Analytical Mechanics Recitation Session of Week 6

Jacob Shapiro

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1 Prologue to HW5

See redacted solutions which in this case contain also some explanations.

2 Epilogue to HW4

2.1 Question 1

- Don't forget that when you write a solution to an ordinary differential equation, the full solution is the sum of the particular solution (which was given to you in the hints) and the homogeneous solution (which ended up being irrelevant in this case).

- In case of the oblate earth, we assume the potential between the two bodies is given by

$$\tilde{V}(x) = -J\frac{M}{2r^3}\left(\frac{R}{r}\right)^2\left(x_1^2+x_2^2-2x_3^2\right)$$

where $r \equiv \|x\|$, M is earth's mass, and J is a parameter measuring the oblateness of earth.
In the second chapter in the script, the 2-body chapter, in the first line, it says that the two bodies are interacting with a *central force*, meaning, a force which depends only on the distance between the two bodies, and whose direction is always along the line connecting them:

$$F(x) = f(\|x\|)\frac{x}{\|x\|} \tag{1}$$

for some $f:\mathbb{R}\to\mathbb{R}$. The force corresponding to $\tilde{V}$ is given by:

$$\begin{aligned} F(x) &\equiv -\left(\nabla\tilde{V}\right)(x) \\ &= \frac{JMR^2}{2}\nabla\left(\frac{\|x\|^2-3(x_3)^2}{\|x\|^5}\right) \\ &= \frac{JMR^2}{2}\nabla\left(\frac{1}{\|x\|^3}-3\frac{(x_3)^2}{\|x\|^5}\right) \end{aligned}$$

The first term gives

$$\begin{aligned}
\nabla \|x\|^{-3} &\equiv \left(\partial_i \|x\|^{-3}\right) e_i \\
&= \left(-3\|x\|^{-4} \partial_i \|x\|\right) e_i \\
&= \left(-3\|x\|^{-4} \frac{x_i}{\|x\|}\right) e_i \\
&= -3 \frac{x}{\|x\|^5} \\
&= \underbrace{\frac{-3}{\|x\|^4}}_{\text{scalar depending only on } \|x\|} \underbrace{\frac{x}{\|x\|}}_{\text{unit vector in the direction of } x}
\end{aligned}$$

This certainly obeys (1). The second term, however, gives

$$\nabla \frac{(x_3)^2}{\|x\|^5} = \frac{2\,(x_3 e_3)\,\|x\|^5 - (x_3)^2\, 5\|x\|^4 \frac{x}{\|x\|}}{\|x\|^{10}}$$

This part certainly does *not* obey (1)! As a result, one may not apply the analysis of chapter 2 of the script to this type of potential.
In the special case that the orbit always obeys $x_3 = 0$, the second part of the force is always zero, and then we *do* have a central force–only along this orbit.