Analysis 1
Colloquium of Week 13
Preparation for Midterm

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1 Exercise Sheet Number 11

1.1 Question 1

- *Claim*: $\lim_{t\to 0}[1+2\sin(t)]^{\cot(t)} = e^2$.
  Proof:
  - Start out by using the fact that $x = e^{\log(x)}$.
  - As such,

$$\begin{aligned}[1+2\sin(t)]^{\cot(t)} &= e^{\log\left([1+2\sin(t)]^{\cot(t)}\right)}\\ &= e^{\cot(t)\log\{[1+2\sin(t)]\}}\\ &= e^{\frac{\log\{[1+2\sin(t)]\}}{\tan(t)}}\end{aligned}$$

  - Thus we have

$$\begin{aligned}\lim_{t\to 0}[1+2\sin(t)]^{\cot(t)} &= \lim_{t\to 0}\exp\left(\frac{\log\{[1+2\sin(t)]\}}{\tan(t)}\right)\\ &\overset{\exp\text{ is continuous}}{=} \exp\left(\lim_{t\to 0}\frac{\log\{[1+2\sin(t)]\}}{\tan(t)}\right)\end{aligned}$$

  - Now we compute $\lim_{t\to 0}\frac{\log\{[1+2\sin(t)]\}}{\tan(t)}$.
  - We have a limit of the form $\frac{0}{0}$, so try to use the Hospital's:

$$\frac{(\log\{[1+2\sin(t)]\})'}{(\tan(t))'} = \frac{\left(\frac{2\cos(t)}{1+2\sin(t)}\right)}{\left(\frac{1}{[\cos(t)]^2}\right)}$$

  and indeed $\lim_{t\to 0}\frac{\left(\frac{2\cos(t)}{1+2\sin(t)}\right)}{\left(\frac{1}{[\cos(t)]^2}\right)} = 2$ exists, so we conclude that $\lim_{t\to 0}\frac{\log\{[1+2\sin(t)]\}}{\tan(t)} = 2$.
  - As a result, $\lim_{t\to 0}[1+2\sin(t)]^{\cot(t)} = e^2$.

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- Note that $\left(a^t\right)' \neq ta^{t-1}$! Use

$$\begin{aligned}a^t &= e^{\log(a^t)}\\ &= e^{t\log(a)}\end{aligned}$$

  and so

$$\begin{aligned}\left(a^t\right)' &= \left(e^{t\log(a)}\right)'\\ &= e^{t\log(a)}\log(a)\\ &= a^t\log(a)\end{aligned}$$

1.2 Question 2

Let $f \in \mathbb{R}^{[a,b]}$ be differentiable and assume further that for some $x \in (a, b)$, f' is differentiable at x.

- *Claim*: $\lim_{h\to 0} \frac{f(x+h)-2f(x)+f(x-h)}{h^2} = f''(x)$.
  Proof:
  - We try to evaluate the limit $\lim_{h\to 0} \frac{f(x+h)-2f(x)+f(x-h)}{h^2}$.
    - Observe that because f is differentiable at x, f is continuous at x and so the limit is of the form $\frac{0}{0}$.
    - Thus we try to employ the Hospital's rule on it (note that the derivative is *with respect to* h because that is the variable of the limit.) and obtain that:

$$\frac{(f(x+h)-2f(x)+f(x-h))'}{(h^2)'} = \frac{f'(x+h)-f'(x-h)}{2h}$$

    - Thus if $\lim_{h\to 0} \frac{f'(x+h)-f'(x-h)}{2h}$ exists then $\lim_{h\to 0} \frac{f(x+h)-2f(x)+f(x-h)}{h^2} = \lim_{h\to 0} \frac{f'(x+h)-f'(x-h)}{2h}$:

$$\begin{aligned}
\lim_{h\to 0} \frac{f'(x+h)-f'(x-h)}{2h} &= \lim_{h\to 0} \frac{f'(x+h)-f'(x)+f'(x)-f'(x-h)}{2h} \\
&= \lim_{h\to 0} \left[\frac{f'(x+h)-f'(x)}{2h} + \frac{f'(x)-f'(x-h)}{2h}\right] \\
&= \frac{1}{2}\left\{\left[\lim_{h\to 0} \frac{f'(x+h)-f'(x)}{h}\right] + \left[\lim_{h\to 0} \frac{f'(x-h)-f'(x)}{-h}\right]\right\} \\
&= \frac{1}{2}\left\{\left[\lim_{h\to 0} \frac{f'(x+h)-f'(x)}{h}\right] + \left[\lim_{\tilde{h}\to 0} \frac{f'\left(x+\tilde{h}\right)-f'(x)}{\tilde{h}}\right]\right\} \\
&= \frac{1}{2}\left\{f''(x)+f''(x)\right\} \\
&= f''(x)
\end{aligned}$$

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2 Partial Review for Midterm

2.1 Cauchy Sequences

- *Definition*: A sequence $(p_n)_{n\in\mathbb{N}}$ in a metric space X is said to be a Cauchy sequence iff $\forall \varepsilon > 0 \exists m(\varepsilon) \in \mathbb{N}$ such that $d_X(p_n, p_l) < \varepsilon$ for all $(n, l) \in \mathbb{N}^2$ such that $\{[n \geqslant m(\varepsilon)] \wedge [l \geqslant m(\varepsilon)]\}$.
- Every convergent sequence is a Cauchy sequence.
- The converse, is in general *not* true:
  - The first counter example that should come to your mind is the metric space $\mathbb{Q}$ defined with the usual Euclidean metric $d_{\mathbb{Q}}(p, q) \equiv |p-q|$.
  - We can define a Cauchy sequence by the decimal expansion of any irrational number, for example, take the decimal expansion of $\sqrt{2}$ and the sequence defined by it:

$$\begin{aligned}
a_1 &:= 1 \\
a_2 &:= 1.4 \\
a_3 &:= 1.41 \\
a_4 &:= 1.414 \\
a_5 &:= 1.4142 \\
&\cdots
\end{aligned}$$

  - Each element of the sequence $a_j \in \mathbb{Q}$.
  - It is very easy to show that $(a_j)_{j\in\mathbb{N}}$ is a Cauchy sequence (it converges in $\mathbb{R}$, so it is Cauchy).
  - However, it does not converge in $\mathbb{Q}$.
  - The property of a metric space where a Cauchy sequence does not converge is called *not complete*.
  - Another handy example is the sequence converging to $e = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$.
- Take-home-message: if you are working in a complete metric space (such as $\mathbb{R}^n$ and in particular $\mathbb{R}$ or $\mathbb{C}$) then sometimes to prove convergence it is enough to prove that the sequence is Cauchy. Especially if you don't know and don't care what the sequence convergences to, but just need to see that it indeed converges.

2.2 Sequences of Functions

Recall from the colloquium of week 8:

- Let $f \in Y^{\mathbb{N} \times X}$ be a family of maps from $X \to Y$, indexed by $\mathbb{N}$. One particular map is denoted by $f_n : X \to Y$.
- Observe that for each $x \in X$, the set of numbers $\{ f_n(x) \mid n \in \mathbb{N} \}$ actually defines a sequence $(f_n(x))_{n\in\mathbb{N}}$ (a whole sequence for each $x \in X$).
- Assume that for each $x \in X$, this sequence $(f_n(x))_{n\in\mathbb{N}}$ indeed converges, to some number which we denote as α_x: $(f_n(x))_{n\in\mathbb{N}} \to \alpha_x$.
- Thus this induces a new function $\varphi : X \to Y$, $x \overset{\varphi}{\mapsto} \alpha_x$, which, at any point x, is defined as the limit of $(f_n(x))_{n\in\mathbb{N}}$. This function is well defined due to our hypothesis that $(f_n(x))_{n\in\mathbb{N}}$ indeed converges for all $x \in X$.
- Under this circumstances, we use a special terminology: we say that $(f_n)_{n\in\mathbb{N}}$ converges *pointwise* to φ. Observe how now, in our notation, the x does not appear. This is because now we are talking about a sequence of *functions* $(f_n)_{n\in\mathbb{N}}$ rather than a sequence of numbers $(f_n(x))_{n\in\mathbb{N}}$, for each $x \in X$.
- Recall that a sequence of functions can converge to a given function in many different "ways", and that *pointwise convergence* is in a way a rather weak type of convergence of functions.
- We say that $(f_n)_{n\in\mathbb{N}} \to \varphi$ uniformly iff
- $\forall x \in X, \forall \varepsilon > 0 \exists m(\varepsilon) \in \mathbb{N} : \forall n \in \mathbb{N} : n \geqslant m(\varepsilon) \Longrightarrow d_Y(f_n(x), \varphi(x))$ where $m(\varepsilon)$ does not depend on $x \in X$.

2.2.1 Cauchy Criterion for Uniform Convergence

- The sequence of functions $(f_n)_{n\in\mathbb{N}}$ converges uniformly on X iff $\forall \varepsilon > 0 \exists m(\varepsilon) \in \mathbb{N}$ such that if $(n, l) \in \mathbb{N}^2$ such that $[n \geqslant m(\varepsilon)] \wedge [l \geqslant m(\varepsilon)]$ and $x \in X$ implies $|f_n(x) - f_l(x)| \leqslant \varepsilon$.
- Example:
  - Define $f_n : [0, 1] \to \mathbb{R}$ by $x \mapsto \frac{x^n}{n}$.
  - Clearly, $f_n \to 0$ pointwise on $\mathbb{R}$.
  - We want to see how $(f_n)_{n\in\mathbb{N}}$ defines a uniformly Cauchy sequence of functions:

$$\begin{aligned} \left| \frac{x^n}{n} - \frac{x^l}{l} \right| &\leqslant \left| \frac{x^n}{n} \right| + \left| \frac{x^l}{l} \right| \\ &\leqslant \frac{1}{n} + \frac{1}{l} \end{aligned}$$

2.3 Integrals

2.3.1 Root Integrals

- With root integrals, most of the time you need to make a trigonometric substitution.
- Example:
  - $\int \frac{t^3}{\sqrt{t^2+1}} dt = ?$
  - Make the substitution $t = \sinh(u)$.
  - Then $\sqrt{t^2+1} = \sqrt{\sinh(u)^2 + 1} = \sqrt{\cosh(u)^2} = \cosh(u)$.
  - In addition, $dt = d[\sinh(u)] = \cosh(u)\, du$.
  - Thus we have

$$\begin{aligned} \int \frac{t^3}{\sqrt{t^2+1}} dt &= \int \frac{[\sinh(u)]^3}{\cosh(u)} \cosh(u)\, du \\ &= \int [\sinh(u)]^3\, du \end{aligned}$$

  - This we can integrate easily by using the fact that $\sinh(u) \equiv \frac{1}{2}(e^u - e^{-u})$ and so

$$\begin{aligned} [\sinh(u)]^3 &= \frac{1}{8}\left(e^u - e^{-u}\right)^3 \\ &= \frac{1}{8}\left(e^{3u} - e^{-3u} + 3e^{-u} - 3e^u\right) \end{aligned}$$

and using the fact that $\int e^{ax}dx = \frac{1}{a}e^{ax}$ so we get:

$$\begin{aligned}
\int [\sinh(u)]^3 \, du &= \int \frac{1}{8}\left(e^{3u} - e^{-3u} + 3e^{-u} - 3e^{u}\right) du \\
&= \frac{1}{8}\left(\frac{1}{3}e^{3u} + \frac{1}{3}e^{-3u} - 3e^{-u} - 3e^{u}\right) \\
&= \frac{1}{8}\left(\frac{2}{3}\cosh(3u) - 6\cosh(u)\right) \\
&= \frac{1}{12}\left(\underbrace{\cosh(3u)}_{4\cosh^3(u) - 3\cosh(u)} - 9\cosh(u)\right) \\
&= \frac{1}{12}\left(4\cosh^3(u) - 12\cosh(u)\right) \\
&= \frac{1}{3}\cosh^3(u) - \cosh(u) \\
&= \frac{1}{3}\left[\sqrt{t^2+1}\right]^3 - \sqrt{t^2+1}
\end{aligned}$$

2.3.2 Integrals of Rational Functions

We are interested in integrals of the form $R(x) := \frac{a_0 + a_1 x + \ldots + a_n x^n}{b_0 + b_1 x + \ldots + b_m x^m}$.

- *Definition*: $R(x)$ is called proper if $n < m$, and improper if $n \geqslant m$.

General Algorithm

1. If $R(x)$ is proper ($n < m$), proceed to step 2 and define $r_0 + r_1 x + \ldots + r_k x^k := a_0 + a_1 x + \ldots + a_n x^n$, otherwise:
   (a) We assume that $n \geqslant m$.
   (b) Use polynomial division (long division) to write

   $$\frac{a_0 + a_1 x + \ldots + a_n x^n}{b_0 + b_1 x + \ldots + b_m x^m} = \left(c_0 + c_1 x + \ldots + c_l x^l\right) + \frac{r_0 + r_1 x + \ldots + r_k x^k}{b_0 + b_1 x + \ldots + b_m x^m}$$

   where necessarily $k < m$. Then $\left(c_0 + c_1 x + \ldots + c_l x^l\right)$ can be integrated easily and we concentrate on $\frac{r_0 + r_1 x + \ldots + r_k x^k}{b_0 + b_1 x + \ldots + b_m x^m}$.
   (c) Example:
2. Thus we have $k < m$.
3. Use partial fraction decomposition on $\frac{r_0 + r_1 x + \ldots + r_k x^k}{b_0 + b_1 x + \ldots + b_m x^m}$ to simplify the denominator (the general algorithm is complicated, usually this will be simple).

- Example:
  - $\int \frac{x^3 - 4}{x^2 - x - 2} dx = ?$
  - $3 > 2$, so we need to do polynomial division:

  $$\frac{x^3 - 4}{x^2 - x - 2} = (x+1) + \frac{3x - 2}{x^2 - x - 2}$$

  - Now we can do partial fraction decomposition on $\frac{3x-2}{x^2-x-2}$ to get:

  $$\frac{x^3 - 4}{x^2 - x - 2} = (x+1) + \frac{\frac{4}{3}}{x-2} + \frac{\frac{5}{3}}{x+1}$$

  - Thus integrating this we get:

  $$\begin{aligned}
  \int \frac{x^3 - 4}{x^2 - x - 2} dx &= \int \left[(x+1) + \frac{\frac{4}{3}}{x-2} + \frac{\frac{5}{3}}{x+1}\right] dx \\
  &= \frac{1}{2}x^2 + x + \frac{4}{3}\log(|x-2|) + \frac{5}{3}\log(|x+1|) + C
  \end{aligned}$$

- Don't forget the result from homework 12 exercise 3:

  $$\int \frac{Ax + B}{x^2 + 2a + b} dx = \frac{A}{2}\log\left(\left|x^2 + 2a + b\right|\right) + \frac{B - aA}{b - a^2}\arctan\left(\frac{x + a}{\sqrt{b - a^2}}\right)$$

  I recommend memorizing this.