Analytical Mechanics Recitation Session of Week 9

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1 What are Eigenfrequencies, or Natural Frequencies?

1 Remark. What follows is a restatement of section 4.2.1 in the lecture's script, which I thought might be good to briefly recollect as the idea is what is behind the solution to the first exercise in homework number eight.

2 Remark. Actually most of this discussion is just a very long way of saying that any two quadratic forms may be simultaneously diagonalized if one of the is positive definite. See `http://math.stackexchange.com/questions/154540/simultaneously-diagonalizing-bilinear-forms`.

Let $f \in \mathbb{N}_{\geq 1}$.

We consider a system whose state may be described by f real parameters–by some point in $\mathbb{R}^f$.

Let T and V be quadratic forms on $\mathbb{R}^f$ (see definition 14). We assume that T is a positive definite form (see definition 15). This makes sense because the kinetic energy is always non-negative, and always strictly positive if the speed is non-zero. Via 16 this induces a positive definite inner product $\langle \cdot, \cdot \rangle_T : \left(\mathbb{R}^f\right)^2 \to \mathbb{R}$. The reason we work with the inner product induced by T is in order to *not* have to pick a basis for $\mathbb{R}^f$. So the point of what follows is a basis-free description of the problem.

3 Claim. There is a unique symmetric linear mapping $\tilde{V} : \mathbb{R}^f \to \mathbb{R}^f$ such that

$$V(x) = \left\langle x, \tilde{V}x \right\rangle_T \quad \forall x \in \mathbb{R}^f$$

Proof. T and V define matrices $\mathscr{T}$ and $\mathscr{V}$ in $Mat_{f\times f}(\mathbb{R})$:

$$T(x) = \langle x, \mathscr{T}x \rangle \, \forall x \in \mathbb{R}^f$$

$$V(x) = \langle x, \mathscr{V}x \rangle \, \forall x \in \mathbb{R}^f$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^f$. Indeed, these matrices are defined via: Let $\{ e_i \}_i$ be the standard basis for $\mathbb{R}^f$.

$$\mathscr{T} := \sum_{i,\,j=1}^{f} \frac{1}{2} \left(T\left(e_i + e_j\right) - T\left(e_i\right) - T\left(e_j\right) \right) e_i \otimes e_j^*$$

and similarly for $\mathscr{V}$.

Note that $\mathscr{T}$ and $\mathscr{V}$ do not have to be symmetric. However, we may define $\mathscr{T}_S := \frac{1}{2}\left(\mathscr{T} + \mathscr{T}^T\right)$, so that $\mathscr{T}_S^T = \mathscr{T}_S$ and

$$\begin{aligned} \langle x,\, \mathscr{T}_S x\rangle &= \left\langle x,\, \frac{1}{2}\left(\mathscr{T} + \mathscr{T}^T\right) x \right\rangle \\ &= \frac{1}{2}\left(\langle x,\, \mathscr{T} x\rangle + \langle x,\, \mathscr{T}^T x\rangle\right) \\ &\quad \left(\text{By } \langle x,\, Ay\rangle = \left\langle A^T x,\, y\right\rangle\right) \\ &= \frac{1}{2}\left(\langle x,\, \mathscr{T} x\rangle + \langle \mathscr{T} x,\, x\rangle\right) \\ &\quad \left(\text{By } \langle x,\, y\rangle = \langle y,\, x\rangle \text{ in } \mathbb{R}^f\right) \\ &= \langle x,\, \mathscr{T} x\rangle \\ &\equiv T(x) \end{aligned}$$

We follow a similar procedure for V to obtain that $\mathscr{V}_S$ is symmetric and

$$\langle x,\, \mathscr{V}_S x\rangle = V(x)$$

Note that because $\mathscr{T}_S$ is positive definite and symmetric, 17 implies that there is some matrix $\mathscr{L} \in Mat_{f\times f}(\mathbb{R})$ such that $\mathscr{T}_S = \mathscr{L}^T\mathscr{L}$. Hence we find:

$$T(x) = \langle \mathscr{L}x,\, \mathscr{L}x\rangle \quad \forall x \in \mathbb{R}^f$$

Moreover, because T is positive definite, $\mathscr{T}_S$ is invertible, so that $\mathscr{L}$ is invertible as well. As a result, $\left\{ \mathscr{L}^{-1}e_i \right\}_i$ is also a basis of $\mathbb{R}^f$ (albeit not necessarily an orthogonal one–since $\mathscr{T}_S$ is not necessarily diagonal). In this basis, the matrix $\mathscr{T}_S$ is given by the components $(i,\, j)$:

$$\begin{aligned} \left\langle \mathscr{L}^{-1}e_i,\, \mathscr{T}_S \mathscr{L}^{-1}e_j \right\rangle &= \left\langle e_i,\, \left(\mathscr{L}^{-1}\right)^T \mathscr{L}^T\mathscr{L}\mathscr{L}^{-1}e_j \right\rangle \\ &= \langle e_i,\, e_j\rangle \\ &= \delta_{ij} \end{aligned}$$

so that in this basis, the matrix $\mathscr{T}_S = \mathbb{1}_{f\times f}$. As a result, taking the usual inner product $\langle\cdot,\,\cdot\rangle$ in the $\left\{ \mathscr{L}^{-1}e_i \right\}_i$ basis is like taking the $\langle\cdot,\,\cdot\rangle_T$ inner product in the standard basis.

We now define (manifestly symmetric)

$$\tilde{\mathscr{V}} := \left(\mathscr{L}^{-1}\right)^T \mathscr{V}_S \mathscr{L}^{-1}$$

which is simply the matrix $\mathscr{V}_S$ in the basis $\left\{ \mathscr{L}^{-1}e_i \right\}_i$. Hence

$$\begin{aligned} V(x) &= \langle x,\, \mathscr{V}_S x\rangle \\ &\quad \left(\text{in the basis } \left\{ \mathscr{L}^{-1}e_i \right\}_i\right) \\ &= \sum_{i,\,j=1}^{f} x_i \left(\left(\mathscr{L}^{-1}\right)^T \mathscr{V}_S \mathscr{L}^{-1}\right)_{ij} x_j \\ &\quad \left(\text{in the basis } \left\{ \mathscr{L}^{-1}e_i \right\}_i \text{ the standard inner product is } \langle\cdot,\,\cdot\rangle_T\right) \\ &= \left\langle x,\, \tilde{\mathscr{V}} x\right\rangle_T \end{aligned}$$

□

We define the system's energy at time $t \in \mathbb{R}$, corresponding to the trajectory $\gamma : \mathbb{R} \to \mathbb{R}^f$ via

$$E_\gamma(t) \quad := \quad T(\dot{\gamma}(t)) + V(\gamma(t))$$

The label t on the left hand-side is actually redundant, because we actually employ the assumption that

4 Assumption. *E_γ does not depend on time.*

So we shall drop that label.

5 Claim. The assumption 4 implies that any trajectory must obey the differential equation

$$\ddot{\gamma} \quad = \quad -\tilde{V}\gamma \tag{1}$$

Proof. Using our notation we may write E_γ as:

$$\begin{aligned} E_\gamma &= T(\dot{\gamma}(t)) + V(\gamma(t)) \\ &= \langle \dot{\gamma}(t)\,,\, \dot{\gamma}(t)\rangle_T + \left\langle \gamma(t)\,,\, \tilde{V}\gamma(t)\right\rangle_T \end{aligned}$$

The fact E_γ is time-independent may be expressed as $\dot{E}_\gamma = 0$. Using the fact that $\dot{\overline{\langle x,\, y\rangle}} = \langle \dot{x},\, y\rangle + \langle x,\, \dot{y}\rangle$ we get

$$\begin{aligned} 0 &= \dot{E}_\gamma \\ &= \langle \ddot{\gamma}(t)\,,\, \dot{\gamma}(t)\rangle_T + \langle \dot{\gamma}(t)\,,\, \ddot{\gamma}(t)\rangle_T + \left\langle \dot{\gamma}(t)\,,\, \tilde{V}\gamma(t)\right\rangle_T + \left\langle \gamma(t)\,,\, \tilde{V}\dot{\gamma}(t)\right\rangle_T \\ &\quad \text{(By symmetry of the forms invovled)} \\ &= 2\langle \dot{\gamma}(t)\,,\, \ddot{\gamma}(t)\rangle_T + 2\left\langle \dot{\gamma}(t)\,,\, \tilde{V}\gamma(t)\right\rangle_T \\ &= 2\left\langle \dot{\gamma}(t)\,,\, \ddot{\gamma}(t) + \tilde{V}\gamma(t)\right\rangle_T \end{aligned}$$

Which readily implies via the positive-definiteness of T that either we have a constant solution (which we are not interested in) or

$$\ddot{\gamma}(t) + \tilde{V}\gamma(t) \quad = \quad 0 \tag{2}$$

$\square$

6 Remark. Recall that real symmetric matrices are orthogonally diagonalizable. Thus we may find some orthonormal basis $\{ e_i \}_{i=1}^{f}$ of $\mathbb{R}^f$ such that

$$\tilde{\mathscr{V}} e_i = \lambda_i e_i \tag{3}$$

for some set of eigenvalues $\{ \lambda_i \}_{i=1}^{f}$. Because $\tilde{\mathscr{V}}$ is real symmetric, $\lambda_i \in \mathbb{R}$.

7 Definition. For each eigenvalue $\lambda_i \in \mathbb{R}$ of $\tilde{\mathscr{V}}$, define $\omega_i := \sqrt{\lambda_i}$. Thus, ω_i may be either real or strictly imaginary. The collection of all ω_i's are called *the natural frequencies of the system defined by T and V*; the name is due to equation (4).

8 Remark. We may also write

$$\gamma = \sum_{i=1}^{f} \langle e_i, \gamma \rangle_T e_i$$

If we define $\xi_i(t) := \langle e_i, \gamma(t) \rangle_T$ we then have the equation of motion (2) equivalent to the following f equations (for each $i \in \{ 1, \ldots, f \}$):

$$\begin{aligned}
\ddot{\xi}_i(t) &= \partial_t^2 \langle e_i, \ddot{\gamma}(t) \rangle_T \\
&= \langle e_i, \ddot{\gamma}(t) \rangle_T \\
&\quad \text{(By the equation of motion)} \\
&= \left\langle e_i, -\tilde{V}\gamma(t) \right\rangle_T \\
&\quad \left(\text{By the fact } \tilde{V} \text{ is symmetric}\right) \\
&= -\left\langle \tilde{V} e_i, \gamma(t) \right\rangle_T \\
&\quad \left(\text{By the fact } e_i \text{ is an eigenbasis for } \tilde{V}\right) \\
&= -\langle \lambda_i e_i, \gamma(t) \rangle_T \\
&= -\lambda_i \xi_i(t)
\end{aligned}$$

We find

$$\ddot{\xi}_i = -{\omega_i}^2 \xi_i \tag{4}$$

The general solution for γ is then easily obtain from (4) as these are simply f uncoupled oscillators. We find:

$$\begin{aligned}
\gamma(t) &= \sum_{i=1}^{f} \xi_i(t) e_i \\
&\quad \text{(Plug in the general solution for an oscillator)} \\
&= \sum_{i=1}^{f} \left[\xi_i(0) \cos(\omega_i t) + \frac{1}{\omega_i} \dot{\xi}_i(0) \sin(\omega_i t) \right] e_i \\
&\equiv \sum_{i=1}^{f} \left[\langle e_i, \gamma(0) \rangle_T \cos(\omega_i t) + \frac{1}{\omega_i} \langle e_i, \dot{\gamma}(0) \rangle_T \sin(\omega_i t) \right] e_i
\end{aligned}$$

9 Definition. A symmetry is a linear map $S : \mathbb{R}^f \to \mathbb{R}^f$ which leaves T and V invariant:

$$T \circ S = T \tag{5}$$

$$V \circ S = S \tag{6}$$

10 Remark. Equation (5) implies that S is an orthogonal map:

$$\begin{aligned} T \circ S &= T \\ &\updownarrow \\ (T \circ S)(x) &= T(x) \quad \forall x \in \mathbb{R}^f \\ &\updownarrow \\ \langle Sx,\, Sx \rangle_T &= \langle x,\, x \rangle_T \quad \forall x \in \mathbb{R}^f \end{aligned}$$

11 Remark. Equation (6) implies that $\left[S,\, \tilde{V}\right] = 0$. Indeed, we have

$$\begin{aligned} \left\langle Sx,\, \tilde{V}Sx \right\rangle_T &= \left\langle x,\, \tilde{V}x \right\rangle_T \quad \forall x \in \mathbb{R}^f \\ &\updownarrow \\ \left\langle x,\, S^T\tilde{V}Sx \right\rangle_T &= \left\langle x,\, \tilde{V}x \right\rangle_T \quad \forall x \in \mathbb{R}^f \\ &\updownarrow \ (S \text{ is orthogonal}) \\ \left\langle x,\, S^{-1}\tilde{V}Sx \right\rangle_T &= \left\langle x,\, \tilde{V}x \right\rangle_T \quad \forall x \in \mathbb{R}^f \\ &\updownarrow \\ S^{-1}\tilde{V}S &= \tilde{V} \end{aligned}$$

Thus, the eigenspaces of S are $\tilde{V}$ invariant: If x is an eigenvector of S with eigenvalue λ then $\tilde{V}x$ is also an eigenvector of S with eigenvector λ. In symbols: $Sx = \lambda x$ then $S\tilde{V}x = \tilde{V}Sx = \tilde{V}\lambda x = \lambda\tilde{V}x$.

12 Algorithm. *In order to solve the eigenvalue problem (3) we can first decompose*

$$\mathbb{R}^f = \bigoplus_{\lambda \in \sigma(S)} im\left(P_\lambda\right)$$

where P_λ is the eigenprojection onto the eigenspace of S corresponding to eigenvalue λ. Since the eigenspaces of S (that is, $\{\, im\left(P_\lambda\right) \}_\lambda$) are $\tilde{V}$ invariant, $\tilde{V}$ will also have a block-diagonal form:

$$\tilde{V} = \bigoplus_{\lambda \in \sigma(S)} \tilde{V}\Big|_{im(P_\lambda)}$$

We then diagonalize each block $\tilde{V}\Big|_{im(P_\lambda)}$ separately, which should be much easier than diagonalizing $\tilde{V}$.

2 Appendix: Forms

13 Definition. A symmetric bilinear form on an $\mathbb{R}$-vector space V is a map $B : V^2 \to \mathbb{R}$ which such that:

$$B = B \circ \mathfrak{s}$$

$$B \circ (\mathfrak{m}_{\mathbb{R}\times V} \times \mathbb{1}_V) = \mathfrak{m}_{\mathbb{R}\times\mathbb{R}} \circ (\mathbb{1}_{\mathbb{R}} \times B)$$

$$B \circ (\mathfrak{a}_{V^2} \times \mathbb{1}_V) = \mathfrak{a}_{\mathbb{R}^2} \circ (B \times B) \circ \mathfrak{h}$$

where the maps $\mathfrak{s}$ and $\mathfrak{h}$ are defined as

$$V^2 \ni (v_1,\, v_2) \overset{\mathfrak{s}}{\mapsto} (v_2,\, v_1) \in V^2$$

$$V^3 \ni (v_1,\, v_2,\, v_3) \overset{\mathfrak{h}}{\mapsto} (v_1,\, v_3,\, v_2,\, v_3) \in V^4$$

$\mathfrak{m}_{\mathbb{R}\times V}$, $\mathfrak{m}_{\mathbb{R}\times\mathbb{R}}$ are the scalar multiplication on V and $\mathbb{R}$ respectively, and $\mathfrak{a}_{V^2}$, $\mathfrak{a}_{\mathbb{R}^2}$ are vector addition on V and $\mathbb{R}$ respectively.

In other words, B is symmetric and $\mathbb{R}$-linear in both its entries.

14 Definition. A *quadratic form* on an $\mathbb{R}$-vector-space V is a map $f : V \to \mathbb{R}$ such that there exists some bilinear (not necessarily symmetric) form $B_f : V^2 \to \mathbb{R}$ with $f = B_f \circ \Delta$ where $\Delta : V \to V^2$ is the co-multiplication, given by $v \mapsto (v, v)$ for all $v \in V$.

15 Claim. A quadratic form $f : V \to \mathbb{R}$ is positive definite iff $f\,(V \backslash \{\, 0 \,\}) \subseteq \mathbb{R}_{>0}$.

16 Claim. Any quadratic form on an $\mathbb{R}$-vector-space V defines a *unique symmetric* bilinear form.

Proof. Let $f : V \to \mathbb{R}$ be any quadratic form. Define $C_f : V^2 \to \mathbb{R}$ via

$$C_f\,(v_1,\, v_2) := \frac{1}{2}\,(B_f\,(v_1,\, v_2) + B_f\,(v_2,\, v_1))$$

where B_f is the bilinear form guaranteed by the definition of f as a quadratic form. By construction, C_f is symmetric, and note that is is also $\mathbb{R}$-linear in both its entries, and hence a symmetric bilinear form.

We find that

$$\begin{aligned} f\,(v) &= B_f\,(v,\, v) \\ &= \frac{1}{2}\,(B_f\,(v,\, v) + B_f\,(v,\, v)) \\ &\equiv C_f\,(v,\, v) \end{aligned}$$

Uniqueness follows by the polarization identity: Let $\tilde{C}_f : V^2 \to \mathbb{R}$ by any other symmetric bilinear form such that $f=\tilde{C}_f\circ\Delta$. Then

$$\begin{aligned}\tilde{C}_f\left(v_1,\,v_2\right) &= \frac{1}{4}\left[\tilde{C}_f\left(v_1,\,v_1\right)+\tilde{C}_f\left(v_2,\,v_1\right)+\tilde{C}_f\left(v_1,\,v_2\right)+\tilde{C}_f\left(v_2,\,v_2\right)\right]\\ &\quad -\frac{1}{4}\left[\tilde{C}_f\left(v_1,\,v_1\right)-\tilde{C}_f\left(v_2,\,v_1\right)-\tilde{C}_f\left(v_1,\,v_2\right)+\tilde{C}_f\left(v_2,\,v_2\right)\right]\\ &= \frac{1}{4}\left[\tilde{C}_f\left(v_1+v_2,\,v_1+v_2\right)-\tilde{C}_f\left(v_1-v_2,\,v_1-v_2\right)\right]\\ &= \frac{1}{4}\left[f\left(v_1+v_2\right)-f\left(v_1-v_2\right)\right]\\ &\quad \text{(By the same calculation in reverse using } C_f\text{)}\\ &= C_f\left(v_1,\,v_2\right)\end{aligned}$$

□

3 Appendix: The Cholesky Decomposition

17 Claim. Iff $A\in Mat_{N\times N}\left(\mathbb{C}\right)$ is Hermitian and positive definite then there exists some $L\in Mat_{N\times N}\left(\mathbb{C}\right)$ such that $A=L^{*}L$.

Proof. See https://en.wikipedia.org/wiki/Cholesky_decomposition. □

18 Remark. This is the analog of the theorem in C-star algebras that says that an element a is positive (that is, $\sigma\left(a\right)\subseteq\left[0,\,\infty\right)$ and self-adjoint) iff it can be written as $a=b^{*}b$ for some other element b.