Analysis 1
Colloquium of Week 12
Taylor Series and Power Series Expansions

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Abstract

We present a few results taken from Rudin's *Principles of Mathematical Analysis*. This may be a repetition of material from the lecture, intended to solidify your knowledge.

1 Taylor's Theorem

- Taylor's theorem allows us to take any sufficiently-well-behaved function and approximate it as a polynomial around some point. This is the bread and butter of physics, and also some forms of math: $\sin(x) \approx x$, $\cos(x) \approx 1 + \frac{1}{2}x^2$ and so on.
- Let $n \in \mathbb{N}\backslash\{0\}$, let $(a, b) \in \mathbb{R}^2$ such that $a < b$ and let $f \in C^{n-1}([a, b], \mathbb{R})$ such that $f^{(n)} \in C^0((a, b))$.
- Recall that means that $f^{(n-1)}$ is continuous (to say this we must implicitly say f is $n-1$-times differentiable) and that $f^{(n)}$ exists and is continuous on the *open* interval (a, b).
- Let $x_0 \in [a, b]$ and take some $\alpha \in \mathbb{R}\backslash\{0\}$ such that $(x_0 + \alpha) \in [a, b]$.
- The point is that we want to make an approximation for the value of $f(x_0 + \alpha)$ (given that we know the value of $f^{(j)}(x_0)$ for any $j \in \mathbb{N} \cup \{0\}$) as some kind of polynomial in powers of α. If α is very small, then the higher powers we take (that is, the larger n) the better the approximation is (assuming the remainder is small).
- Define $P(t) = \sum_{j=0}^{n-1} \frac{f^{(j)}(x_0)}{j!}(t - x_0)^j$ for any $t \in \mathbb{R}$.
- *Claim*: $\exists x \in \mathbb{R}$ such that $x \in [x_0, x_0 + \alpha]$ or $x \in [x_0 + \alpha, x_0]$ (depending on the sign of α) such that

$$\begin{aligned} f(x_0 + \alpha) &= \underbrace{P(x_0 + \alpha)}_{\text{polynomial in } \alpha} + \frac{f^{(n)}(x)}{n!}\alpha^n \\ &= \sum_{j=0}^{n-1} \frac{f^{(j)}(x_0)}{j!}\alpha^j + \frac{f^{(n)}(x)}{n!}\alpha^n \end{aligned}$$

  Note: For $n = 1$ this is just the mean value theorem. In general, if we know the bounds on $\left|f^{(n)}(x)\right|$ we can estimate the deviation of $f(x_0 + \alpha)$ from a polynomial in α.
  Proof:

  - Define $M := \frac{f(x_0+\alpha) - P(x_0+\alpha)}{\alpha^n}$.
  - For any $t \in [a, b]$, define $g(t) := f(t) - P(t) - M(t - x_0)^n$.
  - Our goal is to show that $\exists x$ between x_0 and $x_0 + \alpha$ such that $M = \frac{f^{(n)}(x)}{n!}$.
  - Compute $g^{(n)}(t)$:

$$\begin{aligned} g^{(n)}(t) &= \left(f(t) - P(t) - M(t - x_0)^n\right)^{(n)} \\ &= f^{(n)}(t) - \underbrace{P^{(n)}(t)}_{0 \text{ as } P(t) \propto t^{n-1}} - n!M \\ &\overset{?}{=} 0 \end{aligned}$$

    This is true $\forall t\,(a, b)$ (and not for all $t \in [a, b]$ as we don't assume $f^{(n)}$ exists on the end points).

- *Claim*: $g^{(n)}(x) = 0$ for some x between x_0 and $x_0 + \alpha$ (and thus the proof would be complete).
  Poorf:
  - Note that $\forall k \in \{0, \ldots, n-1\}$ we have

$$\begin{aligned} P^{(k)}(x_0) &= \left(\sum_{j=0}^{n-1} \frac{f^{(j)}(x_0)}{k!} (t - x_0)^j \right)^{(k)} \Bigg|_{t=x_0} \\ &= \left(\sum_{j=0}^{n-1} \frac{f^{(j)}(x_0)}{k!} j(j-1)\ldots(j-(k-1))(t-x_0)^{j-k} \right) \Bigg|_{t=x_0} \\ &= \left(\sum_{j=k}^{n-1} \frac{f^{(j)}(x_0)}{k!} j(j-1)\ldots(j-(k-1))(t-x_0)^{j-k} \right) \Bigg|_{t=x_0} \\ &= f^{(k)}(x_0) \end{aligned}$$

  - Then

$$\begin{aligned} g^{(k)}(x_0) &= f^{(k)}(x_0) - P^{(k)}(x_0) \\ &= 0 \end{aligned}$$

    for all $k \in \{0, \ldots, n-1\}$.
  - Note that

$$\begin{aligned} g(x_0 + \alpha) &= f(x_0 + \alpha) - P(x_0 + \alpha) - \underbrace{M}_{\frac{f(x_0+\alpha) - P(x_0+\alpha)}{\alpha^n}} \alpha^n \\ &= 0 \end{aligned}$$

  - So we have that $g(x_0) = g(x_0 + \alpha) = 0$.
  - Recall the mean value theorem: if g is a real continuous function on $[s, t]$ which is differentiable in (s, t) then $\exists x \in (s, t)$ such that $g(t) - g(s) = (t - s) g'(x)$.
  - Thus we apply the mean value theorem to get that $\exists$ some x_1 between x_0 and $x_0 + \alpha$ such that $g'(x_1) = 0$.
  - But $g'(x_0) = 0$ as well, so we may repeat the process to find some x_2 between x_0 and x_1 such that $g''(x_2) = 0$.
  - After n steps we arrive at the conclusion that $g^{(n)}(x_n) = 0$ for some x_n between x_0 and x_{n-1}. But x_{n-1} was between x_0 and x_{n-2} ... which was between x_0 and $x_0 + \alpha$.

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- Next, let $f : (-R, R) \to \mathbb{R}$ be some map, where $R \in (0, \infty]$, such that $f(x) = \sum_{n=0}^{\infty} c_n x^n$ is a power series expansion of f which converges for $|x| < R$. Recall that functions of this form (which can be written as a power series expansion) are called analytic functions.

- Using the Weierstrass M-test we can prove that the power series expansion in fact converges *uniformly* in $[R - \varepsilon, R + \varepsilon]$ for any $\varepsilon > 0$ and so f is continuous and differentiable in that restricted closed interval.

- Then we can show that (Theorem 8.1 in Rudin):
  - $c_0 = f(0)$, $c_1 = f'(0)$, $c_2 = \frac{1}{2} f''(0)$, and in general $c_n = \frac{1}{n!} f^{(n)}(0)$.

- Thus an analytic function has a power series expansion as

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) x^n$$

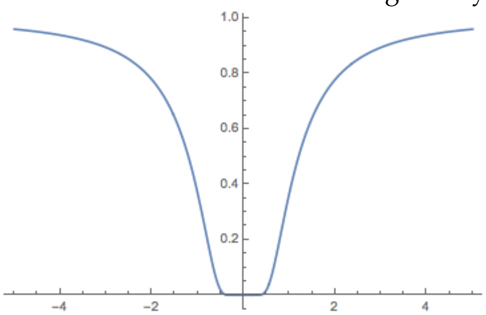
  in its radius of convergence.

- Furthermore, if f is analytic and has a power series expansion given above, and if $a \in (-R, R)$, then f can *also* be expanded in a power series about the point a, and this new power series converges in in $|x - a| < R - |a|$, and

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a) (x - a)^n$$

2 Non-Analytic Smooth Function

- Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $x \mapsto e^{-\frac{1}{x^2}}$



- *Claim*: $f^{(n)}(0) = 0$ for all $n \in \mathbb{N} \cup \{0\}$.
  Proof:
  - *Claim*: For $x \neq 0$ we have $f^{(n)}(x) = \frac{P_{2n-2}(x) e^{-\frac{1}{x^2}}}{x^{3n}}$ where $P_k(x)$ is *some* polynomial in x with degree k.
    Proof:
    - We proceed by induction:
    - For $n = 1$ we have:
      - $f'(x) = \frac{e^{-\frac{1}{x^2}}}{x^3} 2$ so the polynomial is $P_0(X) = 2$.
    - Assume true for some n. So that $f^{(n)}(x) = \frac{P_{2n-2}(x) e^{-\frac{1}{x^2}}}{x^{3n}}$.Check $n+1$:

$$\begin{aligned} f^{(n+1)}(x) &= \left(f^{(n)}(x)\right)' \\ &= \left(\frac{P_{2n-2}(x)\, e^{-\frac{1}{x^2}}}{x^{3n}}\right)' \\ &= \frac{e^{-\frac{1}{x^2}}}{x^{3(n+1)}} \left[\left(2 - 3nx^2\right) P_{2n-2}(x) + \underbrace{x^3 \underbrace{\left(P_{2n-2}(x)\right)'}_{\text{poly. of deg.} 2n-3}}_{\text{poly. of deg. } 2n = 2(n+1)-2} \right] \end{aligned}$$

  - *Claim*: $\lim_{x \to 0^+} \frac{e^{-\frac{1}{x^2}}}{x^m} = 0$ for all $m \in \mathbb{N} \cup \{0\}$.
    Proof:
    - Make a change of variable so that

$$\begin{aligned} \lim_{x \to 0} \frac{e^{-\frac{1}{x^2}}}{x^m} &= \lim_{x \to +\infty} \frac{e^{-x^2}}{x^{-m}} \\ &= \lim_{x \to \infty} \frac{x^m}{e^{x^2}} \end{aligned}$$

    - Now observe that $e^{x^2} \geqslant e^x$ for all $x > 0$ due to the monotone increasing nature of exp so that $0 \leqslant \frac{x^m}{e^{x^2}} \leqslant \frac{x^m}{e^x}$.
    - However, $\lim_{x \to \infty} \frac{x^m}{e^x} = 0$ by m applications of l'Hopital's rule.
    - Thus due to $a_n \leqslant b_n$ implying $\limsup a_n \leqslant \limsup b_n$ we have our result.
    - Also direct proof for the case $m = 0$: we have to compute the limit: $\lim_{x \to 0} e^{-\frac{1}{x^2}} \stackrel{?}{=} 0$.
      - So for any $\varepsilon > 0$, we need to find a $\delta > 0$ such that if $|x| < \delta$ then $e^{-\frac{1}{x^2}} < \varepsilon$.
      - So pick $\delta(\varepsilon) := \sqrt{\left|\frac{1}{\log(\varepsilon)}\right|}$ (assuming $\varepsilon \neq 1$, otherwise the task is easy).
      - Thus if $x < \sqrt{\left|\frac{1}{\log(\varepsilon)}\right|}$ then $x^2 < \left|\frac{1}{\log(\varepsilon)}\right|$. If $\varepsilon < 1$ (which we can assume WLOG, because we are trying to see what happens for *small* ε) then $\log(\varepsilon) < 0$, and so $\left|\frac{1}{\log(\varepsilon)}\right| = -\frac{1}{\log(\varepsilon)}$.
      - Thus we have $x^2 < -\frac{1}{\log(\varepsilon)}$ which implies $-\frac{1}{x^2} < \log(\varepsilon)$.
      - If $a < b$ then $\exp(a) < \exp(b)$ because the exponent is a monotone increasing function.

- Thus we have that $x < \delta(\varepsilon)$ implies that $e^{-\frac{1}{x^2}} < \varepsilon$.

- Now comes the actual proof, which proceeds by induction:
- For the case $n = 1$ we have:

$$\begin{aligned} f'(0) &= \lim_{t\to 0} \frac{f(t) - f(0)}{t} \\ &= \lim_{t\to 0} \frac{e^{-\frac{1}{t^2}} - 0}{t} \\ &= 0 \end{aligned}$$

  using the above.
- Assume $f^{(j)}(0) = 0$ for all $j \leqslant n$ for some $n \in \mathbb{N}$. Check

$$\begin{aligned} f^{(n+1)}(0) &= \lim_{t\to 0} \frac{f^{(n)}(t) - f^{(n)}(0)}{t} \\ &= \lim_{t\to 0} \frac{\frac{P_{2n-2}(t)e^{-\frac{1}{t^2}}}{t^{3n}} - 0}{t} \\ &= \lim_{t\to 0} e^{-\frac{1}{t^2}} \frac{P_{2n-2}(t)}{t^{3n+1}} \\ &= \lim_{t\to\infty} P_{2n-2}\left(\frac{1}{t}\right) \frac{t^{3n+1}}{e^{-t^2}} \\ &= \underbrace{P_{2n-2}(0)}_{\text{const.}} \left(\underbrace{\lim_{t\to\infty} \frac{t^{3n+1}}{e^{-t^2}}}_{\longrightarrow 0} \right) \\ &= 0 \end{aligned}$$

- So the Taylor series of f is given by 0!
- But now it is clear that

$$f(x) \neq \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{f^{(n)}(0)}_{0} x^n$$

  unless $x = 0$!
- Such a function is called non-analytic, as it is not equal to its Taylor series expansion.