Analytical Mechanics Recitation Session of Week 7

Jacob Shapiro

November 2, 2016

1 Natural Units

- To specify physical units, we must specify universal standards for length, mass, time, temperature, and electric charge (in lieu of electric current). The natural unit for temperature is usually determined via Blotzman's constant. The natural unit for electric charge is usually determiend via the electron.
- We usually measure length in meters because the meter is a length of the scale of the human height, which we have experience with. We usually measure mass in kilograms because the kilogram is of the scale of the human weight, which we have experience with. We usually measure time in seconds because the second is of the scale of the human vision system and neural system's response time, which we have experience with.
- However, these scales have nothing special about them from the point of view of abstract laws of physics, and when dealing with other questions, such as planetary motion or elementary particles, it may be more comfortable to pick different universal standards for the units.
- So instead, we choose the length, time and mass using other universal standards. Then we also agree to also drop the units from all equations (write 10 instead of $10kg$ in all equations).
- In some previous question (the perihelion precession question), we were asked to measure mass in units such that $G = m = 1$. What does this mean?
- For the mass we should measure in units of m. We are left with length and time. Let's call these unknown units λ and τ. What should they be so that G and m will drop out of the equations?

  $$\begin{aligned} G &\equiv 6.67 \times 10^{-11} \frac{meter^3}{kg\, sec^2} \\ &\stackrel{!}{=} \frac{\lambda^3}{m\tau^2} \end{aligned}$$

  Since m is already fixed, we have the freedom to choose λ or τ as we please so as to satisfy the above equation. One possible choice, for example, is to continue to use $\tau = sec$ and then

$$\begin{aligned} \lambda &= \left(m\, sec^2 6.67 \times 10^{-11} \frac{meter^3}{kg\, sec^2} \right)^{\frac{1}{3}} \\ &= \left(6.67 \times 10^{-11} \frac{m}{kg} \right)^{\frac{1}{3}} meter \end{aligned}$$

Then when we write expressions that contain G they will drop out if we indeed follow the convention to drop all units:

$$\|F\| = -G\frac{mM}{r^2}$$

so

$$\begin{aligned} \frac{\|F\|}{\left(\frac{m\lambda}{\tau^2}\right)} &= -\frac{\lambda^3}{m\tau^2}\frac{mM}{r^2}\frac{\tau^2}{m\lambda} \\ &= -\frac{M}{m}\left(\frac{r}{\lambda}\right)^{-2} \end{aligned}$$

or simply

$$\left\|\tilde{F}\right\| = -\tilde{M}\tilde{r}^{-2}$$

where the tilde versions denote the numerical values without units.

- Later on there is also a need to "set" the speed of light, $c = 1$. This will mean that we cannot anymore choose $\tau = sec$, but rather, we must solve the system

$$\begin{cases} 6.67 \times 10^{-11} \frac{meter^3}{kg\, sec^2} & \stackrel{!}{=} \frac{\lambda^3}{m\tau^2} \\ 3 \times 10^8 \frac{meter}{sec} & \stackrel{!}{=} \frac{\lambda}{\tau} \end{cases}$$

Which has the solution

$$\begin{cases} \lambda & = \left(6.67 \times 10^{-11} \frac{m}{kg} \left(3 \times 10^8\right)^{-2} \right) meter \\ \tau & = \left(6.67 \times 10^{-11} \frac{m}{kg} \left(3 \times 10^8\right)^{-3} \right) sec \end{cases}$$

Then in an equation like

$$E = Mc^2$$

we find

$$\frac{E}{\left(m\frac{\lambda^2}{\tau^2}\right)} = \frac{M}{m}$$

or simply

$$\tilde{E} = \tilde{M}$$

as desired.

- It is also possible to go back, that is, to go from equations which have only numerical values and no units and restore the units. This is done by dimensional analysis. For example, if we know that

$$\tilde{Q} = \tilde{A}\tilde{B}\tilde{C}$$

holds in natural units, then we'd like to find the corresponding equation with all unis restored. The first step is to find out what are the physical dimensions of $\tilde{Q}$ and then multiply the equation by those units. Then identify what are the units of $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ and distribute units across the expression so that they become dimensionful.

- For example, we know

$$\left\|\tilde{F}\right\| = -\tilde{M}\tilde{r}^{-2}$$

and we know that

$$[\|F\|] = [M]\frac{[L]}{[T]^2}$$

so we multiply the equation by $m\frac{\lambda}{\tau^2}$ to get

$$\begin{aligned}\|F\| &= \left\|\tilde{F}\right\|m\frac{\lambda}{\tau^2}\\ &= -\tilde{M}\tilde{r}^{-2}m\frac{\lambda}{\tau^2}\\ &= -\left(\tilde{M}m\right)m\left(\tilde{r}\lambda\right)^{-2}\frac{\lambda^2}{m^2}m\frac{\lambda}{\tau^2}\\ &= -\left(\tilde{M}m\right)m\left(\tilde{r}\lambda\right)^{-2}\underbrace{\frac{\lambda^3}{m\tau^2}}_{=G}\\ &= -G\left(\tilde{M}m\right)m\left(\tilde{r}\lambda\right)^{-2}\end{aligned}$$

- A second example:

$$d = l^2M^{-1}$$

in natural units. So really,

$$\tilde{d} = \tilde{l}^2\tilde{M}^{-1}$$

and we want to know what's this equation in actual units. We expects some factors of G and m. To get the right expression: we know that $[d] = [L]$, so that we multiply the equation by λ. We also know that

$$[l] = [M][L]^2[T]^{-1}$$

so we get

$$\begin{aligned}d &= \lambda\tilde{d}\\ &= \lambda\underbrace{\left(\tilde{l}m\lambda^2\tau^{-1}\right)^2}_{=l^2}\left(m\lambda^2\tau^{-1}\right)^{-2}\underbrace{\left(\tilde{M}m\right)^{-1}}_{=M^{-1}}m\\ &= l^2M^{-1}m^{-2}\frac{m\tau^2}{\lambda^3}\\ &= G^{-1}l^2M^{-1}m^{-2}\end{aligned}$$

which is exactly equation (2.12) in the script.

2 Conservative versus Central Forces

1 Definition. A central force law $F:\mathbb{R}^3\to\mathbb{R}^3$ is a force such that there exists some $f:\mathbb{R}\to\mathbb{R}$ such that F may be written as

$$F(x) = f(\|x\|)\frac{x}{\|x\|} \quad \forall x\in\mathbb{R}^3$$

2 Example. The gravitational force law, $F(x) = -G\frac{m_1 m_2}{\|x\|^2}\frac{x}{\|x\|}$ with $f(x) = -G\frac{m_1 m_2}{\|x\|^2}$.

3 Example. A charged particle of charge q moves in a homogeneous electric field Ee_3 for some $E\in\mathbb{R}$ and e_3 the standard unit vector in the third direction. Then

$$F(x) = qEe_3 \tag{1}$$

which is clearly not central.

4 Definition. A conservative force law $F:\mathbb{R}^3\to\mathbb{R}^3$ is a force such that there exists some $f:\mathbb{R}\to\mathbb{R}$ such that F can be written as

$$F = -\nabla f$$

5 Claim. The work done by a conservative force along a closed path is zero.

Proof. We have

$$\begin{aligned} W &= \int_\gamma F\cdot dr \\ &\quad \text{(Stokes')} \\ &= \int_A (\nabla\times F)\cdot da \\ &= \int_A (\nabla\times(-\nabla f))\cdot da \\ &\quad (\nabla\times(\nabla f)=0) \\ &= 0 \end{aligned}$$

where we have used

$$\begin{aligned} \nabla\times(\nabla f) &= e_i\varepsilon_{ijk}\partial_j(\nabla f)_k \\ &= e_i\varepsilon_{ijk}\partial_j\partial_k f \\ &= 0 \end{aligned} \tag{2}$$

which follows by the commutativity of partial derivatives on f and because ε_{ijk} is anti-symmetric. □

6 Claim. A force is conservative iff $\nabla \times F = 0$.

Proof. If a force F is conservative then it clearly obeys $\nabla \times F = 0$ by (2). Conversely, if for some F we have $\nabla \times F = 0$, define

$$f(x) \quad := \quad -\int_{\gamma_x} F \cdot dr$$

where γ_x is any curve from the origin to x that doesn't self-intersect.

Claim. f is well-defined (that is, it does not depend on the choice of γ_x).

Proof. If γ_x and λ_x are two possible choices of such curves, then we have

$$\int_{\gamma_x} F \cdot dr - \int_{\lambda_x} F \cdot dr \quad = \quad \int_{\gamma_x \wedge \lambda_x} F \cdot dr$$

where $\gamma_x \wedge \lambda_x$ is the concatenated loop that goes $0 \to x \to 0$ via γ_x first and then the reverse of λ_x. Then using Stokes' we have

$$\int_{\gamma_x \wedge \lambda_x} F \cdot dr \quad = \quad \int_A (\nabla \times F) \cdot da$$

where A is the surface enclosed within the loop $\gamma_x \wedge \lambda_x$. The last expression is then zero by assumption. □

Then we have

$$\begin{aligned} -(\nabla f)(x) &= \nabla \int_{\gamma_x} F \cdot dr \\ &= F(x) \end{aligned}$$

by the fundamental theorem of calculus. □

7 Claim. Every central force is conservative.

Proof. Let F be a central force with scalar f. Using the above claim we verify:

$$
\begin{aligned}
(\nabla \times F)(x) &= e_i \varepsilon_{ijk} (\partial_j F_k)(x) \\
&= e_i \varepsilon_{ijk} \partial_j f(\|x\|) \frac{x_k}{\|x\|} \\
&= e_i \varepsilon_{ijk} \left[(\partial_j f(\|x\|)) \frac{x_k}{\|x\|} + f(\|x\|) \partial_j \frac{x_k}{\|x\|} \right] \\
&= e_i \varepsilon_{ijk} \left[(f'(x) \partial_j \|x\|) \frac{x_k}{\|x\|} + f(\|x\|) \left(\frac{(\partial_j x_k)}{\|x\|} + x_k \partial_j \|x\|^{-1} \right) \right] \\
&= e_i \varepsilon_{ijk} \left[\left(f'(x) \frac{x_j}{\|x\|} \right) \frac{x_k}{\|x\|} + f(\|x\|) \left(\frac{(\partial_j x_k)}{\|x\|} + x_k \partial_j \|x\|^{-1} \right) \right] \\
& \quad \text{(ignore j,k symmetric terms because of } \varepsilon_{ijk}\text{)} \\
&= f(\|x\|) e_i \varepsilon_{ijk} \left[\left(-x_k \|x\|^{-2} \partial_j \|x\| \right) \right] \\
&= f(\|x\|) e_i \varepsilon_{ijk} \left[\left(-x_k \|x\|^{-2} x_j \|x\| \right) \right] \\
&= 0
\end{aligned}
$$

□

8 Example. The example of the charged particle in (1) shows that conservative forces need not be central.

Proof. We must show that (1) is indeed conservative:

$$
\begin{aligned}
(\nabla \times F)(x) &= e_i \varepsilon_{ijk} (\partial_j F_k)(x) \\
&= e_i \varepsilon_{ijk} \partial_j q E \delta_{3k} \\
&= 0
\end{aligned}
$$

□