p.}(2\pi) \\
&= v'(0)+v'_{n.p.}(2\pi) \\
& \quad \left(v'(0)\equiv 0\right) \\
&= v'_{n.p.}(2\pi)
\end{aligned}$$

from which we get

$$\Delta\varphi \quad \approx \quad \frac{d}{\varepsilon}v'_{n.p.}(2\pi)+O\left(\alpha^2\right)$$

and the result readily follows from the hint, by considering only the particular solution corresponding to the second term.

1.1.3 Part iii)

For the third part, compute the force which corresponds to a term in the potential proportional to

$$\frac{x_3^2}{r^5}$$

when evaluated at $x_3 = 0$.

1.1.4 Part iv)

Now we use $\alpha = -Ml^2$ and plug this into the same equation for $\Delta\varphi$ which we obtained.

Figure out the dimensions of the constants to get back to a formula with G and c.

2 Epilogue to HW3

2.1 Question 1

- Talk about signs. Does it make sense that $v(m) \propto \log(m)$? Is this physical?
- Time dependence is implicit! Get v in terms of m, not both in terms of t! (Not wrong, just more work).

2.2 Question 2

In this question, we are doing perturbation theory (that is, expanding a function) around *two* small (dimensionless) parameters: $\omega\sqrt{\frac{l}{g}}$ and $\frac{y_i}{l}$ with $i \in \{1, 2\}$.

The first parameter is small as long as l has a diameter much smaller than the diameter of the sun (simply put in the numbers with the condition $\omega\sqrt{\frac{l}{g}} \ll 1$. The second parameter is simply assumed to be small a-priori (this is a reasonable assumption since in the case $\omega = 0$ and with suitable initial conditions, these parameters are indeed small).

For the unperturbed ($\omega = 0$) problem, we have

$$y_1(t) \approx l\theta(t)$$

$$\ddot{y}_1 = -g\theta$$

so that

$$\frac{y_1(t)}{l} \approx \theta_0 \sin\left(\sqrt{\frac{g}{l}}t\right)$$

and so

$$\dot{y}_1(t) = \theta_0 l\sqrt{\frac{g}{l}}\cos\left(\sqrt{\frac{g}{l}}t\right)$$

so that

$$
\begin{aligned}
O\left(\text{Coriolis}\right) &\propto O\left(\omega y_1\left(t\right)\right) \\
&\propto \omega\sqrt{\frac{g}{l}} \\
&= \underbrace{\omega\sqrt{\frac{l}{g}}}_{\text{linear order in small parameter}} \qquad \frac{g}{l}
\end{aligned}
$$

$$
\begin{aligned}
O\left(\text{Centrifugal}\right) &\propto O\left(\omega^2 y\right) \\
&\propto \omega^2 l \\
&= \underbrace{\omega^2\frac{l}{g}}_{\text{second order in small parameter}} \qquad \frac{g}{l}
\end{aligned}
$$

Finally, (for the last part)

$$
\begin{aligned}
O\left(\omega\dot{y}_1\right) &\propto \omega l\sqrt{\frac{g}{l}} \\
&= \underbrace{\omega\sqrt{\frac{l}{g}}}_{\text{first order in small parameter}} \qquad g
\end{aligned}
$$

so that if we want to solve the third component of the equation of motion order by order, we must neglect this term as well.