Analysis 1
Colloquium of Week 11
Everywhere Continuous Nowhere Differentiable Map

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Abstract

We present Theorem 7.18 from Rudin's Principles of Mathematical Analysis pp. 154.

1 Debt from Last Rectiation Session: Differential Equations

1.1 Ordinary Equations

Ordinary equations specify conditions which are either true or false; as such they are logical statements. For example, the equation $2 = 3$ is false while $2014 = 2014$ is true. Certain equations are *parametrized* by a variable, which ranges over some set. For example,

$$x^2 = 1 \text{ where } x \text{ ranges over } \mathbb{R}$$

is an equation (a condition) parameterized by x. Then we may ask for which values in $\mathbb{R}$ the condition is true. In a way, what the equation (read: "the condition") is telling you is a recipe: Take an element of $\mathbb{R}$, multiply it by itself, and check if the result is equal to 1. If yes, the condition is met and you found a *solution* to your equation.

1.2 Differential Equations

Sometimes, the parameter of the equation ranges over a set not of numbers, but over a set of maps. For example: consider the following equation:

$$[f(x) = 5 \forall x \in \mathbb{R}] \text{ where } f \text{ ranges over } \mathbb{R}^{\mathbb{R}}$$

is an equation parametrized by f. As above, we may ask, for which values in $\mathbb{R}^{\mathbb{R}}$ (now values are whole maps from $\mathbb{R}$ to $\mathbb{R}$) the condition is true. What the equation is saying is: take some element of $\mathbb{R}^{\mathbb{R}}$, evaluate it at some point x (any point), and the result should equal 5. The condition should hold for *every* x. The answer is clear: the solution to the equation is $(x \mapsto 5) \in \mathbb{R}^{\mathbb{R}}$. In a way, this is how we have been defining functions all along: by giving conditions.

But sometimes the conditions may involve more complicated operations–operations we have only studied about in the past few weeks. This is still fine. One such operation is differentiation. Consider the following equation:

$$[f'(x) = 5 \forall x \in \mathbb{R}] \text{ where } f \text{ ranges over } \mathbb{R}^{\mathbb{R}}$$

It is telling you to take an element of $\mathbb{R}^{\mathbb{R}}$, differentiate it, evaluate the result at some x (chosen arbitrarily), and the result should equal 5. I claim that a solution to this equation is given by $f : \mathbb{R} \to \mathbb{R}$ defined by $x \mapsto 5x + C$ where C is *any* real number. To verify this, plug it into the equation:

$$(5x + C)' = 5$$

is indeed true. So we have found a solution. Later on you will study some theorems that prove existence and uniqueness of solutions to differential equations (equations where the unknown is a function, and there is differentiation of the functions).

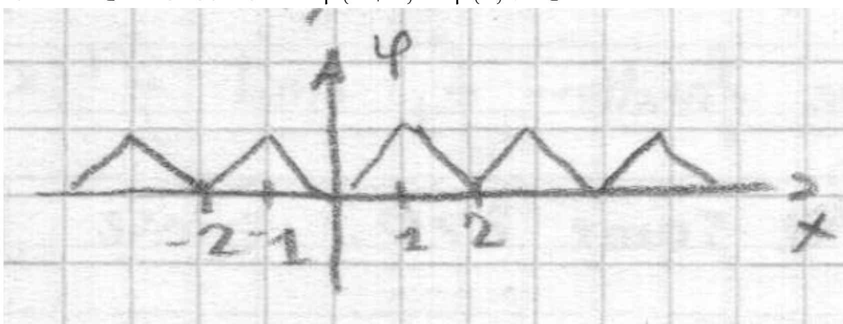
2 A Continuous Function that is Nowhere Differentiable

- *Claim*: $\exists f \in \mathbb{R}^{\mathbb{R}}$ such that f is continuous yet nowhere differentiable.
  Proof:
  - Every number $r \in \mathbb{R}$ may be written *uniquely* as $2n_r + \alpha_r$ for some $n \in \mathbb{Z}$ and some $\alpha \in [-1, 1]$.

– Define

$$\varphi(x) := |\alpha_x|$$

for all $x \in \mathbb{R}$. Observe that $\varphi(x+2) = \varphi(x)\ \forall x \in \mathbb{R}$.



– *Claim*: φ is continuous on $\mathbb{R}$.
Proof:

  * *Claim*: $\forall (s,t) \in \mathbb{R}^2$, $|\varphi(s) - \varphi(t)| \leqslant |s-t|$.
  Proof:
    · Let $(s,\ t) \in \mathbb{R}^2$ be given. Then we know that we may write uniquely $s = 2n_s + \alpha_s$ and $t = 2n_t + \alpha_t$ for some $(n_s,\ n_t) \in \mathbb{Z}^2$ and $(\alpha_s,\ \alpha_t) \in [-1,\ 1]^2$.
    · Then

$$\begin{aligned} |\varphi(s) - \varphi(t)| &= ||\alpha_s| - |\alpha_t|| \\ &\leqslant |\alpha_s - \alpha_t| \end{aligned}$$

    · If $n_s = n_t$ then we are finished: $|\alpha_s - \alpha_t| = |\alpha_s + 2n_s - 2n_t - \alpha_t| = |s-t|$.
    · If $n_s \neq n_t$ then

$$\begin{aligned} |\alpha_s - \alpha_t| &= |\alpha_s + 2n_s - 2n_s + 2n_t - 2n_t - \alpha_t| \\ &= |s - 2n_s + 2n_t - t| \\ &\leqslant |s-t| + |2n_t - 2n_s| \\ &< |s-t| \end{aligned}$$

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– Define a new function, $f : \mathbb{R} \to \mathbb{R}$ by

$$x \overset{f}{\mapsto} \sum_{n \in \mathbb{N} \cup \{0\}} \left(\frac{3}{4}\right)^n \varphi(4^n x)$$

– Using the Weierstrass M-test (Theorem 7.10 in Rudin) we can conclude that $\sum_n^N \left(\frac{3}{4}\right)^n \varphi(4^n x) \overset{N\to\infty}{\longrightarrow} f$ uniformly :

  * Define $M_n := \left(\frac{3}{4}\right)^n$.
  * Then $\left|\left(\frac{3}{4}\right)^n \varphi(4^n x)\right| = \left(\frac{3}{4}\right)^n \varphi(4^n x) \leqslant M_n$ because $\varphi(y) \in [0,\ 1]$ for all $y \in \mathbb{R}$.
  * But $\sum M_n$ converges.

– But then it follows that f is continuous, as $\mathbb{R} \to \mathbb{R}$ given by $x \mapsto \sum_n^N \left(\frac{3}{4}\right)^n \varphi(4^n x)$ is continuous for all $N \in \mathbb{N} \cup \{0\}$ (Theorem 7.12 in Rudin).

– *Claim*: f is not differentiable at any given point on $\mathbb{R}$.
Proof:

  * Let $x \in \mathbb{R}$ be some given point.
  * Define a new numerical sequence $(\delta_m)_{m \in \mathbb{N}}$ by the following rule:

$$\delta_m := \begin{cases} \frac{1}{2} 4^{-m} & \nexists l \in \mathbb{Z} \cap \left(4^m x,\ 4^m x + \frac{1}{2}\right) \\ -\frac{1}{2} 4^{-m} & \nexists l \in \mathbb{Z} \cap \left(4^m x - \frac{1}{2},\ 4^m x\right) \end{cases}$$

  * *Claim*: $(\delta_m)_{m \in \mathbb{N}} \longrightarrow 0$ as $m \to \infty$.
  * Define

$$\gamma_{n,\ m} := \frac{\varphi(4^n (x + \delta_m)) - \varphi(4^n x)}{\delta_m}$$

  for all $(n,\ m) \in \mathbb{N}^2$ and $x \in \mathbb{R}$.

· *Claim*: $|\gamma_{m,m}| = 4^m$.
Proof:

$$\begin{aligned} |\gamma_{m,m}| &= \left| \frac{\varphi(4^m(x+\delta_m)) - \varphi(4^m x)}{\delta_m} \right| \\ &= \frac{|\varphi(4^m(x+\delta_m)) - \varphi(4^m x)|}{|\delta_m|} \end{aligned}$$

now, because as we defined φ, there is no integer between $4^m(x+\delta_m)$ and $4^m x$, $|\varphi(4^m(x+\delta_m)) - \varphi(4^m x)| = \frac{1}{2}$ (we are evaluating the function along one unbroken straight line, so we'll merely get the distance between the two points). As a result,

$$\begin{aligned} |\gamma_{m,m}| &= \frac{\frac{1}{2}}{\left|\pm\frac{1}{2}4^{-m}\right|} \\ &= 4^m \end{aligned}$$

· *Claim*: $\gamma_{n,m} = 0$ for all $n > m$.
Proof:
Observe that if $n > m$, $4^n\delta_m \in 2\mathbb{Z}$ because $4^n\delta_m = \pm 4^n \frac{1}{2} 4^{-m} = \pm 4^{n-m}\frac{1}{2} \in 2\mathbb{Z}$. Thus, when $n > m$ we have

$$\begin{aligned} \gamma_{n,m} &= \frac{\varphi(4^n x + 4^n\delta_m) - \varphi(4^n x)}{\delta_m} \\ &= \frac{\varphi(4^n x) - \varphi(4^n x)}{\delta_m} \\ &= 0 \end{aligned}$$

· *Claim*: When $n \in \{0, \ldots, m\}$, $|\gamma_{n,m}| \leqslant 4^n$.
Proof:
Using the fact that $\forall (s,t) \in \mathbb{R}^2$, $|\varphi(s) - \varphi(t)| \leqslant |s-t|$, we ascertain that

$$\begin{aligned} |\gamma_{n,m}| &\equiv \left| \frac{\varphi(4^n(x+\delta_m)) - \varphi(4^n x)}{\delta_m} \right| \\ &\leqslant \left| \frac{4^n(x+\delta_m) - 4^n x}{\delta_m} \right| \\ &= 4^n \end{aligned}$$

∗ *Claim*: $\left|\frac{f(x+\delta_m) - f(x)}{\delta_m}\right| \geqslant \frac{1}{2}(3^m + 1)$ for all $m \in \mathbb{N}$.
Proof:
· Compute

$$\begin{aligned} \left| \frac{f(x+\delta_m) - f(x)}{\delta_m} \right| &\equiv \left| \frac{\sum_{n\in\mathbb{N}\cup\{0\}} \left(\frac{3}{4}\right)^n \varphi(4^n(x+\delta_m)) - \sum_{n\in\mathbb{N}\cup\{0\}} \left(\frac{3}{4}\right)^n \varphi(4^n x)}{\delta_m} \right| \\ &\overset{\text{uni. conv.}}{=} \left| \frac{\sum_{n\in\mathbb{N}\cup\{0\}} \left(\frac{3}{4}\right)^n [\varphi(4^n(x+\delta_m)) - \varphi(4^n x)]}{\delta_m} \right| \\ &= \left| \sum_{n\in\mathbb{N}\cup\{0\}} \left(\frac{3}{4}\right)^n \gamma_{n,m} \right| \\ &= \left| \sum_{n=0}^{m} \left(\frac{3}{4}\right)^n \gamma_{n,m} \right| \\ &= \left| 3^m - \sum_{n=0}^{m-1} \left(\frac{3}{4}\right)^n \gamma_{n,m} \right| \\ &\geqslant 3^m - \sum_{n=0}^{m-1} \left(\frac{3}{4}\right)^n |\gamma_{n,m}| \\ &\geqslant 3^m - \sum_{n=0}^{m-1} \left(\frac{3}{4}\right)^n 4^n \\ &= 3^m - \sum_{n=0}^{m-1} 3^n \\ &= \frac{1}{2}(3^m + 1) \end{aligned}$$

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* In particular, if the limit existed, then $\lim_{t\to x}\frac{f(t)-f(x)}{t-x} = \lim_{m\to\infty}\frac{f(x+\delta_m)-f(x)}{\delta_m}$ as $\delta_m \to \infty$ and if the limit exists then it doesn't matter how we approach it. But we just showed that $\lim_{m\to\infty}\frac{f(x+\delta_m)-f(x)}{\delta_m} = \infty$. Hence f cannot be differentiable at x.

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