Analysis 1
Colloquium of Week number 5
Moebius Transformations

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Abstract

We describe a few properties of the Moebius transformations.

1 Preface

Define a set of maps (subset of $\mathbb{C}\cup\{\infty\}^{\mathbb{C}\cup\{\infty\}}$) called "Moebius Transformations" by the following:

$$\mathcal{M} := \underbrace{\left\{ z\mapsto\begin{cases}az+b & z\in\mathbb{C}\\ \infty & z=\infty\end{cases}\in\mathbb{C}\cup\{\infty\}^{\mathbb{C}\cup\{\infty\}} \;\middle|\; (a,\,b)\in\mathbb{C}^2 :\, a\neq 0\right\}}_{\mathcal{M}_1}\cup$$
$$\cup\underbrace{\left\{ z\mapsto\begin{cases}\frac{az+b}{cz+d} & z\in\mathbb{C}\backslash\left\{-\frac{d}{c}\right\}\\ \infty & z=\frac{-d}{c}\\ \frac{a}{c} & z=\infty\end{cases}\in\mathbb{C}\cup\{\infty\}^{\mathbb{C}\cup\{\infty\}} \;\middle|\; (a,\,b,\,c,\,d)\in\mathbb{C}^4 :\, (ad-bc\neq 0)\wedge c\neq 0\right\}}_{\mathcal{M}_2}$$

1.1 Remarks

1.1.1 $\mathcal{M}$ Does not Include Constant Maps

- For maps in $\mathcal{M}_1$:
  - When $a=0$ we get a constant map, so we don't want to include that.
- For maps in $\mathcal{M}_2$:
  - If $c\neq 0$, and $a\neq 0$ then:

$$\begin{aligned}\frac{az+b}{cz+d} &= \frac{\frac{a}{c}z+\frac{b}{c}}{z+\frac{d}{c}}\\ &= \frac{a}{c}\cdot\frac{z+\frac{b}{a}}{z+\frac{d}{c}}\end{aligned}$$

- Thus the requirement that $ad - bc \neq 0$ is equivalent to not taking maps such that $z \mapsto \frac{a}{c}$ (a constant map).
- If $c \neq 0$ and $a = 0$, then

$$\frac{az + b}{cz + d} = \frac{\frac{b}{c}}{z + \frac{d}{c}}$$

  and then the requirement that $ad - bc \neq 0$ is equivalent to, again, not taking constant maps.

1.1.2 Moebius Transformations Are Parametrized by Six Real Parameters

- For maps in the second set, $c \neq 0$ and we can thus divide by it to get: $z \mapsto \frac{\frac{a}{c}z + \frac{b}{c}}{z + \frac{d}{c}}$.
- Since a, b, c and d are otherwise unconstrained, all we care about is in fact the ratios $\frac{a}{c}$, $\frac{b}{c}$, $\frac{d}{c}$.
- Each such ratio is a complex number (a pair of real numbers).
- Thus the maps are in general parametrized by six real parameters.

1.1.3 Moebius Transformations Are Bijective

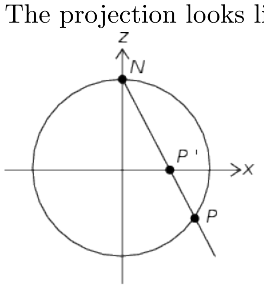
- Proof as Homework!

1.2 Geometric Interpretation

- Play around at (best with Chrome) $a_r \equiv \Re(a)$ and $a_i \equiv \Im(a)$:
  `http://www.phys.ethz.ch/~jshapiro/moebius.html`
- Or a video:
  `https://www.youtube.com/watch?v=0z1fIsUNhO4`
- If you have Mathematica this is a great visualization:
  `https://mathematica.stackexchange.com/questions/59271/mobius-transformations-revealed`

We can think of a Moebius transformation as consisting of the following procedures:

1. Let a point $z \in \mathbb{C}$ on the plane be given.
2. Project the point onto the unit sphere $S^2 \equiv \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \,\middle|\, x^2 + y^2 + z^2 = 1 \right\}$:
   (a) The projection looks like this (from a point $P' \in \mathbb{C}$ onto a point $P \in S^2$):



(b) It is given by the map $z \mapsto \begin{cases} \begin{bmatrix} \frac{2\Re(z)}{|z|^2+1} \\ \frac{2\Im(z)}{|z|^2+1} \\ \frac{|z|^2-1}{|z|^2+1} \end{bmatrix} & z \in \mathbb{C} \\ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & z = \infty \end{cases}$

3. Move the sphere to a new location in space (specified by three real numbers).
4. Rotate the sphere into a new orientation in space (specified by three real numbers–Euler angles).
5. Perform a stereographic projection from the new position of the sphere back to the plane.
6. All together, six real parameters.

2 Moebius Transformations Form a Group

2.1 What's a Group?

- A group G is a mathematical object
  - Consisting of:
    - A nonempty set S.
    - A map $S^2 \xrightarrow{c} S$ called *composition*, denoted by c.
  - Obeying the following conditions:
    - c is associative:
      - $c((a, c((b, c)))) = c((c((a, b)), c))$ for all $(a, b, c) \in S^3$.
    - $\exists e \in S$ such that $c((a, e)) = a$ and $c((e, a)) = a$ for all $a \in S$.
    - $\forall a \in S \exists \tilde{a} \in S$ such that $c((a, \tilde{a})) = e$ and $c((\tilde{a}, a)) = e$.

2.2 Moebius Transformations Indeed Form a Group

With a few (rather obvious) definitions we can "induce" a group out of $\mathcal{M}$.

- Our set is $\mathcal{M}$. It is nonempty because it contains the identity map!
- The composition map c is exactly composition of functions:
  - $c((z \mapsto f(z), z \mapsto g(z))) \equiv z \mapsto g(f(z))$
- Due to the fact that composition of functions is associative, our composition map is immediately associative.
- Is c well defined?
  - c is single-valued: holds because more generally function composition is a single-valued map.

– In order to verify this, we need to check that the range of c is indeed a subset of $\mathcal{M}$: that two Moebius transformation composed result in yet another Moebius transformation. This can either be seen very easily from the geometrical perspective or algebraically by the following proof.
Proof:

  * Let $(f_1,\, f_2) \in \mathcal{M}^2$ be given. We want to show that $f_1 \circ f_2 \in \mathcal{M}$.
  * *Case 1*: $(f_1,\, f_2) \in {\mathcal{M}_1}^2$.
    · If $z = \infty$, $f_2(z) = \infty$, in which case, $f_1 \circ f_2(z) = f_1(\infty) = \infty$.
    · Otherwise, write $f_i(z) = a_i z + b_i$ for all $i \in \{1,\, 2\}$. Because $(f_1,\, f_2) \in \mathcal{M}^2$ we know that $a_i \neq 0 \forall i \in \{1,\, 2\}$. Then for all $z \in \mathbb{C}$ we have:

$$\begin{aligned} f_1 \circ f_2(z) &= a_1(a_2 z + b_2) + b_1 \\ &= a_1 a_2 z + a_1 b_2 + b_1 \end{aligned}$$

    · This clearly defines an element in $\mathcal{M}_1$ because $\mathbb{C}$ is a field and so $a_1 a_2 \in \mathbb{C}$ and $(a_1 b_2 + b_1) \in \mathbb{C}$ and because $a_1 \neq 0$ and $a_2 \neq 0$ then $a_1 a_2 \neq 0$.
    · That is, $f_1 \circ f_2(z) = \begin{cases} a_1 a_2 z + a_1 b_2 + b_1 & z \in \mathbb{C} \\ \infty & z = \infty \end{cases}$ with $a_1 a_2 \neq 0$ and so $f_1 \circ f_2 \in \mathcal{M}_1 \subset \mathcal{M}$.
  * *Case 2*: $f_1 \in \mathcal{M}_1$ and $f_2 \in \mathcal{M}_2$.
    · Write $f_1(z) = \begin{cases} a_1 z + b_1 & z \in \mathbb{C} \\ \infty & z = \infty \end{cases}$ where $a_1 \neq 0$ and $f_2(z) = \begin{cases} \frac{a_2 z + b_2}{c_2 z + d_2} & z \in \mathbb{C} \backslash \left\{ -\frac{d_2}{c_2} \right\} \\ \infty & z = \frac{-d_2}{c_2} \\ \frac{a_2}{c_2} & z = \infty \end{cases}$ where $c_2 \neq 0$ and $a_2 d_2 - b_2 c_2 \neq 0$.
    · If $z = \infty$, $f_2(z) = \frac{a_2}{c_2}$ and so $(f_1 \circ f_2)(z) = a_1 \frac{a_2}{c_2} + b_1$.
    · If $z = -\frac{d_2}{c_2}$, $f_2(z) = \infty$ and so $(f_1 \circ f_2)(z) = f_1(\infty) = \infty$.
    · If $z \in \mathbb{C} \backslash \left\{ -\frac{d_2}{c_2} \right\}$,

$$\begin{aligned} f_1(f_2(z)) &= f_1\left(\frac{a_2 z + b_2}{c_2 z + d_2}\right) \\ &= a_1 \frac{a_2 z + b_2}{c_2 z + d_2} + b_1 \\ &= \frac{a_1 a_2 z + a_1 b_2 + b_1 c_2 z + b_1 d_2}{c_2 z + d_2} \\ &= \frac{(a_1 a_2 + b_1 c_2) z + a_1 b_2 + b_1 d_2}{c_2 z + d_2} \end{aligned}$$

    · We know that $a_1 \neq 0$ and that $a_2 d_2 - b_2 c_2 \neq 0$.
    What do we know about $(a_1 a_2 + b_1 c_2) d_2 - (a_1 b_2 + b_1 d_2) c_2$?

$$\begin{aligned} (a_1 a_2 + b_1 c_2) d_2 - (a_1 b_2 + b_1 d_2) c_2 &= a_1 a_2 d_2 + b_1 c_2 d_2 - a_1 b_2 c_2 - b_1 d_2 c_2 \\ &= a_1(a_2 d_2 - b_2 c_2) \\ &\neq 0 \end{aligned}$$

· Thus we find that $(f_1 \circ f_2)(z) = \begin{cases} a_1 \frac{a_2}{c_2} + b_1 & z = \infty \\ \infty & z = -\frac{d_2}{c_2} \\ \frac{(a_1 a_2 + b_1 c_2) z + a_1 b_2 + b_1 d_2}{c_2 z + d_2} & z \in \mathbb{C} \backslash \left\{ -\frac{d_2}{c_2} \right\} \end{cases}$ with $(a_1 a_2 + b_1 c_2) d_2 - (a_1 b_2 + b_1 d_2) c_2 \neq 0$ and again $c_2 \neq 0$. Due to the fact that $\mathbb{C}$ is a field we find again that $f_1 \circ f_2 \in \mathcal{M}_2$.

∗ *Case 3*: $f_1 \in \mathcal{M}_2$ and $f_2 \in \mathcal{M}_1$.

· Write $f_2(z) = \begin{cases} a_2 z + b_2 & z \in \mathbb{C} \\ \infty & z = \infty \end{cases}$ where $a_2 \neq 0$ and $f_1(z) = \begin{cases} \frac{a_1 z + b_1}{c_1 z + d_1} & z \in \mathbb{C} \backslash \left\{ -\frac{d_1}{c_1} \right\} \\ \infty & z = \frac{-d_1}{c_1} \\ \frac{a_1}{c_1} & z = \infty \end{cases}$ where $c_1 \neq 0$ and $a_1 d_1 - b_1 c_1 \neq 0$.

· If $z = \infty$, $f_2(z) = \infty$ and so $(f_1 \circ f_2)(z) = \frac{a_1}{c_1}$.

· If $z = -\frac{c_1 b_2 + d_1}{c_1 a_2}$, $f_2(z) = a_2 \left(-\frac{c_1 b_2 + d_1}{c_1 a_2} \right) + b_2 = -\frac{a_2 c_1 b_2 + a_2 d_1}{c_1 a_2} + b_2 = -\frac{d_1}{c_1}$ and so $(f_1 \circ f_2)(z) = f_1(\infty) = \infty$.

· If $z \in \mathbb{C} \backslash \left\{ -\frac{c_1 b_2 + d_1}{c_1 a_2} \right\}$,

$$\begin{aligned} f_1(f_2(z)) &= f_1(a_2 z + b_2) \\ &= \frac{a_1 (a_2 z + b_2) + b_1}{c_1 (a_2 z + b_2) + d_1} \\ &= \frac{a_1 a_2 z + a_1 b_2 + b_1}{c_1 a_2 z + c_1 b_2 + d_1} \end{aligned}$$

· We know that $a_2 \neq 0$ and that $a_1 d_1 - b_1 c_1 \neq 0$.
What do we know about $a_1 a_2 (c_1 b_2 + d_1) - (a_1 b_2 + b_1) c_1 a_2$?

$$\begin{aligned} a_1 a_2 (c_1 b_2 + d_1) - (a_1 b_2 + b_1) c_1 a_2 &= a_1 a_2 c_1 b_2 + a_1 a_2 d_1 - a_1 b_2 c_1 a_2 - b_1 c_1 a_2 \\ &= a_2 (a_1 d_1 - b_1 c_1) \\ &\neq 0 \end{aligned}$$

· Thus we find that $(f_1 \circ f_2)(z) = \begin{cases} \frac{a_1}{c_1} & z = \infty \\ \infty & z = -\frac{c_1 b_2 + d_1}{c_1 a_2} \\ \frac{a_1 a_2 z + a_1 b_2 + b_1}{c_1 a_2 z + c_1 b_2 + d_1} & z \in \mathbb{C} \backslash \left\{ -\frac{c_1 b_2 + d_1}{c_1 a_2} \right\} \end{cases}$ with $a_1 a_2 (c_1 b_2 + d_1) - (a_1 b_2 + b_1) c_1 a_2 \neq 0$ and $c_1 a_2 \neq 0$ (because $c_1 \neq 0$ and $a_2 \neq 0$). Due to the fact that $\mathbb{C}$ is a field we find again that $f_1 \circ f_2 \in \mathcal{M}_2$.

∗ *Case 4*: $(f_1, f_2) \in {\mathcal{M}_2}^2$:

· Write $f_i(z) = \begin{cases} \frac{a_i z + b_i}{z + d_i} & z \in \mathbb{C} \backslash \{-d_i\} \\ \infty & z = -d_i \\ a_i & z = \infty \end{cases}$ $a_i d_i \neq b_i$ for all $i \in \{1, 2\}$. We can do this, because, as we mentioned above, for $\mathcal{M}_2$, the maps are parametrized by six real parameters exactly.

· *Case 4.1*: $a_2 = -d_1$:

1. If $z = \infty$, then $f_2(z) = a_2 = -d_1$ and so $(f_1 \circ f_2)(z) = \infty$.

2. Otherwise if $z \in \mathbb{C}$ then

$$\begin{aligned} f_1(f_2(z)) &= f_1\left(\frac{a_2 z + b_2}{z + d_2}\right) \\ &= \frac{a_1 \frac{a_2 z + b_2}{z + d_2} + b_1}{\frac{a_2 z + b_2}{z + d_2} + d_1} \\ &= \frac{a_1 a_2 z + a_1 b_2 + z b_1 + d_2 b_1}{a_2 z + b_2 + z d_1 + d_2 d_1} \\ &= \frac{(a_1 a_2 + b_1) z + a_1 b_2 + d_2 b_1}{\underbrace{(a_2 + d_1)}_{0} z + b_2 + d_2 d_1} \\ &= \frac{(a_1 a_2 + b_1) z + a_1 b_2 + d_2 b_1}{b_2 + d_2 d_1} \\ &= \frac{(-a_1 d_1 + b_1) z + a_1 b_2 + d_2 b_1}{b_2 + d_2 d_1} \end{aligned}$$

3. First, we need to show that this makes sense, that is, that under these circumstances it's impossible that $b_2 + d_2 d_1 = 0$. To that end, assume otherwise, that is, that $b_2 + d_2 d_1 = 0$. Then $b_2 = -d_2 d_1$. But we know that $b_2 \neq a_2 d_2$ so that we know that $a_2 d_2 \neq -d_2 d_1$. Since we know that $a_2 = -d_1$ this can never happen.
4. In addition we also know that $-a_1 d_1 + b_1 \neq 0$ by assumption on $f_1 \in \mathcal{M}_2$ and so $f_1 \circ f_2 \in \mathcal{M}_1 \subset \mathcal{M}$ as desired.

· *Case 4.2:* $a_2 \neq d_1$:

1. If $z = \infty$, then $f_2(z) = a_2$ and so $(f_1 \circ f_2)(z) = \frac{a_1 a_2 + b_1}{a_2 + d_1}$.
2. If $z = -\frac{b_2 + d_2 d_1}{a_2 + d_1}$, then

$$\begin{aligned} f_2(z) &= \frac{a_2\left(-\frac{b_2 + d_2 d_1}{a_2 + d_1}\right) + b_2}{\left(-\frac{b_2 + d_2 d_1}{a_2 + d_1}\right) + d_2} \\ &= \frac{-a_2 b_2 - a_2 d_2 d_1 + b_2 a_2 + b_2 d_1}{-b_2 - d_2 d_1 + d_2 a_2 + d_2 d_1} \\ &= \frac{-a_2 d_2 d_1 + b_2 d_1}{-b_2 + d_2 a_2} \\ &= -d_1 \end{aligned}$$

and so $f_1(f_2(z)) = f_1(-d_1) = \infty$.

3. If $z \in \mathbb{C} \backslash \left\{-\frac{b_2 + d_2 d_1}{a_2 + d_1}\right\}$ then

$$\begin{aligned} f_1(f_2(z)) &= f_1\left(\frac{a_2 z + b_2}{z + d_2}\right) \\ &= \frac{a_1 \frac{a_2 z + b_2}{z + d_2} + b_1}{\frac{a_2 z + b_2}{z + d_2} + d_1} \\ &= \frac{a_1 a_2 z + a_1 b_2 + z b_1 + d_2 b_1}{a_2 z + b_2 + z d_1 + d_2 d_1} \\ &= \frac{(a_1 a_2 + b_1) z + a_1 b_2 + d_2 b_1}{(a_2 + d_1) z + b_2 + d_2 d_1} \end{aligned}$$

4. We know that $a_2 \neq -d_1$ which already satisfies one condition for $f_1 \circ f_2$ being in $\mathcal{M}_2$.
5. Next, we would like to ascertain that $(a_1a_2+b_1)(b_2+d_2d_1)-(a_1b_2+d_2b_1)(a_2+d_1) \neq 0$:

$$\begin{aligned}(a_1a_2+b_1)(b_2+d_2d_1)-(a_1b_2+d_2b_1)(a_2+d_1) &= a_1a_2b_2+a_1a_2d_2d_1+b_1b_2+b_1d_2d_1\\ &\quad -a_1b_2a_2-a_1b_2d_1-d_2b_1a_2-d_2b_1d_1\\ &= a_1a_2d_2d_1+b_1b_2-a_1b_2d_1-d_2b_1a_2\\ &= a_1d_1(a_2d_2-b_2)+b_1(b_2-a_2d_2)\\ &= (a_2d_2-b_2)(a_1d_1-b_1)\\ &\neq 0\end{aligned}$$

   which works out beautifully.
6. As a result, we have fulfilled all the requirements for $f_1 \circ f_2$ to be in $\mathcal{M}_2$.

■

- Intuitively we expect that the identity element e would be $e \equiv z \mapsto z \equiv \mathbb{1}_{\mathbb{C}\cup\{\infty\}}$. Let us indeed verify that:
  - Let $f \in \mathcal{M}$ be given. $c\left(\left(\mathbb{1}_{\mathbb{C}\cup\{\infty\}},\, f\right)\right) = \mathbb{1}_{\mathbb{C}\cup\{\infty\}} \circ f = f$ and $c\left(\left(f,\, \mathbb{1}_{\mathbb{C}\cup\{\infty\}}\right)\right) = f \circ \mathbb{1}_{\mathbb{C}\cup\{\infty\}} = f$.
- The last remaining property to show that we indeed defined a group is to find inverses:
  - Let $f \in \mathcal{M}$ be given.
  - *Case 1*: $f \in \mathcal{M}_1$.
    * Write $f(z) = \begin{cases} az+b & z \in \mathbb{C} \\ \infty & z = \infty \end{cases}$ with $a \neq 0$. Then define $\tilde{f}(z) := \begin{cases} \frac{z-b}{a} & z \in \mathbb{C} \\ \infty & z = \infty \end{cases}$. Because $a \neq 0$ then $\frac{1}{a} \neq 0$ and so $\tilde{f} \in \mathcal{M}_1$.
    * If $z \in \mathbb{C}$ then $\left(f \circ \tilde{f}\right)(z) = f\left(\frac{z-b}{a}\right) = a\left(\frac{z-b}{a}\right) + b = z$ and $\left(\tilde{f} \circ f\right)(z) = \tilde{f}(az+b) = \frac{(az+b)-b}{a} = z$.
    * If $z = \infty$ then $\left(f \circ \tilde{f}\right)(z) = f(\infty) = \infty$ and $\left(\tilde{f} \circ f\right)(z) = \tilde{f}(\infty) = \infty$.
  - *Case 2*: $f \in \mathcal{M}_2$.
    * Write $f(z) = \begin{cases} \frac{az+b}{z+d} & z \in \mathbb{C}\setminus\{-d\} \\ \infty & z = -d \\ a & z = \infty \end{cases}$ where $ad \neq b$. Then define $\tilde{f}(z) := \begin{cases} \frac{dz-b}{-z+a} & z \in \mathbb{C}\setminus\{a\} \\ \infty & z = a \\ -d & z = \infty \end{cases}$. Because $-1 \neq 0$ the first condition for $\mathcal{M}_2$ is fulfilled. For the second condition we would want that $ad - (-b)(-1) \neq 0$ which is true because $f \in \mathcal{M}_2$.

∗ If $z \in \mathbb{C}\setminus\{a\}$ then

$$\begin{aligned}\left(f\circ\tilde{f}\right)(z) &= f\left(\frac{dz-b}{-z+a}\right)\\ &\overset{*}{=} \frac{a\left(\frac{dz-b}{-z+a}\right)+b}{\left(\frac{dz-b}{-z+a}\right)+d}\\ &= \frac{adz-ab-bz+ab}{dz-b-dz+da}\\ &= \frac{adz-bz}{-b+da}\\ &= z\end{aligned}$$

(observe that ∗ we justified because $ad\neq b$ implies $\frac{dz-b}{-z+a}\neq -d$ for any $z\in\mathbb{C}\setminus\{a\}$, and so we have used the right line in the application of f)

∗ If $z=a$ then $\left(f\circ\tilde{f}\right)(z)=f(\infty)=a$.

∗ If $z \in \mathbb{C}\setminus\{-d\}$ then

$$\begin{aligned}\left(\tilde{f}\circ f\right)(z) &= \tilde{f}\left(\frac{az+b}{z+d}\right)\\ &\overset{*}{=} \frac{d\left(\frac{az+b}{z+d}\right)-b}{-\left(\frac{az+b}{z+d}\right)+a}\\ &= \frac{daz+db-bz-bd}{-az-b+az+ad}\\ &= \frac{daz-bz}{-b+ad}\\ &= z\end{aligned}$$

(where similarly ∗ was justified because we assume that $\frac{az+b}{z+d}\neq a$ which follows from $ad\neq b$ again).

∗ If $z=\infty$ then $\left(f\circ\tilde{f}\right)(z)=f(-d)=\infty$ and $\left(\tilde{f}\circ f\right)(z)=\tilde{f}(a)=\infty$.

∗ Thus $f\circ\tilde{f}=\mathbb{1}_{\mathbb{C}\cup\{\infty\}}$.

3 Moebius Transformations Send Circles and Straight Lines to Circles and Straight Lines

- In the fourth homework sheet we have seen that $z\mapsto\frac{1}{z}$ maps circles and straight lines into circles and straight lines.
- Assuming this fact, we can build any Moebius transformation as a composition of simpler Moebius transformations, each of which respects this fact, and so the total composition must also respect this fact.
- First, it is clear that elements in $\mathcal{M}_1$ obey the condition because they merely correspond to translation, scaling, and rotation, all of which preserve the geometric shapes.

- If we are given a map in $\mathcal{M}_2$, $f(z) = \begin{cases} \frac{az+b}{z+d} & z \in \mathbb{C}\setminus\{-d\} \\ \infty & z = -d \\ a & z = \infty \end{cases}$ where $ad \neq b$, then we can construct it as:

  1. Translation by d (a map in $\mathcal{M}_1$): $z \mapsto z + d$
  2. Inversion (a map in $\mathcal{M}_2$, but a special one which we dealt with in the homework!): $z + d \mapsto \frac{1}{z+d}$.
  3. Scaling and rotation by $b - ad \neq 0$ (a map in $\mathcal{M}_1$): $\frac{1}{z+d} \mapsto (b - ad)\frac{1}{z+d}$.
  4. Translation by a (a map in $\mathcal{M}_1$):

$$\begin{aligned} (b-ad)\frac{1}{z+d} & \mapsto & (b-ad)\frac{1}{z+d} + a \\ & = & \frac{az+b}{z+d} \end{aligned}$$

4 Moebius Transformations Determined Completely from their Values on Just Three Points of $\mathbb{C}$

Claim: Given $(z_1, z_2, z_3, w_1, w_2, w_3) \in (\mathbb{C} \cup \{\infty\})^6$ such that $z_1 \neq z_2 \wedge z_1 \neq z_3 \wedge z_2 \neq z_3$ and $w_1 \neq w_2 \wedge w_1 \neq w_3 \wedge w_2 \neq w_3$ there exists a *unique* element of $\mathcal{M}$ which maps $z_i \to w_i$ for all $i \in \{1, 2, 3\}$.
Proof:

- Define $f(z) = \begin{cases} \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} & z \in \mathbb{C}\setminus\{z_3\} \\ \infty & z = z_3 \\ \frac{z_2-z_3}{z_2-z_1} & z = \infty \end{cases}$ for all $z \in \mathbb{C} \cup \{\infty\}$.
- *Claim:* $f \in \mathcal{M}_2 \subset \mathcal{M}$.
  Proof:
  - Note that

$$\begin{aligned} f(z) & \equiv & \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} \\ & = & \frac{(z_2-z_3)z - z_1(z_2-z_3)}{(z_2-z_1)z - z_3(z_2-z_1)} \end{aligned}$$

  - Because z_i are all distinct, $z_2 - z_1 \neq 0$, and so we already fulfill the first condition for a map in $\mathcal{M}_2$.
  - For the second condition, $(z_2 - z_3)[-z_3(z_2 - z_1)] - [-z_1(z_2 - z_3)(z_2 - z_1)]$, note that

$$\begin{aligned} (z_2-z_3)[-z_3(z_2-z_1)] - [-z_1(z_2-z_3)(z_2-z_1)] & = & (z_2-z_3)(z_2-z_1)(-z_3) + z_1(z_2-z_3)(z_2-z_1) \\ & = & (z_2-z_3)(z_2-z_1)(z_1-z_3) \\ & \neq & 0 \end{aligned}$$

- *Claim:* Under f we have: $z_1 \overset{f}{\mapsto} 0$, $z_2 \overset{f}{\mapsto} 1$ and $z_3 \overset{f}{\mapsto} \infty$.
  Proof:

  - The last condition is true by definition.
  - When we feed z_1 we clearly get 0.
  - When we feed z_2, we clearly get 1.

- There is a similar map in $\mathcal{M}_2$, g, which maps $w_1 \overset{g}{\mapsto} 0$, $w_2 \overset{g}{\mapsto} 1$ and $w_3 \overset{g}{\mapsto} \infty$.
- Because the Moebius transformations are a group, $g^{-1} \circ f$ will map $z_1 \overset{f}{\mapsto} 0 \overset{g^{-1}}{\mapsto} w_1$. and so on: $z_i \overset{g^{-1} \circ f}{\mapsto} w_i$ for all $i \in \{1,\, 2,\, 3\}$.
- So we have shown that the sought after map exists! What about uniqueness?
- Assume we found another map $h \in \mathcal{M}$ which maps $z_i \overset{h}{\mapsto} w_i$ for all $i \in \{1,\, 2,\, 3\}$.
- Then $g \circ h \circ f^{-1}$ will map $0 \overset{f^{-1}}{\mapsto} z_1 \overset{h}{\mapsto} w_1 \overset{g}{\mapsto} 0$, $1 \overset{f^{-1}}{\mapsto} z_2 \overset{h}{\mapsto} w_2 \overset{g}{\mapsto} 1$ and $\infty \overset{f^{-1}}{\mapsto} z_3 \overset{h}{\mapsto} w_3 \overset{g}{\mapsto} \infty$.
- Recall that $h \in \mathcal{M}$ so we could write it as $\left(g \circ h \circ f^{-1}\right)(z) = \begin{cases} \frac{az+b}{z+d} & z \in \mathbb{C} \setminus \{-d\} \\ \infty & z = -d \\ a & z = \infty \end{cases}$ where $ad \neq b$ or $\left(g \circ h \circ f^{-1}\right)(z) = \begin{cases} az+b & z \in \mathbb{C} \\ \infty & z = \infty \end{cases}$ with $a \neq 0$.
- We know that $\infty \mapsto \infty$ so that it must be the second possibility (i.e. $g \circ h \circ f^{-1} \in \mathcal{M}_1$).
- We know that $0 \mapsto 0$ so that $b = 0$ necessarily.
- We know that $1 \mapsto 1$ so that $a = 1$ necessarily.
- As a result, $g \circ h \circ f^{-1} = \mathbb{1}_{\mathbb{C} \cup \{\infty\}}$, which means $h = g^{-1} \circ f$, that is, h is the same map we constructed ourselves.
- Because h was aribitrary, all such maps will be the very same map we constructed, and hence, $g^{-1} \circ f$ is unique.

■

5 Moebius Transformations Retain the Cross-Ratio

Claim: Let $f \in \mathcal{M}$ be given and let $z_i \in \mathbb{C}$ for all $i \in \{1,\, 2,\, 3,\, 4\}$ such that z_i are all distinct and $f(z_i) \neq \infty \forall i \in \{1,\, 2,\, 3,\, 4\}$. Then

$$\frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)} = \frac{[f(z_1) - f(z_3)][f(z_2) - f(z_4)]}{[f(z_2) - f(z_3)][f(z_1) - f(z_4)]}$$

Proof:

- *Case 1:* $f \in \mathcal{M}_1$.
  - Write $f(z) = \begin{cases} az+b & z \in \mathbb{C} \\ \infty & z = \infty \end{cases}$ with $a \neq 0$.
  - Left as homework exercise.

- *Case 2:* $f \in \mathcal{M}_2$.
  - Write $f(z) = \begin{cases} \frac{az+b}{z+d} & z \in \mathbb{C}\setminus\{-d\} \\ \infty & z = -d \\ a & z = \infty \end{cases}$ where $ad \neq b$.
  - *Claim:* $f(z_i) - f(z_j) = \frac{(z_i - z_j)(ad-b)}{(z_i+d)(z_j+d)}$ if $z_i \in \mathbb{C}\setminus\{-d\}$ and $z_j \in \mathbb{C}\setminus\{-d\}$.
    Proof:
    * Calculate

$$\begin{aligned} f(z_i) - f(z_j) &= \frac{az_i + b}{z_i + d} - \frac{az_j + b}{z_j + d} \\ &= \frac{(az_i + b)(z_j + d) - (az_j + b)(z_i + d)}{(z_i + d)(z_j + d)} \\ &= \frac{az_i z_j + adz_i + bz_j + bd - az_j z_i - adz_j - bz_i - bd}{(z_i + d)(z_j + d)} \\ &= \frac{(z_i - z_j)(ad - b)}{(z_i + d)(z_j + d)} \end{aligned}$$

  - Insert this calculation into the cross-ratio to obtain:

$$\begin{aligned} \frac{[f(z_1) - f(z_3)][f(z_2) - f(z_4)]}{[f(z_2) - f(z_3)][f(z_1) - f(z_4)]} &= \frac{\left[\frac{(z_1-z_3)(ad-b)}{(z_1+d)(z_3+d)}\right]\left[\frac{(z_2-z_4)(ad-b)}{(z_2+d)(z_4+d)}\right]}{\left[\frac{(z_2-z_3)(ad-b)}{(z_2+d)(z_3+d)}\right]\left[\frac{(z_1-z_4)(ad-b)}{(z_1+d)(z_4+d)}\right]} \\ &= \frac{\left[\frac{(z_1-z_3)}{(z_1+d)(z_3+d)}\right]\left[\frac{(z_2-z_4)}{(z_2+d)(z_4+d)}\right]}{\left[\frac{(z_2-z_3)}{(z_2+d)(z_3+d)}\right]\left[\frac{(z_1-z_4)}{(z_1+d)(z_4+d)}\right]} \\ &= \frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)} \end{aligned}$$

■

6 Pretty Examples (Homework)

- Maps which send the unit circle to the upper half plane.
- Maps which send the upper halfplane into itself.