Analysis 1
Colloquium of the Sixth Week
Metric Spaces on S^2 and $\mathbb{R}P^2$

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1 Preface

1.1 Recall What a Metric Is

Let X be a set. A metric d on X is a map $d: X \times X \to \mathbb{R}$ such that $\forall (p,\, q) \in X^2$:

1. $p \neq q \Longrightarrow d(p,\, q) > 0$
2. $d(p,\, p) = 0$
3. $d(p,\, q) = d(q,\, p)$
4. $d(p,\, q) \leq d(p,\, r) + d(r,\, q)$ for any $r \in X$.

- Examples:

1. *Claim:* Let X be a set and let d be a metric on X. Let $A \subseteq X$ be a subset of X. Then d is also a metric on A.
2. *Claim:* If $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$ (that is, $\|v\| \equiv \sqrt{\sum_{j=1}^n |v_j|^2}$) then $d(u,\, v) := \|u - v\|$ is a metric on $\mathbb{R}^n$, that is, $d(u,\, v) = \sqrt{\sum_{j=1}^n |u_j - v_j|^2}$
   Proof: homework.
3. Let X be the set of all words in English. Each English word can be encoded as a finite sequence of digits $(w_i)_{i=1}^{N_w}$ where $w_i \in \{a,\, b,\, c,\, \dots,\, x,\, y,\, z\}$ for all $i \in \{1,\, \dots, N_w\}$. For example, the word "apple" will be the sequence $w_1 =' a',\, w_2 =' p',\, w_3 =' p',\, w_4 =' l',\, w_5 =' e'$ and we would also have $N_w = 5$. The Levenshtein metric measures the minimum number of single-character edits (i.e. insertions, deletions or substitutions) required to change one word into the other. Define for any two words $(w_i)_{i=1}^{N_w}$ and $(z_i)_{i=1}^{N_z}$ their distance as:

$$d_{Lev}\left((w_i)_{i=1}^{N_w},\, (z_i)_{i=1}^{N_z}\right) := \begin{cases} \max(\{N_w,\, N_z\}) & \text{if } \min(\{N_w,\, N_z\}) = 0 \\ \min\left(\left\{\begin{array}{c} d_{Lev}\left((w_i)_{i=1}^{N_w-1},\, (z_i)_{i=1}^{N_z}\right) + 1, \\ d_{Lev}\left((w_i)_{i=1}^{N_w},\, (z_i)_{i=1}^{N_z-1}\right) + 1, \\ d_{Lev}\left((w_i)_{i=1}^{N_w-1},\, (z_i)_{i=1}^{N_z-1}\right) + \left(1 - \delta_{w_{N_w},\, z_{N_z}}\right) \end{array}\right\}\right) & \text{otherwise} \end{cases}$$

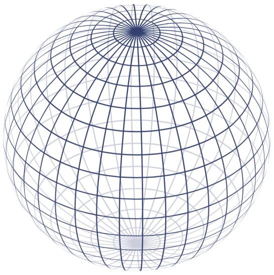
   A few examples are in order:
   - The distance between any word of length N and the empty word is just the length of the first word, namely, N.
   - The distance between "book" and "back" is 2, because be had to change two characters to get from one to the other.
   - The distance between two identical words is 0 (as you could verify by using the formula), and two non-identical strings will always greater longer than zero distance.

Proof: homework.

1.2 Two Important Sets

1.2.1 The 2-Sphere S^2

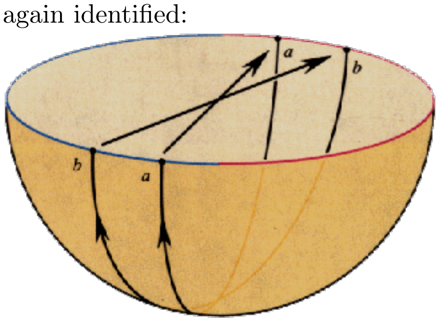
We define a subset of $\mathbb{R}^3$, the two-sphere, defined as $S^2 \equiv \left\{\, x \in \mathbb{R}^3 \;\middle|\; d(x,\, 0) = 1 \,\right\}$, where d is the Euclidean metric as defined above.



- We can think of the two-sphere as the product of a "one point compactification" of $\mathbb{C}$ or $\mathbb{R}^2$, $\mathbb{C} \cup \{\infty\}$.
- The set S^2 is compact in the sense that it is closed and bounded: Closed in the sense that its complement $\mathbb{R}^3 \backslash S^2$ is open (because given any point not on the surface of the two-sphere in $\mathbb{R}^3$, we can always find a small enough open ball around that point in $\mathbb{R}^3$ that will not touch the two-sphere), and bounded in the sense that we can always put it in a large enough box and it will be completely contained inside of it.
- Note how this was not true for the set we started with $\mathbb{R}^2$: it was not bounded–you couldn't put it inside any large enough box.
- This "compactification" is done via the stereographic projection: $(X, Y) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$ where X and Y are now coordinates in $\mathbb{R}^2$ where as x, y and z are coordinates in $\mathbb{R}^3$ or $(x, y, z) = \left(\frac{2X}{1+X^2+Y^2}, \frac{2Y}{1+X^2+Y^2}, \frac{-1+X^2+Y^2}{1+X^2+Y^2}\right)$ going the other way around. This is almost a bijection, except that the point $z = 1$ gets sent outside of $\mathbb{R}^2$ as you can see.

1.2.2 The Real Projective Plane $\mathbb{R}P^2$

- The real projective plane $\mathbb{R}P^2$ is equivalently as either one of the following:
  - The set of all straight lines going through the origin in the plane $\mathbb{R}^3$ (so we care only about the direction of the lines: that is the information that is being retained by the elements of the set).
  - The two-sphere S^2 with antipodal points identified (that means, if two points are antipodal then they are the same element).
  - All the points in the southern hemisphere of S^2 union with its boundary (a circle), but on the boundary, antipodal points again identified:



  - All points in the interior of the two-disk, union with the boundary, but the boundary (a circle) having antipodal points identified.
- See how this is again a compactification of $\mathbb{R}^2$ or $\mathbb{C}$, but this time it's certainly not a "one-point compactification": we add all the points on a circle at the boundary at infinity, that is, all possible directions in which a line can go to infinity, which is infinitely many points at infinity.
- This is part of a larger topic called projective geometry: a way to do geometry where all lines intersect. This means parallel lines must intersect somewhere: they intersect at infinity.

2 Metrics on $\mathbb{R}^2$ Induced by Metrics on S^2 and $\mathbb{R}P^2$

2.1 Metrics Induced from S^2

We will define some metrics on S^2 and see what kind of distance they correspond to in $\mathbb{R}^2$ when performing the stereographic projection back.

2.1.1 Euclidean Metric on $\mathbb{R}^3$ and Thus on S^2–Woodworm Metric

The Euclidean metric on $\mathbb{R}^3$, $d_{Euc}(u, v) \equiv \sqrt{(u_x - v_x)^2 + (u_y - v_y)^2 + (u_z - v_z)^2}$ induces a metric on S^2, because $S^2 \subset \mathbb{R}^3$.

- What kind of metric does this induce on $\mathbb{R}^2$ via the stereographic projection back?

2.1.2 Arclength Metric on S^2–Ant Metric

The arclength on the sphere of radius 1 is just the angle spanned by two given unit-vectors.

$$\begin{aligned} d_{arcl}(u, v) &:= \arccos(u \cdot v) \\ &\equiv \arccos(u_x v_x + u_y v_y + u_z v_z) \end{aligned}$$

. This is because $u \cdot v = \|u\| \|v\| \cos(\alpha)$ where α is the angle between u and v. However, as u and v lie on S^2, they have norm 1.

- *Claim:* This indeed defines a metric on S^2.
- What kind of metric does this induce on $\mathbb{R}^2$ via the stereographic projection back?

2.1.3 Discrete Metric on S^2–Flea Metric

Define

$$d_{discr}(u, v) := \begin{cases} 0 & u = v \\ 1 & u \neq v \end{cases}$$

- *Claim:* This indeed defines a metric on S^2.
- What kind of metric does this induce on $\mathbb{R}^2$ via the stereographic projection back?

2.1.4 The Floor Arclength Metric on S^2–Small Flea Metric

Define

$$d_{farclength}(u, v) := \lfloor d_{arcl}(u, v) \rfloor$$

where $\lfloor x \rfloor$ is the largest integer larger than or equal to x: $\lfloor 3.14 \rfloor = 3$ and $\lfloor 0.9 \rfloor = 0$.

- *Claim:* This indeed defines a metric on S^2.
- What kind of metric does this induce on $\mathbb{R}^2$ via the stereographic projection back?

2.1.5 Post Office Metric on S^2

Define

$$d_{post\,office}(u, v) := \begin{cases} d_{arcl}(0, u) + d_{arcl}(0, v) & u \neq v \\ 0 & u = v \end{cases}$$

- *Claim:* This indeed defines a metric on S^2.
- What kind of metric does this induce on $\mathbb{R}^2$ via the stereographic projection back?

2.2 Metrics Induced from $\mathbb{R}P^2$

For each metric d on S^2, we can define a metric on $\mathbb{R}P^2$ via the following:

- If we employ the characterization of $\mathbb{R}P^2$ as the set of all points of S^2 where antipodal points are identified, then we can write $\mathbb{R}P^2 = \left\{ \{x, -x\} \subset \mathbb{R}^3 \,\middle|\, x \in S^2 \right\}$.
- Using this characterization, define the distance between two "points" $\{u, -u\}$ and $\{v, -v\}$ as:

$$d_{\mathbb{R}P^2}(\{u, -u\}, \{v, -v\}) := \min(d(u, v), d(-u, v), d(u, -v), d(-u, -v))$$

- *Claim:* This indeed defines a metric on $\mathbb{R}P^2$, given that d is a bonafide metric on S^2.
- For each of the examples above given for S^2, what kind of metric does this induce on $\mathbb{R}^2$?