Analysis 1
Colloquium of Week 10
Compactness

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1 Preface

In this class we have been flirting with the notion of topology, even though for analysis one really can do enough only with metric spaces. As was mentioned already, the notion of compactness is historically a concept that mathematicians struggled with (they were not sure exactly how to define it, so that all properties they intuitively expected from "compact" sets would be fulfilled, and so that all theorems would have so-to-speak minimal proofs).

In the very end we arrived at the definition of compactness as we know it today, and the rivaling notions got slightly more cumbersome names, such as "paracompact", "precompact", "sequentially-compact" and so on. For metric spaces (which are one example of the more general type of "topological spaces" mathematicians were dealing with, notable among them is Hausdorff) most of these notions converge to one (which agrees with compactness), and so, until you take a class in topology, you don't actually have to worry about the difference between all the various notions of compactness.

Despite this, it's still quite useful to be familiar with the *actual* definition of compactness in its most general sense and then see how the other notions of compactness will be identical to it for our setting of metric spaces. This is the topic of today's colloquium.

2 Topology on a Set

- Let a topological space $(X, Open(X))$ be given (think of a metric space, and from it, the induced topology; however for this definition only the induced topology will play a role).
- That means that X is a set (just any plain old set: a collection of elements which we call points) and $Open(X) \subseteq 2^X$ (a set of subsets of X) which obeys the following conditions:
  1. $X \in Open(X)$
  2. $\varnothing \in Open(X)$
  3. $\left(\bigcup_{\alpha\in A} S_\alpha\right) \in Open(X)$ if $S_\alpha \in Open(X)$ for all $\alpha \in A$.
  4. $\left(\bigcap_{j=1}^{n} S_j\right) \in Open(X)$ if $S_j \in Open(X)$ for all $j \in \{1, \ldots, n\}$ where $n \in \mathbb{N}$.
- Examples:
  - Let $X = \{\text{Charlie, Miles, John}\}$.
    - Possible topologies on X would be:
      1. $Open(X) = \{\varnothing, X\}$ (this topology exists no matter which set you are looking at, in a way, it's the crudest topology possible).
      2. $Open(X) = 2^X \equiv \{\varnothing, \{Charlie\}, \{Miles\}, \{John\}, \{Charlie, Miles\}, \{Charlie, John\}, \{Miles, John\}, X$ (this topology also exists no matter what set you look at. It's the finest topology you could think of, where every single point is discerned).
      3. Let trivial example: $Open(X) = \{\varnothing, \{Charlie, Miles\}, \{Miles, John\}, \{Miles\}, X\}$. You can get more by permuting the elements.
    - Subsets of 2^X which cannot be topologies:
      1. $\mathcal{S} = \{\varnothing, \{Charlie\}, \{Miles\}, X\}$.
      2. $\mathcal{S} = \{\varnothing, \{Charlie, Miles\}, \{Miles, John\}, X\}$
  - Let X be a metric space with the metric d.
    - A topology on X (the topology induced from the metric d) is defined as

$$\boxed{Open(X) = \left\{ U \in 2^X \;\middle|\; \forall x \in U \exists \varepsilon > 0 : B_\varepsilon(x) \subseteq U \right\}}$$

Prove to yourself that this is indeed a topology.

* There are other topologies possible on X which can coexists with the one induced from the metric, for example, 2^X (called the discrete topology). In this sense the notation $Open(X)$ is misleading, because it implies that given X, there is a function (denoted by $Open$ from $X \to 2^X$) which gives you the topology of X. However, this is clearly false, as given X, *many different topologies are all possible, and so the map notation is improper* as it implies a multi-valued map. We use it anyway because it is a nice shortcut (many times we need to talk about topologies of different sets simultaneously, then we would write $Open(X)$, $Open(Y)$ etc.

3 Compactness

3.1 Cover of a Set

- Given a set X, a cover of X is a set $\mathcal{A} \subseteq 2^X$ such that

$$\left(\bigcup_{A \in \mathcal{A}} A \right) = X$$

3.2 An Open Cover

- Given a set X with *a* topology $Open(X)$ defined on it, an open cover is a cover $\mathcal{A}$ (as defined just above) such that

$$A \in Open(X) \, \forall A \in \mathcal{A}$$

3.3 Compactness

- Compactness is a property of subsets of X: A subset $F \in 2^X$ can be either compact or not compact.
- The subset $F \in 2^X$ is compact (with respect to $Open(X)$) iff:
  - For every open covering of X, $\mathcal{A}$, there exists a *finite* subset of $\mathcal{A}$, $\{ A_1, \ldots, A_n \} \subseteq \mathcal{A}$ such that

$$\left(\bigcup_{j=1}^{n} A_j \right) = X$$

    (that is, every open cover contains a finite subcover)
- Every finite set is automatically compact.
- Examples with $\mathbb{R}$ and its topology induced by the Euclidean metric:
  - $\mathbb{R}$ is not compact:
    * To show this, we merely have to find *one* example of an open cover which does not have an finite sub-cover.
    * To this end, let

$$\begin{aligned} \mathcal{A} &:= \left\{ (n,\, n+2) \in 2^{\mathbb{R}} \,\middle|\, n \in \mathbb{Z} \right\} \\ &= \{ (0,\,2),\, (1,\,3),\, (2,\,4),\, \ldots,\, (-1,\,1),\, (-2,\,0),\, \ldots \} \end{aligned}$$

    * Clearly, $\mathcal{A}$ is a cover of $\mathbb{R}$ due to the overlap of the intervals, and also, each interval is open, so it is an open subcover and thus would have to be checked if we want to see if $\mathbb{R}$ is compact.
    * However, $\mathcal{A}$ does not admit a finite subcover:
      · Assume otherwise.
      · Then there is a largest element n in the collections of intervals of the form $(n,\, n+2)$ where $n \in \mathbb{Z}$.
      · Then the point $n + 500$ is not covered (as well as many other points).
    * This isn't to say that there aren't *any* open covers or $\mathbb{R}$ which admit a finite subcover. The most silly example if $\{ \mathbb{R} \}$, which is clearly a cover of $\mathbb{R}$, it is also open (as $\mathbb{R} \in Open(\mathbb{R})$) and it is also finite! But that's not enough for compactness. For compactness, *every single open cover* must admit a finite subcover.
  - $(0,\,1)$ is not compact:
    * Again, we really only need one counter-example.
    * Take the cover

$$\begin{aligned} \mathcal{A} &:= \left\{ \left(\frac{1}{n},\, 1 \right) \,\middle|\, n \in \mathbb{N} \backslash \{0,\,1\} \right\} \\ &= \left\{ \left(\frac{1}{2},\, 1 \right),\, \left(\frac{1}{3},\, 1 \right),\, \left(\frac{1}{4},\, 1 \right),\, \ldots \right\} \end{aligned}$$

  * It might seem counter intuitive that this is indeed a cover because we are not covering points near 0.
  * This is in fact false: we are covering all points near zero except zero:
    - Let $x \in (0,\, 1)$ be given (arbitrarily close to 0 as you wish).
    - Then we can always find $n \in \mathbb{N}$ big enough such that $\frac{1}{n} < x$.
    - Then $\left(\frac{1}{n},\, 1\right)$ is an element of $\mathcal{A}$ and of course, $x \in \left(\frac{1}{n},\, 1\right)$ as $\frac{1}{n} < x$.
  * Note that the point 0 is not covered and it doesn't *need* to be covered: $0 \notin (0,\, 1)$.
  * It is also an open cover as all these intervals are open.
  * It is also clear that no finite subcover will work. If you pick one, let n be the minimum element of all intervals of the form $\left(\frac{1}{n},\, 1\right)$. Then just pick some $x < \frac{1}{n}$ which is always possible as there is always a number in $\left(0,\, \frac{1}{n}\right)$.
  * Something in this example is fishy: it is exactly the end points that make this interval set fail to be compact.
- $[0,\, 1]$ is compact:
  * Going from the previous example, we already know that covers like $\left\{ \left(\frac{1}{n},\, 1\right) \,\middle|\, n \in \mathbb{N} \backslash \{0,\, 1\} \right\}$ will not work for $[0,\, 1]$ because 0 now *does need* to be covered.
  * Let $\mathcal{A}$ be some open covering of $[0,\, 1]$.
  * Define

$$S \quad := \quad \{\, x \in [0,\, 1] \mid [0,\, x] \text{ can be covered by finitely many elements of } \mathcal{A} \,\}$$

  * Observe, $S \subseteq \mathbb{R}$ and $\mathbb{R}$ has the supremum proprety (every non-empty bounded from above set has a supremum).
  * Thus, define $\alpha := \sup(S)$.
  * If $\alpha = 1$ we are done (right?).
  * α cannot be bigger than 1 because by the approximation property of the supremum, $\forall \varepsilon > 0 \exists y \in S$ such that $\alpha - \varepsilon < y \leq \alpha$. Since $\alpha > 1$, and $y \in S$ (hence $y \leq 1$) we actually have $\alpha - \varepsilon < y \leq \alpha$. So just pick any $\varepsilon \in (0,\, \alpha - 1)$ (so that $\alpha - \varepsilon > 1$) to reach a contradiction: $\alpha - \varepsilon < y < \alpha$ and so $y > 1$.
  * α can also not be smaller than 1:
    - Assume otherwise.
    - Then $\exists A_* \in \mathcal{A}$ such $\alpha \in A_*$ and $A_* \in \mathit{Open}(\mathbb{R})$.
    - Because $A_* \in \mathit{Open}(\mathbb{R})$, $\exists$ some $\varepsilon > 0$ such that $B_\varepsilon(\alpha) \subseteq A_*$, where $B_\varepsilon(\alpha) \equiv (\alpha - \varepsilon,\, \alpha + \varepsilon)$.
    - Because $\alpha = \sup(S)$, any element in $[0,\, 1]$ smaller than α in $[0,\, 1]$ is in S.
    - In particular, $\left(\alpha - \frac{1}{2}\varepsilon\right) \in S$, and so $\left[0,\, \alpha - \frac{1}{2}\varepsilon\right]$ is covered by finitely many elements of $\mathcal{A}$, call it $\{\, A_1,\, A_2,\, \ldots,\, A_n \,\}$ for some $n \in \mathbb{N}$.
    - But then, $\{\, A_1,\, A_2,\, \ldots,\, A_n \,\} \cup \{\, A_* \,\}$ covers $\left[0,\, \alpha + \frac{1}{2}\varepsilon\right]$.
    - But this is a contradiction because $\{\, A_1,\, A_2,\, \ldots,\, A_n \,\} \cup \{\, A_* \,\}$ is a finite cover of $\left[0,\, \alpha + \frac{1}{2}\varepsilon\right]$ and so $\left(\alpha + \frac{1}{2}\varepsilon\right) \in S$. But $\alpha + \frac{1}{2}\varepsilon > \alpha \equiv \sup(S)$!

3.4 Heine-Borel Theorem

- In class you have shown the Heine-Borel theorem, which gives a very convenient way to prove that a set is compact if we are working in $\mathbb{R}^n$:
  - $F \in 2^{\mathbb{R}^n}$ is compact if and only if $F \in \mathit{Closed}(\mathbb{R}^n)$ AND F is bounded.
- This if and only if relation breaks down for general metric spaces. It holds for $\mathbb{R}^n$ (a special type of metric space) and there are other examples of metric spaces for which the Heine-Borel theorem works.
- A metric space is said to have *the Heine–Borel property* if every closed and bounded subset is compact. Many metric spaces fail to have the Heine–Borel property. For instance, the metric space of rational numbers (or indeed any incomplete metric space) fails to have the Heine–Borel property. Complete metric spaces may also fail to have the property!
- Using the theorem we immediately know that $(0,\, 1)$ is not bounded, and the long-winded proof above is not neessary.
- Using the theorem we also immediately know that the Cantor set is compact, as it is the (infinite) union of closed intervals (and so it is closed) and of course it is bounded (by 0 and 1). It would have been horrible to try to prove that from the definition.

3.5 Sequential Compactness

- A subset $F \in 2^X$ is called sequentially compact iff every infinite sequence taking values in F has a convergent subsequence (convering to a point in F).
- *Claim*: For a metric space, sequential compactness if equivalent to compactness.
  Proof:
  - Theorem 28.2 in Munkres *Topology* second edition.

- *Claim*: There are topological spaces for which these two notions *do not agree*.
  As we are not dealing with general topological spaces yet, examples of this are not needed now. Just know that this is not the same thing, and it *happens* to be the same thing for metric spaces.

- How to see that $\mathbb{R}$ is not sequentially compact:
  - Take the infinite sequence $(n)_{n\in\mathbb{N}}$ which has no convergent subsequence.

- How to see that $[0, 1]$ is sequentially compact:
  - Let $(a_n)_{n\in\mathbb{N}}$ be some infinite sequence taking values in $[0, 1]$.
  - As such, $(a_n)_{n\in\mathbb{N}}$ is a bounded infinite sequence.
  - So we may use the Bolzano-Weierstrass theorem which says exactly that every bounded infinite sequence has a convergent subsequence.
  - Proof of the Bolzano-Weierstrass theorem may be found as Theorem 2.42 in Rudin's *PMA* page 40.

- How to see that $(0, 1)$ is *not* sequentially compact:
  - Define $\left(1 - \frac{1}{n}\right)_{n\in\mathbb{N}\setminus\{0\}}$.
  - This is an infinite sequence, which clearly converges to 1 (a point outside of $(0, 1)$).
  - Because it converges, every subsequence also converges to 1.
  - Thus, there is no subsequence that converges to a point *inside* of $(0, 1)$.