Analysis 1
Colloquium of Week 14
Post-Midterm Review

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Abstract

Even though many of the questions in the exam could have been solved easily using smart "tricks", in what follows I attempt to present the most naive, straight forward solution that a student could have been expected to come up with during the exam.

1 Taylor Expansions

- In question 5 of the open section, you were asked to compute the Taylor expansion of a function at 0 up to order 4.
- The general recipe to do this is as follows:
- Suppose $f : \mathbb{R} \to \mathbb{R}$ is given which is sufficiently many times differentiable.
- Then we have

$$f(x) \approx f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \frac{1}{6}f^{(3)}(0)x^3 + \frac{1}{24}f^{(4)}(0)x^4 + \mathcal{O}\left(x^5\right)$$

- Thus the proble is reduced to computing derivatives of a function and evaluating those derivatives at 0.
- Let's take the particular example that was given on the exam:
- $f(x) = \sqrt{1-2x^2}$:

$$\begin{cases} f(x) & = \sqrt{1-2x^2} \\ f^{(1)}(x) & = \frac{1}{2}\left(1-2x^2\right)^{-\frac{1}{2}}(-2)(2x) = \frac{-2x}{\sqrt{1-2x^2}} \\ f^{(2)}(x) & = -2\left(\frac{1}{\sqrt{1-2x^2}} + x\left(-\frac{1}{2}\right)\left(1-2x^2\right)^{-\frac{3}{2}}(-2)(2x)\right) = \frac{-2}{\sqrt{1-2x^2}} - \frac{4x^2}{(1-2x^2)^{\frac{3}{2}}} \\ f^{(3)}(x) & = -2\left(-\frac{1}{2}\right)\left(1-2x^2\right)^{-\frac{3}{2}}(-2)(2x) - 4\left(\frac{2x}{(1-2x^2)^{\frac{3}{2}}} + x^2\left(-\frac{3}{2}\right)\left(1-2x^2\right)^{-\frac{5}{2}}(-2)(2x)\right) = \frac{-12x}{(1-2x^2)^{\frac{3}{2}}} - \frac{24x^3}{(1-2x^2)^{\frac{5}{2}}} \\ f^{(4)}(x) & = -12\left(\frac{1}{(1-2x^2)^{\frac{3}{2}}} + x\left(-\frac{3}{2}\right)\left(1-2x^2\right)^{-\frac{5}{2}}(-2)(2x)\right) - 24\left(3x^2\frac{1}{(1-2x^2)^{\frac{5}{2}}} + x^3\left(-\frac{5}{3}\right)\left(1-2x^2\right)^{-\frac{7}{2}}(-2)(2x)\right) = \\ & = -\frac{12}{(1-2x^2)^{\frac{3}{2}}} - \frac{144x^2}{(1-2x^2)^{\frac{5}{2}}} - \frac{240x^4}{(1-2x^2)^{\frac{7}{2}}} \end{cases}$$

$$\begin{cases} f(0) & = 1 \\ f^{(1)}(0) & = 0 \\ f^{(2)}(0) & = -2 \\ f^{(3)}(0) & = 0 \\ f^{(4)}(0) & = -12 \end{cases}$$

- This was expected, because f is an even function, so its Taylor expansion should contain only even powers and if f were an odd function its Taylor expansion would have contained only odd powers.
- Thus we have

$$\begin{aligned} f(x) &\approx f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \frac{1}{6}f^{(3)}(0)x^3 + \frac{1}{24}f^{(4)}(0)x^4 + \mathcal{O}\left(x^5\right) \\ &= 1 + \frac{1}{2}(-2)x^2 + \frac{1}{24}(-12)x^4 + \mathcal{O}\left(x^5\right) \\ &= 1 - x^2 - \frac{1}{2}x^4 + \mathcal{O}\left(x^5\right) \end{aligned}$$

2 Integrals

2.1 Sinus Squared

- We would like to compute
$$I := \int_0^{\pi} [\sin(x)]^2 \, dx$$
- Write $[\sin(x)]^2 = \frac{1}{2} - \frac{1}{2}\cos(2x)$ using the formula $\cos(2x) = 1 - 2[\sin(x)]^2$.
- Thus we have
$$\begin{aligned} I &= \int_0^{\pi} \left[\frac{1}{2} - \frac{1}{2}\cos(2x)\right] dx \\ &= \frac{1}{2}\int_0^{\pi} [1-\cos(2x)] \, dx \\ &= \frac{1}{2}\left\{\int_0^{\pi} dx - \int_0^{\pi} \cos(2x)\, dx\right\} \\ &= \frac{1}{2}\left\{ x|_0^{\pi} - \underbrace{\int_0^{\pi} \cos(2x)\, dx}_{u=2x} \right\} \\ &= \frac{1}{2}\left\{ x|_0^{\pi} - \int_0^{2\pi} \cos(u)\,\frac{1}{2} du \right\} \\ &= \frac{1}{2}\left\{ x|_0^{\pi} - \frac{1}{2}\sin(u)|_0^{2\pi} \right\} \\ &= \left\{\frac{1}{2}x - \frac{1}{4}\sin(2x)\right\}\Bigg|_0^{2\pi} \\ &= \frac{\pi}{2} \end{aligned}$$

2.2 Logs

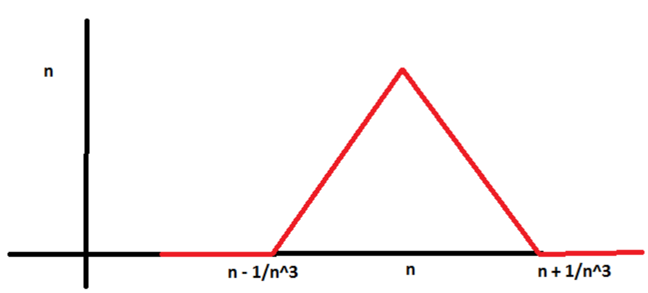
- We want to evaluate
$$I := \int_0^{\frac{1}{2}} \frac{1}{1-x^2} dx$$
- The first step is to perform partial fraction decomposition. We know that $\left(1-x^2\right) = (1-x)(1+x)$, so we expect $\frac{1}{1-x^2}$ to decompose as $\frac{A}{1-x} + \frac{B}{1+x}$ where A and B are unknown.
- To find A and B, we find the common denominator and get
$$\begin{aligned} \frac{1}{1-x^2} &\overset{!}{=} \frac{A}{1-x} + \frac{B}{1+x} \\ &= \frac{A(1+x) + B(1-x)}{1-x^2} \\ &= \frac{(A-B)x + A + B}{1-x^2} \end{aligned}$$
from which it must follow that $\begin{cases} A - B & \overset{!}{=} 0 \\ A + B & \overset{!}{=} 1 \end{cases}$ so that $A = \frac{1}{2}$ and $B = \frac{1}{2}$:
$$\frac{1}{1-x^2} = \frac{\frac{1}{2}}{1-x} + \frac{\frac{1}{2}}{1+x}$$

- Thus we have

$$
\begin{aligned}
I &= \int_0^{\frac{1}{2}} \left[\frac{\frac{1}{2}}{1-x} + \frac{\frac{1}{2}}{1+x}\right] dx \\
&= \frac{1}{2}\int_0^{\frac{1}{2}} \left[\frac{1}{1-x} + \frac{1}{1+x}\right] dx \\
&= \frac{1}{2}\left[\int_0^{\frac{1}{2}} \frac{1}{1-x} dx + \int_0^{\frac{1}{2}} \frac{1}{1+x} dx\right] \\
&= \frac{1}{2}\left[\int_0^{\frac{1}{2}} \frac{1}{1-x} dx + \int_0^{\frac{1}{2}} \frac{1}{1+x} dx\right] \\
&= \frac{1}{2}\left[\underbrace{\int_0^{\frac{1}{2}} \frac{1}{1-x} dx}_{u=1-x} + \underbrace{\int_0^{\frac{1}{2}} \frac{1}{1+x} dx}_{v=1+x}\right] \\
&= \frac{1}{2}\left[\int_1^{\frac{1}{2}} \frac{1}{u}(-du) + \int_1^{\frac{3}{2}} \frac{1}{v} dv\right] \\
&= \frac{1}{2}\left[-\log(|u|)|_1^{\frac{1}{2}} + \log(|v|)|_1^{\frac{3}{2}}\right] \\
&= \frac{1}{2}\left[-\log(|1-x|)|_0^{\frac{1}{2}} + \log(|1+x|)|_0^{\frac{1}{2}}\right] \\
&= \frac{1}{2}\left[-\log(|1-x|) + \log(|1+x|)\right]|_0^{\frac{1}{2}} \\
&= \frac{1}{2}\left[\log\left(\frac{|1+x|}{|1-x|}\right)\right]\Big|_0^{\frac{1}{2}} \\
&= \frac{1}{2}\left[\log\left(\frac{\frac{3}{2}}{\frac{1}{2}}\right) - \underbrace{\log(1)}_{0}\right] \\
&= \frac{1}{2}\log(3)
\end{aligned}
$$

2.3 Some Facts

- Define a function by this graph (at every $n \in \mathbb{N}$):



- This function is then continuous on $[0, \infty)$, and is unbounded!
- Yet the integral of this function must exist, because the area of the triangle is bounded by $n \times \frac{2}{n^3} = \frac{2}{n^2}$ and $\sum \frac{2}{n^2} < \infty$.
- $\int_0^\infty \frac{1}{x+1} dx$ is an integral of a continuous function, which converges to zero at infinity, yet the integral does not exist.
- There are differentiable functions whose derivative is not integrable! The best recipe to reach non-integrability is to look at unbounded functions, because we define Riemann integrability exactly on bounded functions. Note that this wouldn't work for improper integrals, but only for integrals on a closed interval. For example: $f(x) := \begin{cases} x^2 \sin\left(\frac{1}{x^2}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$ is differentiable on $[-1, 1]$, yet f' is not bounded on $[-1, 1]$: $f'(x) = 2x\sin\left(\frac{1}{x^2}\right) - \frac{2\cos\left(\frac{1}{x^2}\right)}{x}$.
- The last statement is exactly Theorem 6.20 in Rudin.

3 The Continuum Hypothesis

- The continuum hypothesis says that there is no cardinality strictly between $|\mathbb{N}|$ and $|\mathbb{R}|$.
- An uncountable subset must have cardinality bigger than $|\mathbb{N}|$.
- A subset of $\mathbb{R}$ must have cardinality smaller than or equal to $|\mathbb{R}|$.
- Thus necessarily the continuum hypothesis leads to the fact that every uncountable subset of $\mathbb{R}$ has cardinality $|\mathbb{R}|$.
- Thus the third choice must be correct.
- Even though it *is* true that $|\mathbb{R}| = |2^{\mathbb{N}}|$, this is not the hypothesis (it is a simple result of the binary expansion of real numbers!)
- The fourth option is exactly the converse of the statement.

4 Infinite Series

4.1 Question About Series in General.

(aside: series is an English word which is the same in both singular and plural form)

- The first option cannot be true because we know
$$\sum_{n\in\mathbb{N}\backslash\{0\}} \frac{(-1)^n}{n}$$
converges.
- The second option cannot hold because we know that
$$\sum_{n\in\mathbb{N}} \frac{1}{n}$$
diverges.
- False again by penultimate example.
- The last option must hold then, which indeed it does.

4.2 Question about $\sum_{n\in\mathbb{N}\backslash\{0\}} \frac{(-1)^n}{n}$

- Absolute convergence is when $\sum |a_n|$ converges, which, this one doesn't. But it does converge, and thus, not absolutely.
- It *is* true that if we re-arrange the order of the series, it could be made to converge to any other number (Theorem 3.54 in Rudin). However, written as $\sum_{n\in\mathbb{N}\backslash\{0\}} \frac{(-1)^n}{n}$ this series has only *one* limit.

5 Intermediate Value Theorem

Let $f : [a, b] \to \mathbb{R}$ be continuous. Then if u is a number between $f(a)$ and $f(b)$ then $\exists c \in (a, b) : f(c) = u$.

6 Continuity

- Sequential continuity must hold *for every* sequence. (Theorem 4.2 in Rudin).