Analytical Mechanics Recitation Session of
Week 9
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1 What are Eigenfrequencies, or Natural Frequencies?

1 Remark. What follows is a restatement of section 4.2.1 in the lecture’s script, which I thought might be good to briefly recollect as the idea is what is behind the solution to the first exercise in homework number eight.

2 Remark. Actually most of this discussion is just a very long way of saying that any two quadratic forms may be simultaneously diagonalized if one of the is positive definite. See http://math.stackexchange.com/questions/154540/simultaneously-diagonalizing-bilinear-forms.

Let $f \in \mathbb{N}_{\geq 1}$.

We consider a system whose state may be described by $f$ real parameters—by some point in $\mathbb{R}^f$.

Let $T$ and $V$ be quadratic forms on $\mathbb{R}^f$ (see definition 14). We assume that $T$ is a positive definite form (see definition 15). This makes sense because the kinetic energy is always non-negative, and always strictly positive if the speed is non-zero. Via 16 this induces a positive definite inner product $\langle \cdot, \cdot \rangle_T : (\mathbb{R}^f)^2 \to \mathbb{R}$. The reason we work with the inner product induced by $T$ is in order to not have to pick a basis for $\mathbb{R}^f$. So the point of what follows is a basis-free description of the problem.

3 Claim. There is a unique symmetric linear mapping $\tilde{V} : \mathbb{R}^f \to \mathbb{R}^f$ such that

\[ V(x) = \langle x, \tilde{V}x \rangle_T \quad \forall x \in \mathbb{R}^f \]

Proof. $T$ and $V$ define matrices $T$ and $V$ in $Mat_{f \times f}(\mathbb{R})$:

\[ T(x) = \langle x, T x \rangle \forall x \in \mathbb{R}^f \]

\[ V(x) = \langle x, V x \rangle \forall x \in \mathbb{R}^f \]

where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^f$. Indeed, these matrices are
defined via: Let \( \{ e_i \} \) be the standard basis for \( \mathbb{R}^f \).

\[
\mathcal{T} := \sum_{i,j=1}^{f} \frac{1}{2} (T(e_i + e_j) - T(e_i) - T(e_j)) e_i \otimes e_j^*
\]

and similarly for \( \mathcal{V} \).

Note that \( \mathcal{T} \) and \( \mathcal{V} \) do not have to be symmetric. However, we may define \( \mathcal{T}_S := \frac{1}{2} (\mathcal{T} + \mathcal{T}^T) \), so that \( \mathcal{T}_S^T = \mathcal{T}_S \) and

\[
\langle x, \mathcal{T}_S x \rangle = \left\langle x, \frac{1}{2} (\mathcal{T} + \mathcal{T}^T) x \right\rangle
\]

(We follow a similar procedure for \( \mathcal{V} \) to obtain that \( \mathcal{V}_S \) is symmetric and

\[
\langle x, \mathcal{V}_S x \rangle = V(x)
\]

Note that because \( \mathcal{T}_S \) is positive definite and symmetric, 17 implies that there is some matrix \( \mathcal{L} \in \text{Mat}_{f \times f}(\mathbb{R}) \) such that \( \mathcal{T}_S = \mathcal{L}^T \mathcal{L} \). Hence we find:

\[
T(x) = \langle \mathcal{L} x, \mathcal{L} x \rangle \quad \forall x \in \mathbb{R}^f
\]

Moreover, because \( T \) is positive definite, \( \mathcal{T}_S \) is invertible, so that \( \mathcal{L} \) is invertible as well. As a result, \( \{ \mathcal{L}^{-1} e_i \} \) is also a basis of \( \mathbb{R}^f \) (albeit not necessarily an orthogonal one—since \( \mathcal{T}_S \) is not necessarily diagonal). In this basis, the matrix \( \mathcal{T}_S \) is given by the components \((i, j)\):

\[
\langle \mathcal{L}^{-1} e_i, \mathcal{T}_S \mathcal{L}^{-1} e_j \rangle = \left\langle e_i, (\mathcal{L}^{-1})^T \mathcal{L}^T \mathcal{L} \mathcal{L}^{-1} e_j \right\rangle = \langle e_i, e_j \rangle = \delta_{ij}
\]

so that in this basis, the matrix \( \mathcal{T}_S = \mathbb{1}_{f \times f} \). As a result, taking the usual inner product \( \langle \cdot, \cdot \rangle \) in the \( \{ \mathcal{L}^{-1} e_i \} \) basis is like taking the \( \langle \cdot, \cdot \rangle_T \) inner product in the standard basis.

We now define (manifestly symmetric)

\[
\tilde{\mathcal{V}} := (\mathcal{L}^{-1})^T \mathcal{V}_S \mathcal{L}^{-1}
\]
which is simply the matrix $\mathcal{V}$ in the basis $\{L^{-1}e_i\}_i$. Hence

$$V(x) = \langle x, \mathcal{V}x \rangle$$

(in the basis $\{L^{-1}e_i\}_i$)

$$= \sum_{i,j=1}^{f} x_i \left((L^{-1})^T \mathcal{V}L^{-1}\right)_{ij} x_j$$

(in the basis $\{L^{-1}e_i\}_i$, the standard inner product is $\langle \cdot, \cdot \rangle_T$)

We define the system’s energy at time $t \in \mathbb{R}$, corresponding to the trajectory $\gamma : \mathbb{R} \rightarrow \mathbb{R}^f$ via

$$E_\gamma(t) := T(\dot{\gamma}(t)) + V(\gamma(t))$$

The label $t$ on the left hand-side is actually redundant, because we actually employ the assumption that

4 **Assumption.** $E_\gamma$ does not depend on time.

So we shall drop that label.

5 **Claim.** The assumption 4 implies that any trajectory must obey the differential equation

$$\ddot{\gamma} = -\tilde{V}\gamma \quad (1)$$

**Proof.** Using our notation we may write $E_\gamma$ as:

$$E_\gamma = T(\dot{\gamma}(t)) + V(\gamma(t)) = \langle \dot{\gamma}(t) , \dot{\gamma}(t) \rangle_T + \langle \gamma(t) , \tilde{V}\gamma(t) \rangle_T$$

The fact $E_\gamma$ is time-independent may be expressed as $\dot{E}_\gamma = 0$. Using the fact that $\langle x, y \rangle = \langle \dot{x}, \dot{y} \rangle + \langle x, \dot{y} \rangle$ we get

$$0 = \dot{E}_\gamma$$

$$= \langle \ddot{\gamma}(t) , \ddot{\gamma}(t) \rangle_T + \langle \dot{\gamma}(t) , \ddot{\gamma}(t) \rangle_T + \langle \dot{\gamma}(t) , \tilde{V}\gamma(t) \rangle_T + \langle \gamma(t) , \tilde{V}\dot{\gamma}(t) \rangle_T$$

(By symmetry of the forms involved)

$$= 2 \langle \ddot{\gamma}(t) , \ddot{\gamma}(t) \rangle_T + 2 \langle \dot{\gamma}(t) , \tilde{V}\gamma(t) \rangle_T$$

$$= 2 \langle \ddot{\gamma}(t) , \ddot{\gamma}(t) + \tilde{V}\gamma(t) \rangle_T$$

Which readily implies via the positive-definiteness of $T$ that either we have a constant solution (which we are not interested in) or

$$\ddot{\gamma}(t) + \tilde{V}\gamma(t) = 0 \quad (2)$$

3
6 Remark. Recall that real symmetric matrices are orthogonally diagonalizable. Thus we may find some orthonormal basis \{e_i\}_{i=1}^f of \mathbb{R}^f such that
\[ \tilde{V} e_i = \lambda_i e_i \] (3)
for some set of eigenvalues \{\lambda_i\}_{i=1}^f. Because \tilde{V} is real symmetric, \lambda_i \in \mathbb{R}.

7 Definition. For each eigenvalue \lambda_i \in \mathbb{R} of \tilde{V}, define \omega_i := \sqrt{\lambda_i}. Thus, \omega_i may be either real or strictly imaginary. The collection of all \omega_i’s are called the natural frequencies of the system defined by \(T\) and \(V\); the name is due to equation (4).

8 Remark. We may also write
\[ \gamma = \sum_{i=1}^f \langle e_i, \gamma \rangle_T e_i \]
If we define \(\xi_i(t) := \langle e_i, \gamma(t) \rangle_T\), we then have the equation of motion (2) equivalent to the following \(f\) equations (for each \(i \in \{1, \ldots, f\}\)):
\[ \ddot{\xi}_i(t) = \partial^2_T \langle e_i, \gamma(t) \rangle_T \\
= \langle e_i, \dot{\gamma}(t) \rangle_T \] (By the equation of motion)
\[ = \langle e_i, -\tilde{V} \gamma(t) \rangle_T \] (By the fact \(\tilde{V}\) is symmetric)
\[ = -\langle \tilde{V} e_i, \gamma(t) \rangle_T \] (By the fact \(e_i\) is an eigenbasis for \(\tilde{V}\))
\[ = -\langle \lambda_i e_i, \gamma(t) \rangle_T \]
\[ = -\lambda_i \xi_i(t) \]
We find
\[ \ddot{\xi}_i = -\omega_i^2 \xi_i \] (4)
The general solution for \(\gamma\) is then easily obtain from (4) as these are simply \(f\) uncoupled oscillators. We find:
\[ \gamma(t) = \sum_{i=1}^f \xi_i(t) e_i \] (Plug in the general solution for an oscillator)
\[ = \sum_{i=1}^f \left[ \xi_i(0) \cos(\omega_i t) + \frac{1}{\omega_i} \dot{\xi}_i(0) \sin(\omega_i t) \right] e_i \]
\[ = \sum_{i=1}^f \left[ \langle e_i, \gamma(0) \rangle_T \cos(\omega_i t) + \frac{1}{\omega_i} \langle e_i, \dot{\gamma}(0) \rangle_T \sin(\omega_i t) \right] e_i \]
9 Definition. A symmetry is a linear map $S: \mathbb{R}^f \to \mathbb{R}^f$ which leaves $T$ and $V$ invariant:

$$T \circ S = T$$ \hfill (5)

$$V \circ S = S$$ \hfill (6)

10 Remark. Equation (5) implies that $S$ is an orthogonal map:

$$T \circ S = T$$

$$\implies (T \circ S)(x) = T(x) \quad \forall x \in \mathbb{R}^f$$

$$\implies \langle Sx, Sx \rangle_T = \langle x, x \rangle_T \quad \forall x \in \mathbb{R}^f$$

11 Remark. Equation (6) implies that $[S, \tilde{V}] = 0$. Indeed, we have

$$\langle Sx, \tilde{V}Sx \rangle_T = \langle x, \tilde{V}x \rangle_T \quad \forall x \in \mathbb{R}^f$$

$$\implies \langle x, S^T \tilde{V}Sx \rangle_T = \langle x, \tilde{V}x \rangle_T \quad \forall x \in \mathbb{R}^f$$

$$\implies \langle x, S^{-1} \tilde{V}Sx \rangle_T = \langle x, \tilde{V}x \rangle_T \quad \forall x \in \mathbb{R}^f$$

$$\implies S^{-1} \tilde{V}S = \tilde{V}$$

Thus, the eigenspaces of $S$ are $\tilde{V}$ invariant: If $x$ is an eigenvector of $S$ with eigenvalue $\lambda$ then $\tilde{V}x$ is also an eigenvector of $S$ with eigenvector $\lambda$. In symbols: $Sx = \lambda x$ then $S\tilde{V}x = \tilde{V}Sx = \tilde{V}\lambda x = \lambda \tilde{V}x$.

12 Algorithm. In order to solve the eigenvalue problem (3) we can first decompose

$$\mathbb{R}^f = \bigoplus_{\lambda \in \sigma(S)} \text{im}(P_\lambda)$$

where $P_\lambda$ is the eigenprojection onto the eigenspace of $S$ corresponding to eigenvalue $\lambda$. Since the eigenspaces of $S$ (that is, $\{ \text{im}(P_\lambda) \}_\lambda$) are $\tilde{V}$ invariant, $\tilde{V}$ will also have a block-diagonal form:

$$\tilde{V} = \bigoplus_{\lambda \in \sigma(S)} \tilde{V}\big|_{\text{im}(P_\lambda)}$$

We then diagonalize each block $\tilde{V}\big|_{\text{im}(P_\lambda)}$ separately, which should be much easier than diagonalizing $\tilde{V}$. 

5
2 Appendix: Forms

13 Definition. A symmetric bilinear form on an \( \mathbb{R} \)-vector space \( V \) is a map 
\[ B : V^2 \to \mathbb{R} \] 
which such that:
\[ B = B \circ s \]
\[ B \circ (\mathbb{R} \times V) = \mathbb{R} \times (1 \times B) \]
\[ B \circ (a \times 1) = a \times (B \times B) \circ h \]
where the maps \( s \) and \( h \) are defined as
\[ V^2 \ni (v_1, v_2) \mapsto (v_2, v_1) \in V^2 \]
\[ V^3 \ni (v_1, v_2, v_3) \mapsto (v_1, v_3, v_2, v_3) \in V^4 \]
\( m_{\mathbb{R} \times V}, m_{\mathbb{R} \times R} \) are the scalar multiplication on \( V \) and \( \mathbb{R} \) respectively, and \( a_{\mathbb{R} \times V}, a_{\mathbb{R} \times R} \) are vector addition on \( V \) and \( \mathbb{R} \) respectively.
In other words, \( B \) is symmetric and \( \mathbb{R} \)-linear in both its entries.

14 Definition. A quadratic form on an \( \mathbb{R} \)-vector-space \( V \) is a map \( f : V \to \mathbb{R} \) such that there exists some bilinear (not necessarily symmetric) form \( B_f : V^2 \to \mathbb{R} \) with 
\[ f = B_f \circ \Delta \]
where \( \Delta : V \to V^2 \) is the co-multiplication, given by \( v \mapsto (v, v) \) for all \( v \in V \).

15 Claim. A quadratic form \( f : V \to \mathbb{R} \) is positive definite iff \( f(V \setminus \{0\}) \subseteq \mathbb{R}_{>0} \).

16 Claim. Any quadratic form on an \( \mathbb{R} \)-vector-space \( V \) defines a unique symmetric bilinear form.

Proof. Let \( f : V \to \mathbb{R} \) be any quadratic form. Define \( C_f : V^2 \to \mathbb{R} \) via
\[ C_f(v_1, v_2) := \frac{1}{2} (B_f(v_1, v_2) + B_f(v_2, v_1)) \]
where \( B_f \) is the bilinear form guaranteed by the definition of \( f \) as a quadratic form. By construction, \( C_f \) is symmetric, and note that is is also \( \mathbb{R} \)-linear in both its entries, and hence a symmetric bilinear form.
We find that
\[ f(v) = B_f(v, v) = \frac{1}{2} (B_f(v, v) + B_f(v, v)) \equiv C_f(v, v) \]
Uniqueness follows by the polarization identity: Let \( \tilde{C}_f : V^2 \to \mathbb{R} \) by any
other symmetric bilinear form such that \( f = \tilde{C}_f \circ \Delta \). Then

\[
\tilde{C}_f (v_1, v_2) = \frac{1}{4} \left[ \tilde{C}_f (v_1, v_1) + \tilde{C}_f (v_2, v_1) + \tilde{C}_f (v_1, v_2) + \tilde{C}_f (v_2, v_2) \right] \\
- \frac{1}{4} \left[ \tilde{C}_f (v_1, v_1) - \tilde{C}_f (v_2, v_1) - \tilde{C}_f (v_1, v_2) + \tilde{C}_f (v_2, v_2) \right] \\
= \frac{1}{4} \left[ \tilde{C}_f (v_1 + v_2, v_1 + v_2) - \tilde{C}_f (v_1 - v_2, v_1 - v_2) \right] \\
= \frac{1}{4} \left[ f (v_1 + v_2) - f (v_1 - v_2) \right] \\
(\text{By the same calculation in reverse using } C_f) \\
= C_f (v_1, v_2)
\]

3 Appendix: The Cholesky Decomposition

17 Claim. If \( A \in \text{Mat}_{N \times N} (\mathbb{C}) \) is Hermitian and positive definite then there exists some \( L \in \text{Mat}_{N \times N} (\mathbb{C}) \) such that \( A = L^* L \).


18 Remark. This is the analog of the theorem in C-star algebras that says that an element \( a \) is positive (that is, \( \sigma (a) \subseteq [0, \infty) \) and self-adjoint) iff it can be written as \( a = b^* b \) for some other element \( b \).